Most human things go in pairs.
Alcmaeon, $\sim$ 450 BC

true	false		good	bad
right	left		up	down
front	back		future	past
light	dark		hot	cold
matter	antimatter		boson	fermion

How can we formalize a *general* concept of duality? The Chinese tried yin-yang theory, which inspired Leibniz to develop binary notation, which in turn underlies digital computation!

But what's the state of the art *now?*

In category theory the fundamental duality is the act of reversing an arrow:

$$\bullet \longrightarrow \bullet \qquad \rightsquigarrow \qquad \bullet \longleftarrow \bullet$$

We use this to model switching past and future, false and true, small and big...

Every category $\mathcal{C}$ has an *opposite* $\mathcal{C}^{\mathrm{op}}$, where the arrows are reversed. This is a symmetry of the category of categories:

$$\mathrm{op} \colon \mathrm{Cat} \to \mathrm{Cat}$$

and indeed the only nontrivial one:

$$\mathrm{Aut}(\mathrm{Cat}) = \mathbb{Z}/2$$

In logic, the simplest duality is negation. It's order-reversing:

$$\text{if } P \text{ implies } Q \text{ then } \neg Q \text{ implies } \neg P$$

and—ignoring intuitionism!—it's an involution:

$$\neg\neg P = P$$

Thus if $\mathcal{C}$ is a category of propositions and proofs, we expect a functor:

$$\neg \colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$$

with a natural isomorphism:

$$\neg^2 \cong 1_{\mathcal{C}}$$

This has *two* analogues in quantum theory.

One shows up already in the category of finite-dimensional vector spaces, FinVect. Every vector space V has a dual V^*. Taking the dual is contravariant:

$$\text{if } f\colon V \to W \text{ then } f^*\colon W^* \to V^*$$

and—ignoring infinite-dimensional spaces!—it's an involution:

$$V^{**} \cong V$$

This kind of duality is captured by the idea of a $*$-autonomous category.

Recall that a *symmetric monoidal category* is roughly a category $\mathcal{C}$ with a unit object $I \in \mathcal{C}$ and a tensor product

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

that is unital, associative and commutative up to coherent natural isomorphisms.

A symmetric monoidal category $\mathcal{C}$ is *closed* if for every object $A \in \mathcal{C}$ the functor

$$X \mapsto A \otimes X$$

has a right adjoint, called the *internal hom*:

$$X \mapsto A \multimap X$$

In other words, there's a natural bijection

$$\hom(A \otimes X, Y) \cong \hom(X, A \multimap Y)$$

In FinVect, $A \otimes X$ is the usual tensor product, and $A \multimap Y$ is the vector space of linear maps from A to Y.

In classical logic, $A \otimes X$ is the proposition $A \& X$, and $A \multimap Y$ is the proposition $A \Rightarrow Y$.

A symmetric monoidal closed category $\mathcal{C}$ is $*$-*autonomous* if it has an object $\perp$ such that if we set

$$X^* = X \multimap \perp$$

then the canonical morphism $X \to X^{**}$ is an isomorphism. We call $\perp$ a *dualizing object*.

Any $*$-autonomous category comes with a functor:

$$*\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$$

and a natural isomorphism:

$$*^2 \cong 1_{\mathcal{C}}$$

In FinVect we take $\perp = \mathbb{C}$, so

$$X^* = X \multimap \perp$$

is the usual dual of the vector space X. Here $\perp$ is the unit for the tensor product. Whenever this happens we say $\mathcal{C}$ is *compact*. In a compact category, $A \multimap X \cong A^* \otimes X$.

In classical logic we take $\perp$ = FALSE, so

$$X^* = X \multimap \perp$$

is the usual negation $\neg X$. Here $\perp$ is not the unit for the tensor product: the unit for & is TRUE.

In the category of finite-dimensional Hilbert spaces, we also have a second kind of duality. Besides the duals for objects with

$$\hom(A \otimes X, Y) \cong \hom(X, A^* \otimes Y)$$

we have duals for morphisms: given $f\colon X \to Y$ we have $f^\dagger\colon Y \to X$ with

$$\langle fx, y\rangle = \langle x, f^\dagger y\rangle$$

for all $x \in X$ and $y \in Y$.

In short, *a Hilbert space is like a miniature category*, with an amplitude to go from x to y instead of a set of ways to go from X to Y.

Duality for morphisms is captured by the concept of a *dagger category*: one where every morphism

$$f\colon X \to Y$$

gives a morphism

$$f^\dagger\colon Y \to X$$

with

$$(ST)^\dagger = T^\dagger S^\dagger \qquad T^{\dagger\dagger} = T$$

Any dagger category comes with a functor

$$\dagger\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$$

and a natural isomorphism

$$\dagger^2 \cong 1_{\mathcal{C}}$$

Duality for objects and morphisms fit together in the concept of a *dagger-compact category*.

This is a compact dagger category for which all relevant natural isomorphisms are *unitary* ($u^\dagger = u^{-1}$), and the dagger of the *unit*

$$X^* \otimes X \to I$$

is the *counit*

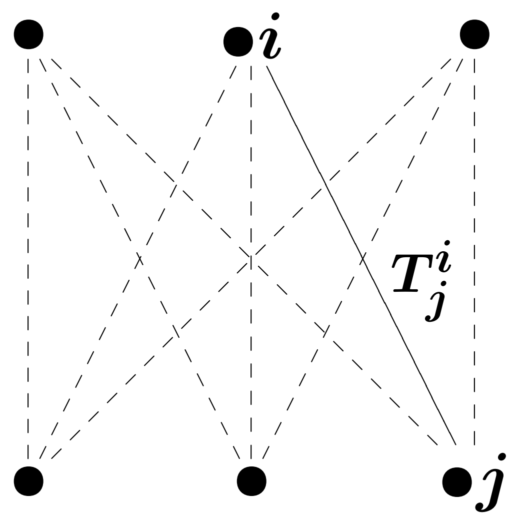
$$I \to X \otimes X^*$$

followed by the isomorphism

$$X \otimes X^* \xrightarrow{\sim} X^* \otimes X$$

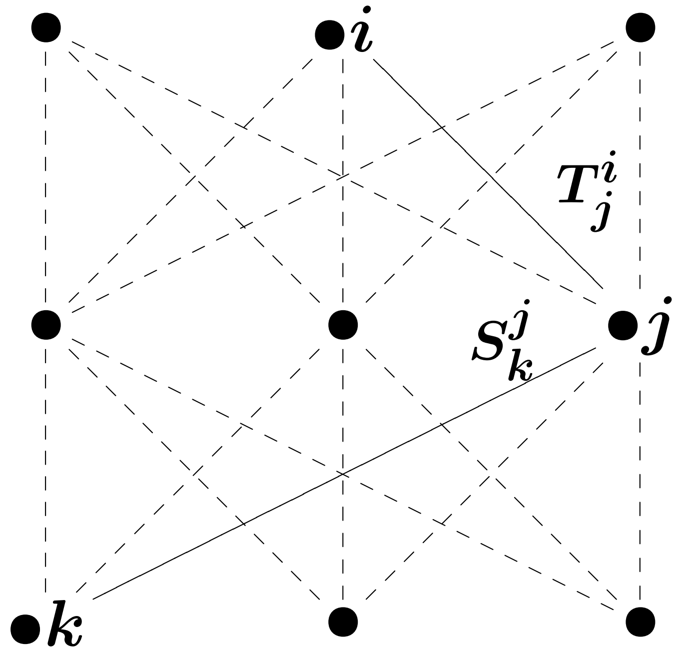
Dagger-compact categories are deeply related to 'matrix mechanics'.

Heisenberg used matrices with complex entries to describe processes in quantum mechanics:



For each input state i and output state j, the process T gives a complex number $T^i_j \in \mathbb{C}$, the *amplitude* to go from i to j.

To compose processes, we sum over paths:



$$(ST)^i_k = \sum_j S^j_k \times T^i_j$$

In the continuum limit, such sums become *path integrals*.

Matrix mechanics also works with other rigs (= rings without negatives):

- **$[0, \infty)$ with its usual $+$ and $\times$**

  Now T^i_j gives the *probability* to go from i to j.

- **$\{\mathrm{TRUE}, \mathrm{FALSE}\}$ with $\vee$ as $+$ and & as $\times$**

  Now T^i_j gives the *possibility* to go from i to j.

- **$\mathbb{R}^{\mathrm{MIN}} = \mathbb{R} \cup \{+\infty\}$ with MIN as $+$ and $+$ as $\times$**

  Now T^i_j gives the *action* to go from i to j.

In the continuum limit, matrix mechanics using $\mathbb{R}^{\mathrm{MIN}}$ gives the *principle of least action* in classical mechanics — see work on 'idempotent analysis' and 'tropical algebra'.

For any commutative rig R, there is a compact category Mat(R) where:

- **objects are natural numbers;**
- **morphisms $T\colon m \to n$ are $n \times m$ matrices with entries in R.**

If R is also a $*$-*rig*:

$$(a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*, \qquad a^{**} = a$$

then Mat(R) is dagger-compact, with

$$(T^\dagger)^i_j = (T^j_i)^*$$

Every rig becomes a $*$-rig with $a^* = a$. So, Mat($\mathbb{R}^{\mathrm{MIN}}$) is dagger-compact and *we can apply dagger-compact categories to <u>classical</u> mechanics!*

$\mathrm{Mat}(\mathbb{R})$ is equivalent to the category of finite-dimensional real Hilbert spaces.

$\mathrm{Mat}(\mathbb{C})$ is equivalent to the category of finite-dimensional complex Hilbert spaces.

$\mathrm{Mat}(\mathbb{H})$ is equivalent to the category of finite-dimensional quaternionic Hilbert spaces.

The first two are dagger-compact. The third is not monoidal but still a dagger-category. *Why does Nature love complex Hilbert spaces best?*

My answer:

In fact Nature loves all three, each in its own way!

To understand this, it helps to equip our Hilbert spaces with more structure.

Start with a compact topological group G. Let Rep(G) be the category of continuous unitary representations of G on finite-dimensional *complex* Hilbert spaces.

This category is dagger-compact. It's crucial in quantum physics, where G describes symmetries!

But Freeman Dyson's 'Threefold Way' reveals that continuous unitary representations of G on finite-dimensional *real* and *quaternionic* Hilbert spaces are *also* contained in Rep(G)!

We can ask if an object $X \in \mathrm{Rep}(G)$ is its own dual. If X is *irreducible* — not a direct sum of other objects in a non-trivial way — there are three mutually exclusive choices:

- **$X \ncong X^*$.**

- **$X \cong X^*$ and X is *real*: it comes from a representation of G on a real Hilbert space H:**

$$X = \mathbb{C} \otimes_{\mathbb{R}} H$$

- **$X \cong X^*$ and X is *quaternionic*: it comes from a representation of G on a quaternionic Hilbert space H:**

$$X = \text{the underlying complex space of } H$$

Why? If X is irreducible there's a 1-dimensional space of morphisms $f\colon X \to X$. So if $X \cong X^*$ there's a 1d space of morphisms $f\colon X \to X^*$, and thus a 1d space of morphisms

$$g\colon X \otimes X \to \mathbb{C}$$

But

$$X \otimes X \cong S^2 X \oplus \Lambda^2 X$$

so either there exists a nonzero g that is *symmetric*:

$$g(x, y) = g(y, x)$$

or one that is *skew-symmetric*:

$$g(x, y) = -g(y, x)$$

One or the other, not both!

Either way, we can write

$$g(x, y) = \langle jx, y \rangle$$

for some real-linear operator $j\colon X \to X$ with

$$ji = -ij$$

If g is symmetric, we can rescale j to achieve

$$j^2 = 1$$

Then j acts like complex conjugation, so

$$H = \{x \in X\colon\; jx = x\}$$

is a real representation of G with:

$$X = \mathbb{C} \otimes_{\mathbb{R}} H$$

If g is skew-symmetric, we can rescale j to achieve

$$j^2 = -1$$

Then i, j and $k = ij$ act like the quaternions:

$$i^2 = j^2 = k^2 = ijk = -1$$

so we obtain a representation of G on a quaternionic Hilbert space H with:

$$X = \text{the underlying complex space of } H$$

Why can we rescale j to achieve either $j^2 = \pm 1$ but not both?

Suppose g is symmetric, so that

$$\langle jx, y\rangle = g(x, y) = g(y, x) = \langle jy, x\rangle$$

Then

$$\langle j^2 x, x\rangle = \langle jx, jx\rangle \geq 0$$

so $j^2 > 0$. Letting $J = \alpha j$ with $\alpha \in \mathbb{R}$ we have

$$J^2 = \alpha^2 j^2$$

so by correctly choosing α we can achive

$$J^2 = 1$$

Similarly, if g is skew-symmetric $j^2 < 0$ and we can rescale j to achieve $J^2 = -1$.

An example: $G = \mathrm{SU}(2)$. Every object in $\mathrm{Rep}(\mathrm{SU}(2))$ is self-dual. There is one irreducible representation for each 'spin' $j = 0, \frac{1}{2}, 1, \ldots$

If $j \in \mathbb{Z}$, the spin-j rep is real.

If $j \in \mathbb{Z} + \frac{1}{2}$, the spin-$j$ rep is quaternionic.

For example: the spin-1 or *vector* rep on $\mathbb{C}^3$ is the complexification of a real rep on $\mathbb{R}^3$.

But the spin-$\frac{1}{2}$ or *spinor* rep on $\mathbb{C}^2$ is the underlying complex rep of a quaternionic rep on $\mathbb{H}$.

In short: spin-$\frac{1}{2}$ particles are quaternions!

But what is the physical meaning of j? For any $iJ \in \mathfrak{su}(2)$:

$$j \, \exp(i\theta J) = \exp(i\theta J) \, j$$

and thus

$$j \, iJ = iJ \, j$$

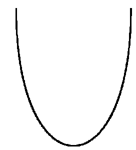
but $ij = -ji$ so

$$jJj^{-1} = -J$$

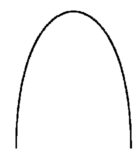
So: j reverses the angular momentum J.

This calculation works for all reps of SU(2). Thus the meaning of j is *time reversal.*

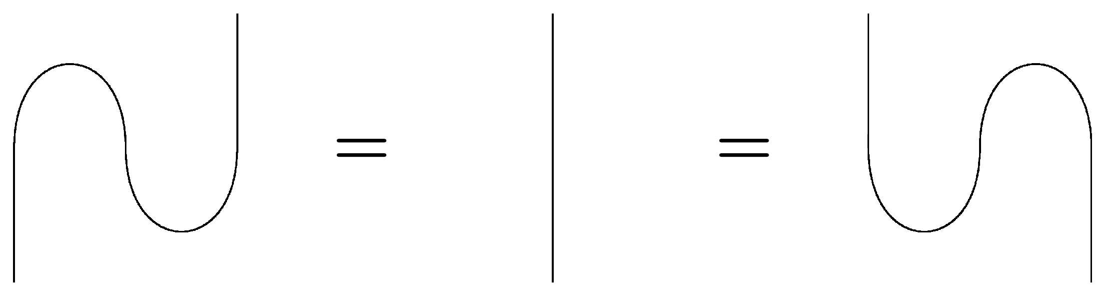
For any self-dual $X \in \mathrm{Rep}(G)$ we can write $g\colon X \otimes X \to \mathbb{C}$ as a 'cup':



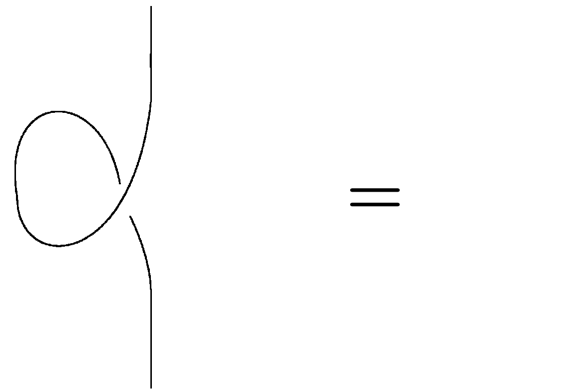
Since g defines an isomorphism $X \cong X^*$ we have a corresponding 'cap':



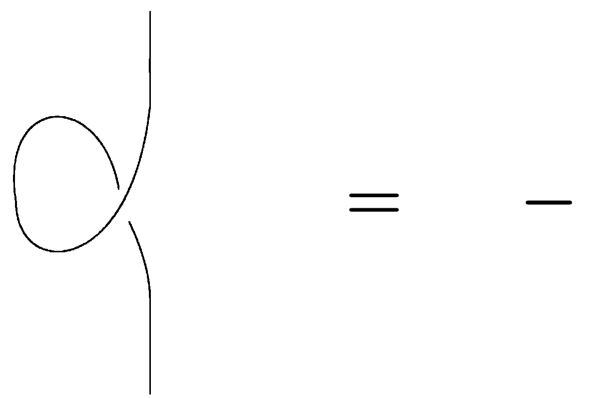
satisfying the *zig-zag identities:*



For a real self-dual object ($j^2 = 1$) we have



but for a quaternionic one ($j^2 = -1$) we have



This is the behavior we expect from *bosons* and *fermions*, respectively.

THE THREEFOLD WAY

complex	$X \not\cong X^*$	unitary	$\mathrm{U}(n)$
real	$X \cong X^*$ $j^2 = 1$	orthogonal	$\mathrm{O}(n)$
quaternionic	$X \cong X^*$ $j^2 = -1$	symplectic	$\mathrm{Sp}(n)$

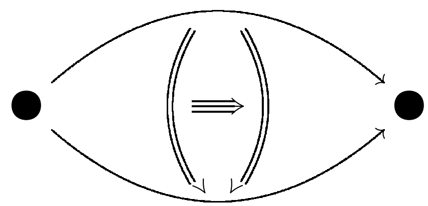
This accounts for the 'classical groups', leaving 5 exceptional groups related to the octonions and 'triality'. These are important in superstring theory...

...but that's a story for another day.

When it comes to duality, categories are just the beginning. We also have n-categories:

$$\bullet \longrightarrow \bullet$$

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$



and so on...

An $(n+k)$-category with only one j-morphism for $j<k$ is called a *k-monoidal n-category*. We have guesses about these...

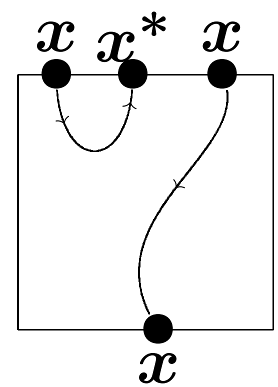
THE PERIODIC TABLE

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	sets	categories	2-categories
$k = 1$	monoids	monoidal categories	monoidal 2-categories
$k = 2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories
$k = 3$	‘’	symmetric monoidal categories	sylleptic monoidal 2-categories
$k = 4$	‘’	‘’	symmetric monoidal 2-categories
$k = 5$	‘’	‘’	‘’
$k = 6$	‘’	‘’	‘’

Back in 1995, James Dolan and I proposed studying n-categories with *duals at all levels*, in order to make precise and prove:

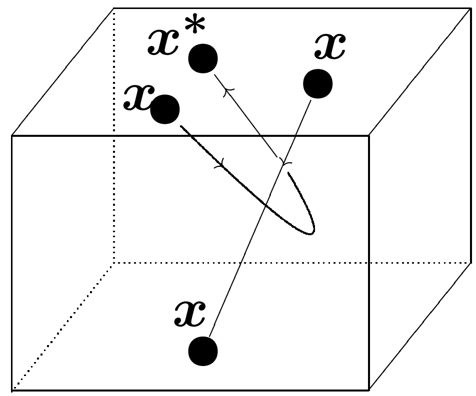
THE TANGLE HYPOTHESIS: The n-category of framed n-dimensional tangles in a $(n+k)$-dimensional cube is the free k-monoidal n-category with duals on one object: the postively oriented point.

When $n = 1$, $k = 1$ we get a monoidal category with duals:



Morphisms here model worldlines of particles in 2d spacetime.

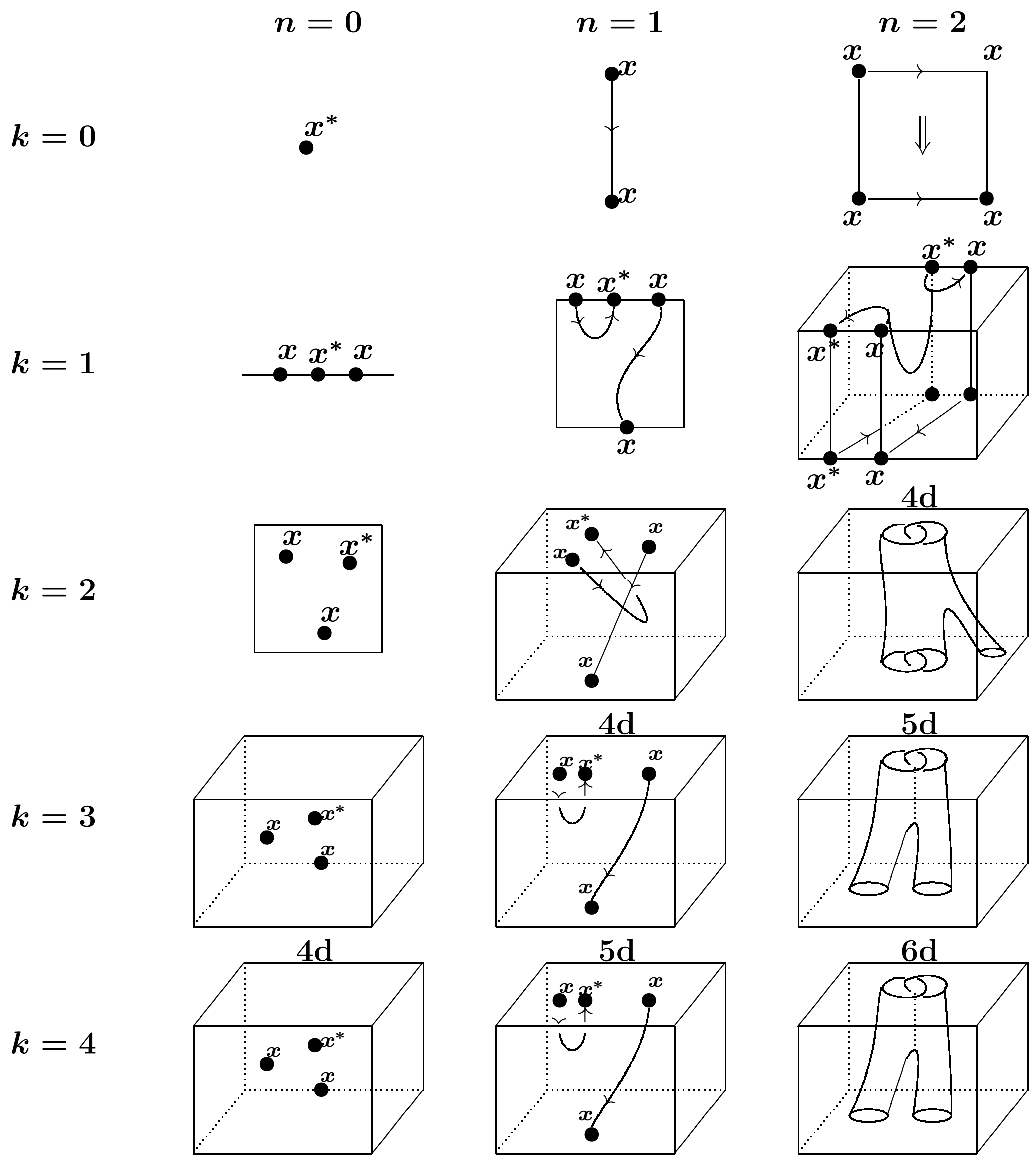
When $n = 1$, $k = 2$ we get a braided monoidal category with duals:



Morphisms here model worldlines of particles in 3d spacetime.

When $n = 1$, $k = 3$ we get a symmetric monoidal category with duals—i.e. a dagger-compact category. Morphisms here model models worldlines of particles in 4d spacetime. ***Dagger-compact categories seem natural in logic and physics because we're used to particles in 4d spacetime!***

PERIODIC TABLE OF TANGLES



'Categorified matrix mechanics' gives examples of symmetric monoidal 2-categories with duals.

For example, there's a symmetric monoidal 2-category Prof where:

- an object is a small category $\mathcal{C}$;
- a morphism T from $\mathcal{C}$ to $\mathcal{D}$ is a functor

  $$T\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$$

  also known as a *profunctor*;
- a 2-morphism is a natural transformation between profunctors.

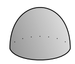
Prof has duals for objects: the dual of $\mathcal{C}$ is $\mathcal{C}^{\mathrm{op}}$. It doesn't have duals (adjoints) for all morphisms—but a functor gives a profunctor with a right adjoint. It doesn't have daggers for 2-morphisms.

We can do better by replacing Set here by a dagger-compact category.

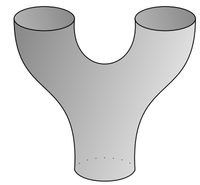
Furthermore, the symmetric monoidal 2-category of 'finite-dimensional 2-Hilbert spaces' has duals at all levels!

But let's look at Prof....

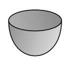
A 2-dimensional topological quantum field theory gives a *Frobenius monoid* in FinVect:



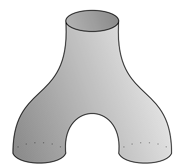
unit



multiplication

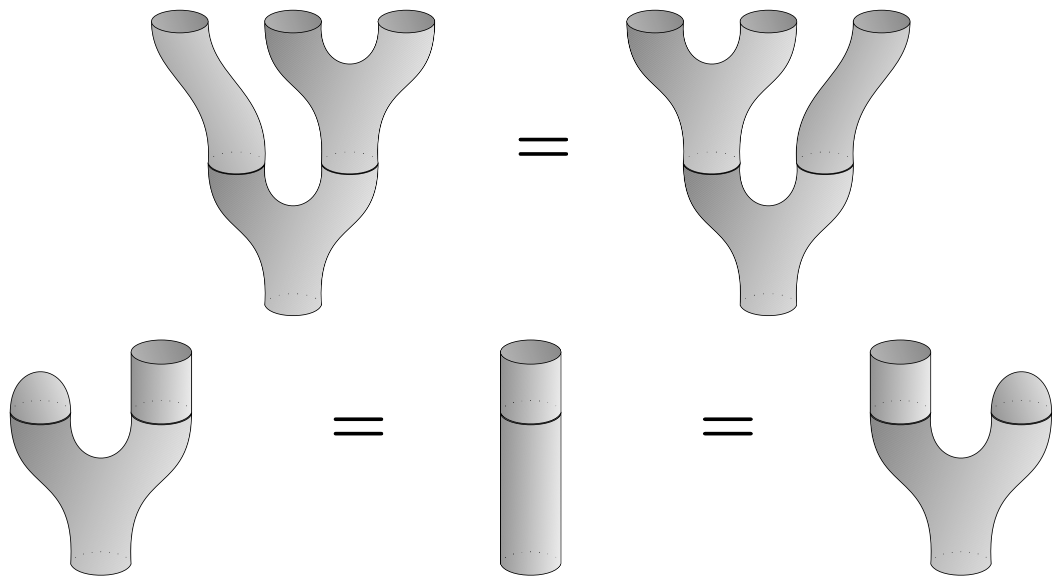


counit

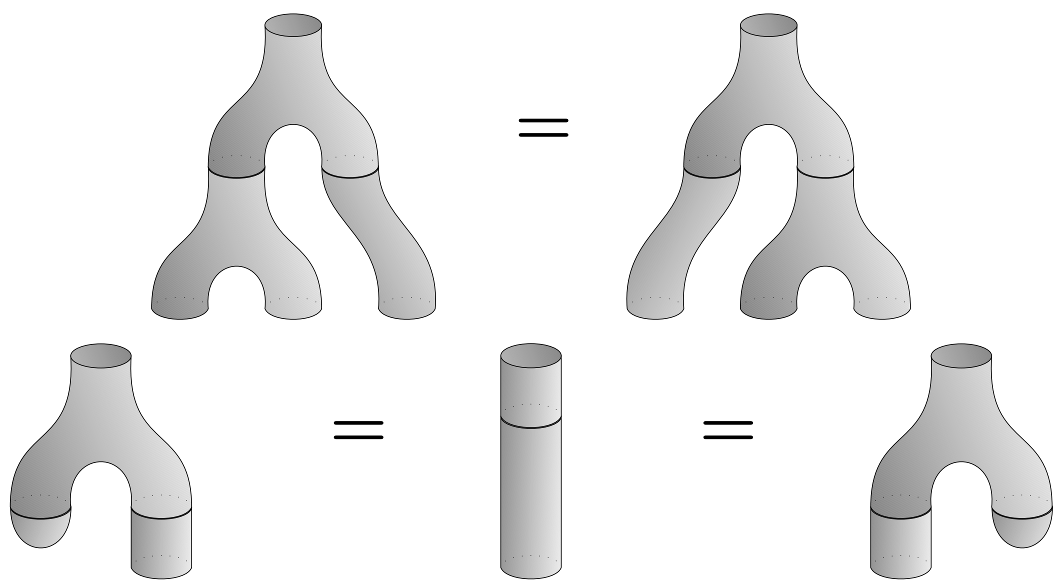


comultiplication

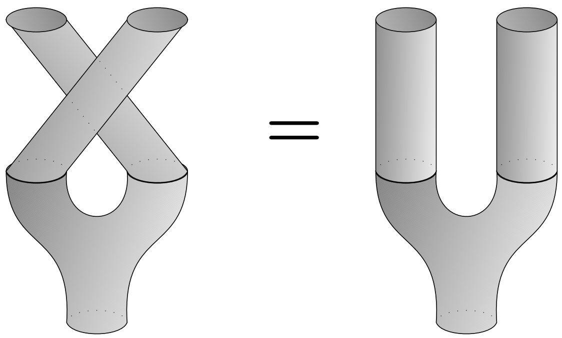
Associativity and unit laws:



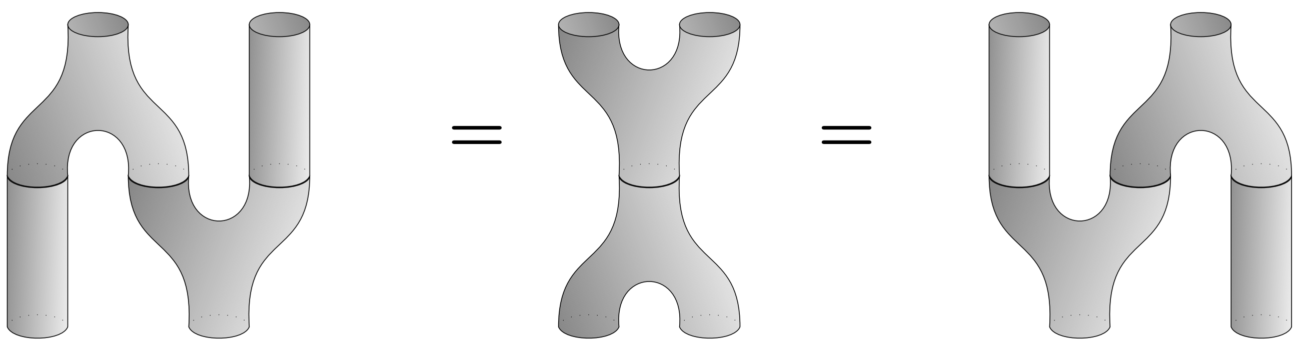
Coassociativity and counit laws:



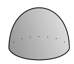
Commutativity:



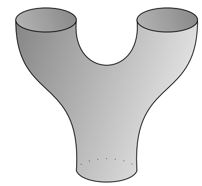
The Frobenius law:



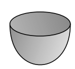
Similarly, Ross Street showed that a $*$-autonomous category $\mathcal{C}$ gives a 'weak Frobenius monoid' in Prof!



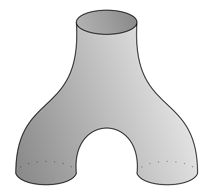
$\mathrm{hom}(\mathrm{TRUE}, A)$



$\mathrm{hom}(A \& B, C)$



$\mathrm{hom}(A, \mathrm{FALSE})$



$\mathrm{hom}(A, B \vee C)$

The Frobenius law is the 'cut rule' in logic!

SUMMARY

'Matrix mechanics' over commutative $*$-rigs describes quantum *and classical* physics. See Aaron Fenyes' no-cloning theorem for classical mechanics.

The study of duality unifies real, complex and quaternionic quantum mechanics into a single theory *which is already implicit* in standard physics.

Dagger-compact categories are the $n = 1$, $k = 3$ example of 'k-monoidal n-categories with duals'—the case most relevant to particles in 4d spacetime, but just one of many.

Treating Prof as a categorified version of FinVect relates *propositional logic*—i.e. $*$-autonomous categories—to *categorified 2d topological quantum field theories!*