Exotic Statistics and Particle Types in 3- and 4d BF Theory

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For references and further details, see my paper with John Baez & Alissa Crans:

Exotic statistics for strings in 4d BF theory
gr-qc/0603085

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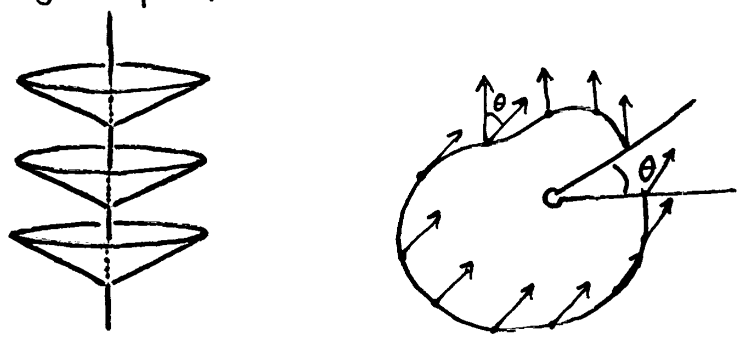
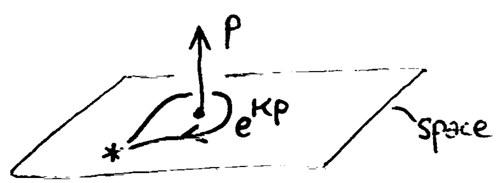
Momentum in 3D Gravity

A point particle in 3d GR bends space into a cone with deficit angle proportional to its mass



Using this fact and arguing in the particle's rest frame, one can show that the holonomy of the flat Lorentz connection around the particle is (up to conjugation)

$$e^{\kappa p} \in SO_0(2,1)$$



where

κ is (proportional to) Newton's constant

p is the particle's energy-momentum, thought of as an element of

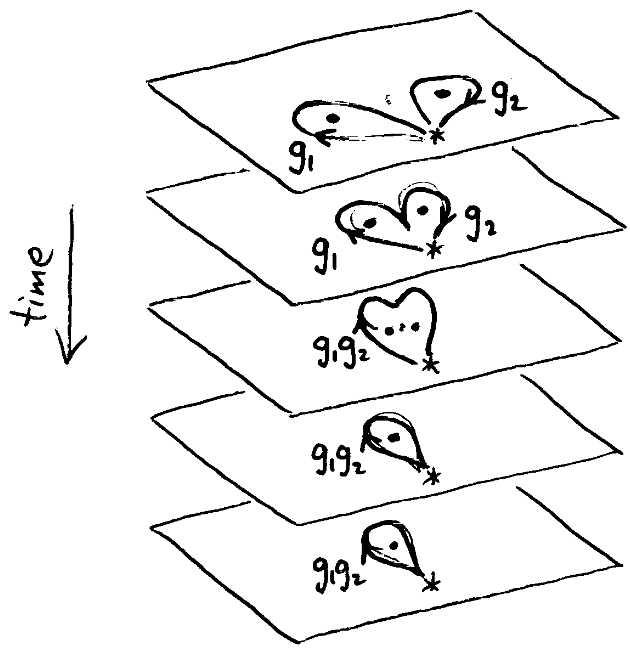
$$\mathfrak{so}(2,1) = \left\{ \begin{pmatrix} 0 & p_y & p_x \\ p_y & 0 & E \\ p_x & -E & 0 \end{pmatrix} : p_x, p_y, E \in \mathbb{R} \right\} \cong \mathbb{R}^{2,1}$$

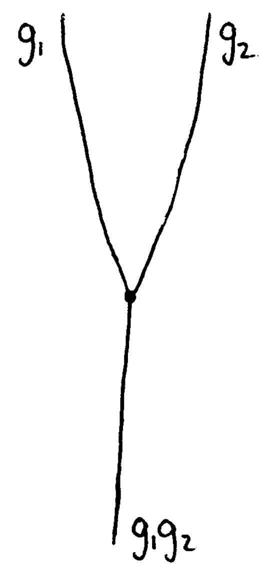
with metric $\langle p, p' \rangle = \frac{1}{2} \operatorname{tr}(pp')$

In fact, it's good to think of the <u>group element</u> $e^{\kappa p}$ as the momentum in 3d gravity! Why? ...

Conservation of Group-Valued Energy-Momentum

In a collision:



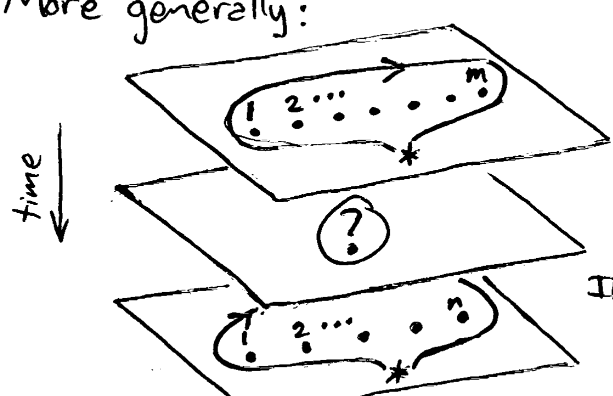


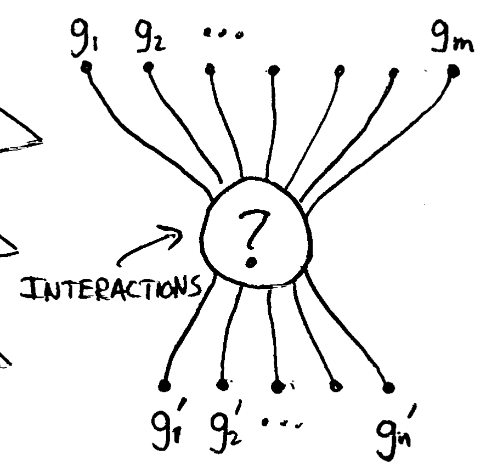
so 'group valued momenta' get conserved in a multiplicative sense. By the Baker-Campbell-Hausdorff formula

$$g_1 g_2 = e^{\kappa p_1} e^{\kappa p_2} = e^{\kappa(p_1+p_2) + \frac{\kappa^2}{2}[p_1, p_2] + \cdots}$$

this conservation of group-valued momenta can be viewed as a correction (cf. DSR) to addition of vector valued momenta which is small when κ is small or p_1, p_2 are nearly proportional.

More generally:

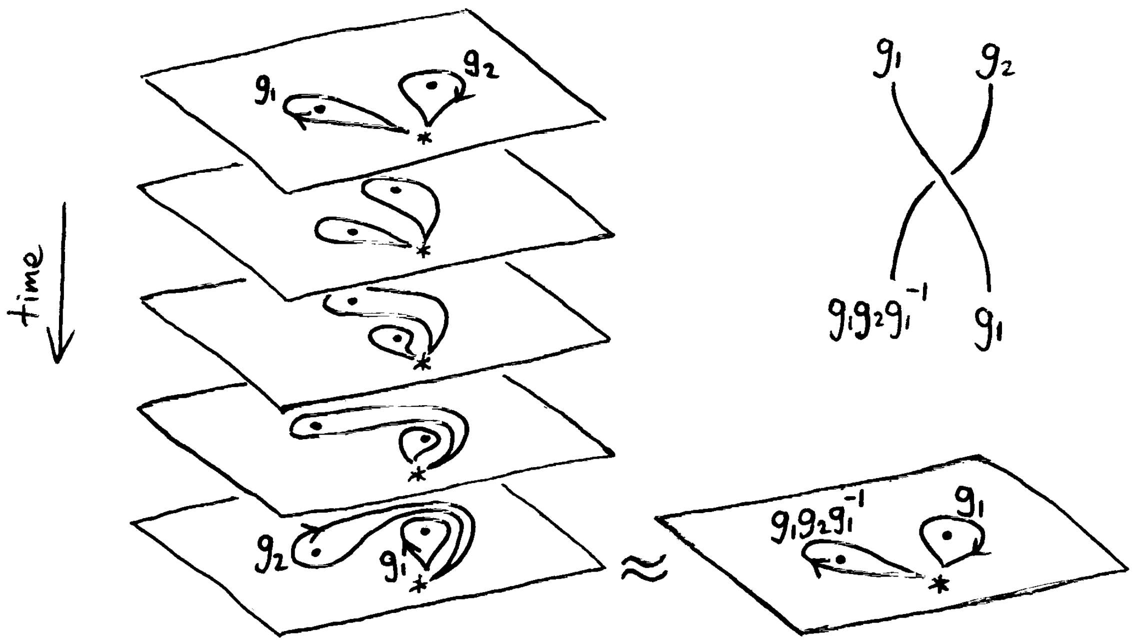




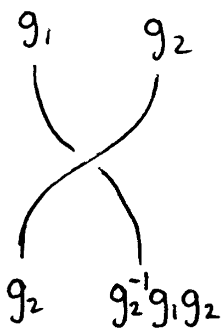
$$g_1 g_2 \cdots g_m = g_1' g_2' \cdots g_n'$$

Consequences of Group-Valued Energy-Momentum

1) Since multiplication in $SO(2,1)$ is not commutative:



and similarly:

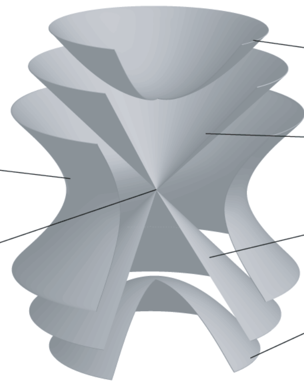


So we get an action of the <u>braid group</u> B_n on the group valued energy-momenta of a collection of n particles. This leads to 'exotic statistics' for point particles in 3d quantum gravity.

2) In 3d special relativity, identifying Minkowski vector space $\mathbb{R}^{2,1}$ with $so(2,1)$, the action of $SO(2,1)$ as Lorentz transformations

$$SO(2,1) \times so(2,1) \longrightarrow so(2,1)$$

is just the adjoint action of $SO(2,1)$. Mass shells in 3d SR are thus precisely the adjoint orbits in $so(2,1)$:



When we turn on gravity the space of energy-momenta becomes $SO(2,1)$ and "mass shells" become <u>conjugacy classes</u> in $SO(2,1)$. Near the identity — i.e. for small energy-momentum — these conjugacy classes look just like the above picture. But for high energies we see a modification of particle types!

For example:

a tardyon in its rest frame has momentum

$$p = (m, 0, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m \\ 0 & -m & 0 \end{pmatrix} \in \mathfrak{so}(2,1)$$

and hence group-valued momentum

$$e^{\kappa p} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \kappa m & \sin \kappa m \\ 0 & -\sin \kappa m & \cos \kappa m \end{pmatrix} \in SO(2,1)$$

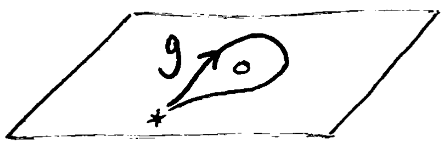
But this does not change if we increase m by $\frac{2\pi}{\kappa}$! Masses for tardyons are thus not elts of $[0,\infty)$ but of $\mathbb{R}/\frac{2\pi}{\kappa}\mathbb{Z}$ — there's a circle's worth of tardyon "mass shells", i.e. conjugacy classes of elements of the form $e^{\kappa p}$ for timelike p.

In the paper with Baez & Crans, we classify all of the spin-0 particle types in 3d quantum gravity, by describing the conjugacy classes of $SO_0(2,1)$ (and its double cover $SL(2,\mathbb{R})$).

Generalizing to 3d BF Theory

All of what we've discussed so far generalizes immediately to 3d BF theory with arbitrary gauge group G, with "particles" as punctures in space ($\cong \mathbb{R}^2$):

- we <u>define</u> the group-valued momentum to be the holonomy of the flat connection around a particle:



- we still get exotic statistics, provided G is nonabelian.
- "particle types" (ignoring 'spin') are conjugacy classes in G.

Particularly interesting examples: 3d gravity with cosmological constant has a BF formulation (related to MacDowell-Mansouri gravity) with gauge group

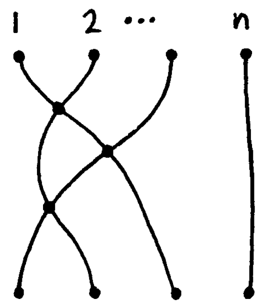
$$G = \begin{cases} SO(3,1) & \Lambda > 0 \\ ISO(2,1) & \Lambda = 0 \\ SO(2,2) & \Lambda < 0 \end{cases}$$

(or Riemannian analogs)

But we would like to generalize all of this to BF theories in higher dimensions, beginning with generalizing the idea of "exotic statistics"...

Permutation Group Statistics

Regular fermionic / bosonic statistics are governed by representations of the symmetric groups S_n.

$S =$ 

permutation diagram

gives an operator

$$\begin{array}{c} \mathcal{H}^{\otimes n} \\ \big\downarrow \rho(s) \\ \mathcal{H}^{\otimes n} \end{array}$$

We get the Fock space

$$\mathcal{F} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}^{\otimes n} / S_n$$

where S_n acts either by the bosonic rep

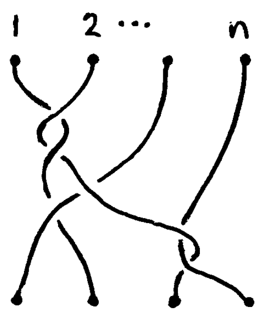
$$\rho(s) : \psi_1 \otimes \cdots \otimes \psi_n \longmapsto \psi_{s(1)} \otimes \cdots \otimes \psi_{s(n)}$$

or the fermionic rep

$$\rho(s) : \psi_1 \otimes \cdots \otimes \psi_n \longmapsto \text{sign}(s)\, \psi_{s(1)} \otimes \cdots \otimes \psi_{s(n)}$$

Braid Group Statistics

When space is 2d, "exotic statistics" are possible because it makes sense to ask <u>in which direction</u> we switched two particles – clockwise or counterclockwise. We get statistics governed by the braid groups B_n.



braid diagram

gives

$$\mathcal{H}^{\otimes n} \xrightarrow{\rho(\sigma)} \mathcal{H}^{\otimes n}$$

We get a "braided Fock space"

$$\mathcal{F} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}^{\otimes n} / B_n$$

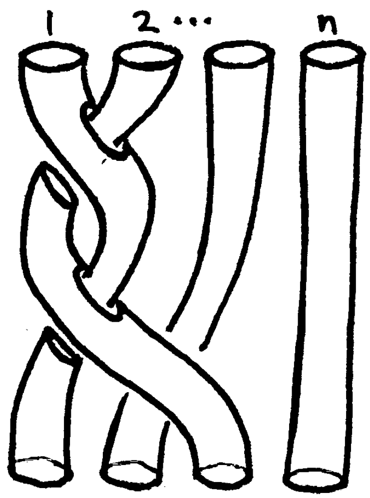
where B_n has more interesting representations than just the bosonic / fermionic reps of the symmetric group.

(simplest case: 'abelian anyonic reps', which amount to multiplication by a phase which need not be ± 1)

Loop Braid Group Statistics

The key idea behind the Baez-Crans-Wise paper (gr-qc/0603085) is that exotic statistics are still possible in 3d space *if* we consider "closed strings" instead of point particles. These strings can exchange places in topologically nontrivial ways, giving rise to statistics governed by the "loop braid groups" LB_n.

σ



loop braid diagram

gives

$$\begin{array}{c} \mathcal{H}^{\otimes n} \\ \Big\downarrow \rho(\sigma) \\ \mathcal{H}^{\otimes n} \end{array}$$

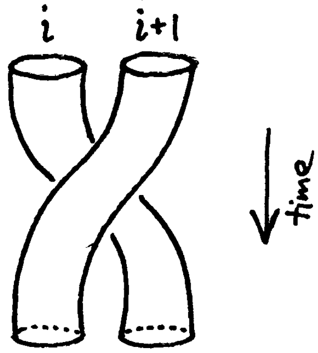
We get "loop braided Fock space"

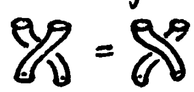
$$\mathcal{F} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}^{\otimes n} / LB_n$$

where LB_n may act by any of a number of interesting reps...

The Loop Braid Group

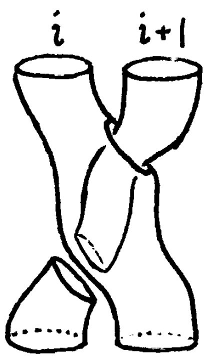
LB_n is the group of topologically distinct ways to exchange the positions of n oriented, unlinked circles in $\mathbb{R}^3$. It has generators that move one circle around the next:

$s_i =$ 

- one spatial dimension suppressed in the diagram
- note  so the generators s_i act like permutations 

In fact, the s_i generate a copy of S_n.

but also generators that pass the $(i+1)$st circle over and then down through the ith as they trade places:

$\sigma_i =$ 

- the σ_i act like braid group generators 

In fact, they generate a copy of B_n.

The loop braid group, braid group, and symmetric group are all cases of the more general concept of a "motion group"...

Motion Groups & Statistics

The idea of a "motion group" provides a framework for describing 'statistics' of matter more general than point particles:

$S \sim$ a smooth oriented manifold ("space")

$\Sigma \sim$ smooth oriented submanifold of S ("matter")

$$\begin{array}{l} \mathrm{Diff}(S) := \text{gp. of orientation-preserving diffeos of } S \\ \quad \cup\!\shortmid \\ \mathrm{Diff}(S,\Sigma) := \text{subgp. whose elts restrict to elts of } \mathrm{Diff}(\Sigma) \end{array}$$

A <u>motion</u> of Σ in S is a path

$$f : [0,1] \longrightarrow \mathrm{Diff}(S)$$

such that $(\exists \varepsilon)$

$$f(t) = Id \qquad \forall t \in [0,\varepsilon)$$

$$f(t) \in \mathrm{Diff}(S,\Sigma) \text{ and is independent of } t \quad \forall t \in (1-\varepsilon, 1]$$

<u>Composition</u> of motions:

$$(f\cdot g)(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1)\circ f(1) & \frac{1}{2} \le t \le 1 \end{cases}$$

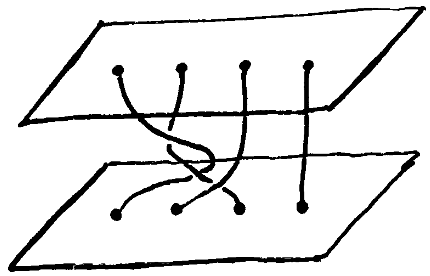
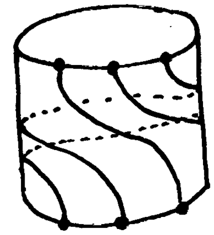
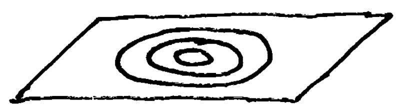
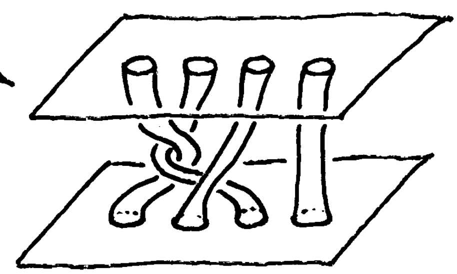
<u>Reverse</u> of a motion

$$\bar{f}(t) = f(1-t)\circ f(1)^{-1}$$

The <u>motion group</u>

$$Mo(S,\Sigma) = \left[\begin{matrix}\text{motions of} \\ \Sigma \text{ in } S\end{matrix}\right] \Big/ \begin{matrix}\text{motions } f \,\&\, g \text{ are equivalent} \\ \text{if } \bar{f}\cdot g \text{ is homotopic to a} \\ \text{path in } \mathrm{Diff}(S,\Sigma)\end{matrix}$$

EXAMPLES OF MOTION GROUPS

S	Σ	$M_0(S,\Sigma)$	
$\mathbb{R}^1$	n points	$\langle 1 \rangle$	
$\mathbb{R}^2$	n points	B_n	
$\mathbb{R}^d$ $d>2$	n points	S_n	
S^1	n points $(n>0)$	$\mathbb{Z}$	
S^2	n points		
$\vdots$			
$\mathbb{R}^2$	n nested circles	$\langle 1 \rangle$	
$\mathbb{R}^2$	n distant circles	B_n	
$\mathbb{R}^3$	n circles	$\boxed{LB_n}$ LOOP BRAID GROUP	
$\vdots$			
$\mathbb{R}^4$	n spheres	"S^2B_n"	some "sphere braid group"

The point for physics is that statistics of indistinguishable "particles" (point particles, strings, ...) depends on both the topology of space & the topology of the particles!

Next, let's see how exotic statistics arise for generalized "particles" in BF theory. For this, we first need the classical configuration space...

The Moduli Space of Flat Bundles

A G-bundle on X with flat connection A gives a homomorphism

$$\mathrm{hol}(A) \colon \pi_1(X) \longrightarrow G$$

and applying a gauge transformation just conjugates this holonomy map by an element of G. Moreover, every homo. $\pi_1(X) \longrightarrow G$ comes from a flat connection on some principal G-bundle, so we call

$\hom(\pi_1(X), G)/G$ the "moduli space of flat G-bundles on X"

where $g \in G$ acts on $f \colon \pi_1(X) \to G$ by conjugation:

$$(gf)(\gamma) = g f(\gamma) g^{-1}.$$

The naive configuration space (ignoring boundary conditions) for BF theory when

S is "space"
$\Sigma \subset S$ is "matter"

is just this moduli space

$$\hom(\pi_1(X), G)/G$$

where $X = S - \Sigma$

$Mo(S,\Sigma)$ - STATISTICS IN BF THEORY

The motion group $Mo(S,\Sigma)$ acts on

$$\hom(\pi_1(X), G) \qquad\qquad (X = S-\Sigma)$$

— the space of "group-valued momenta for the 'matter' Σ, according to a chosen observer" — since a motion induces a diffeomorphism that distorts the loops in the fundamental group. Hence $Mo(S,\Sigma)$ acts also on

$$L^2(\hom(\pi_1(X), G))$$

If we define this L^2 using a measure on $\hom(\pi_1(X), G)$ which is invariant under the actions of both $Mo(S,\Sigma)$ and G on $\hom(\pi_1(X), G)$, then we get a unitary action of $Mo(S,\Sigma)$ on

$$L^2(\hom(\pi_1(X), G)/G),$$

which is the Hilbert space for BF theory. In other words, we get exotic statistics governed by the motion group!

STRINGS IN 4D BF THEORY

Though the machinery is quite general, in the paper we ultimately specialize to the case where

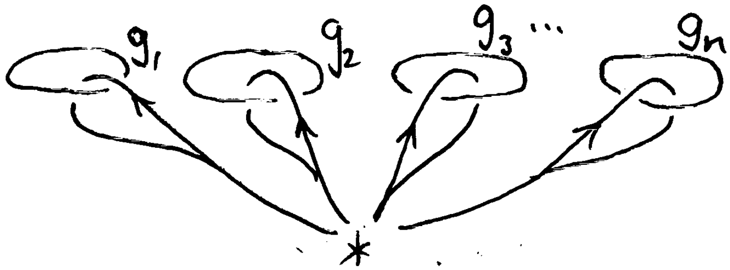
$S = \mathbb{R}^3$

Σ = n unknotted unlinked circles ("strings")

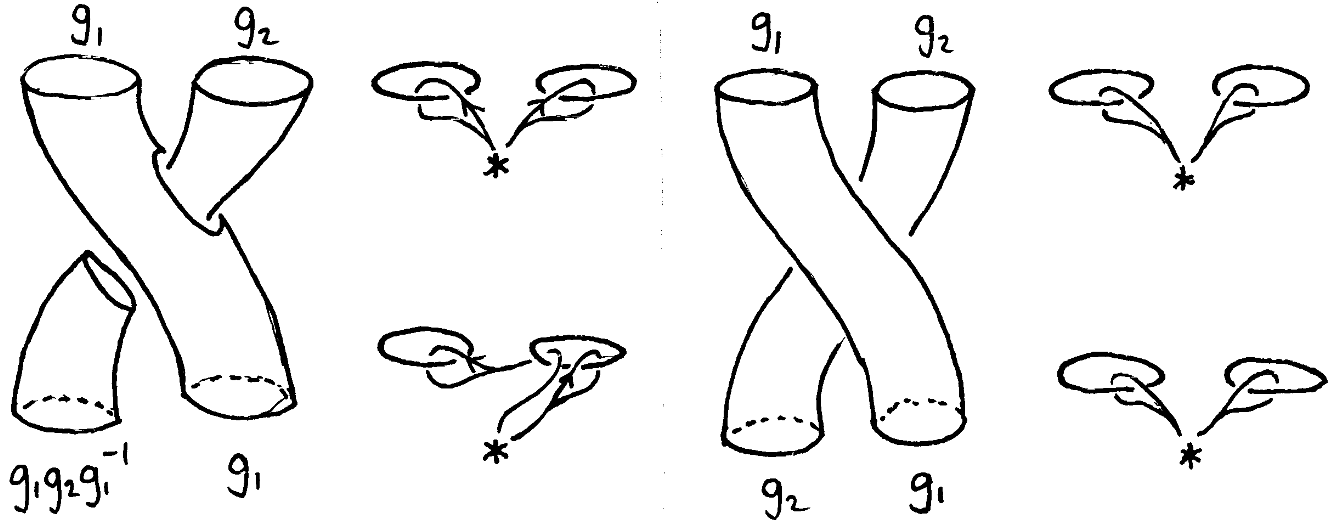
$X = S - \Sigma$

So $\pi_1(X)$ = the free group on n generators

& $\hom(\pi_1(X), G) \cong G^n$ ~ "momentum space"



Here $Mo(S,\Sigma) = LB_n$, the loop braid group, and LB_n acts on G^n as follows:



When G has a measure invariant under conjugation, our general discussion shows we can quantize to get "quantum BF theory with "closed strings" obeying loop braid group statistics." (Here we are treating these "strings" merely as topological defects, and ignoring the possibility of interations. For a proposed dynamics for these strings, see the follow-up paper by Baez & Perez, gr-qc/0605087)

When $G = SO_0(3,1)$ we also classify the "string types" (by conjugacy classes). The result: we get analogs of familiar particle types such as

- "tardyons" of mass $0 < m < \frac{\pi}{k}$
- "luxons" of mass $m = 0$
- "tachyons" of mass im $\quad 0 < m < \infty$

but also

- strings of "complex mass" $im_1 + m_2$

  with $0 < m_1 < \infty \quad 0 < m_2 < \frac{2\pi}{k}$!

GOAL:

Use some of these ideas as ways of talking about matter in formulations of full-fledged GR that are related to BF theory, e.g.:

- Palatini (constrained BF)
- perturbative expansion around MacDowell-Mansouri-type BF theory

  (foundations laid by M-M, Smolin, Freidel, Starodubtsev, ...)