There's no cloning in symplectic mechanics

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1 Background

1.1 Motivation

The fact that you can't clone a quantum system is closely related to the fact that the tensor product in the category of Hilbert spaces is non-Cartesian. At the end of his 2008 classical mechanics course, John Baez pointed out that the tensor product in the category of Poisson manifolds is also non-Cartesian, which should mean there's a classical analogue of the no-cloning theorem [4]! This turns out to be true in symplectic mechanics, and I don't see any reason it shouldn't be true in more generalized settings as well.

1.2 Symplectic mechanics

In symplectic mechanics, physical systems are represented by symplectic manifolds, and physical processes are represented by symplectomorphisms. If M and N are symplectic manifolds with symplectic forms ω and σ, respectively, the product manifold $M \times N$ has a natural symplectic form Ω given by

$$\Omega[(u, s), (v, t)] = \omega(u, v) + \sigma(s, t),$$

where u and v are tangent vectors at the same point in M, and s and t are tangent vectors at the same point in N [3, page 7]. I'll denote the resulting symplectic manifold $M \otimes N$ to emphasize that it's not a Cartesian product in the category of symplectic manifolds. I'll often write a point in $M \otimes N$ as an ordered pair of points in M and N, just as I previously wrote a tangent vector on $M \otimes N$ as an ordered pair of tangent vectors on M and N.

2 Argument

2.1 What's a cloning machine?

A *cloning machine* for a symplectic manifold M has two parts:

- A special state $b \in M$, called the *blank* state.
- A symplectomorphism $\phi\colon M \otimes M \to M \otimes M$ with the property that $\phi(x, b) = (x, x)$ for any $x \in M$.

2.2 There's no such thing as a cloning machine

Suppose the state b and the symplectomorphism ϕ give a cloning machine for M. Obviously, $\phi(b, b) = (b, b)$. Let u and v be tangent vectors on M at b. If you start at (b, b) and move along the submanifold $\{(x, b) \mid x \in M\} \subset M \otimes M$ with velocity $(u, 0)$, your image in ϕ will move along $M \otimes M$ with velocity (u, u). Therefore, $d\phi_{(b,b)}(u, 0) = (u, u)$. By the same token, $d\phi_{(b,b)}(v, 0) = (v, v)$.

Let Ω be the symplectic form that comes with $M \otimes M$. Since ϕ is a symplectomorphism,

$$\begin{aligned}\Omega_{(b,b)}[(u,0),(v,0)] &= \Omega_{\phi(b,b)}[d\phi_{(b,b)}(u,0), d\phi_{(b,b)}(v,0)] \\ &= \Omega_{(b,b)}[(u,u),(v,v)].\end{aligned}$$

We can use the definition of Ω to rewrite this equation in terms of ω, the symplectic form that comes with M:

$$\omega_b(u, v) + \omega_b(0, 0) = \omega_b(u, v) + \omega_b(u, v).$$

Simplifying, we see that

$$0 = \omega_b(u, v)$$

for all tangent vectors u and v at b, contradicting the fact that ω is nondegenerate.

3 Related reading

3.1 No-cloning in statistical mechanics

In a classic 2002 paper, Andreas Daffertshofer, Angel Ricardo Plastino, and Angelo Plastino showed that you can't set up a dynamical system whose evolution will copy an arbitrary probability distribution (over system states) from one subsystem (the "source") to another (the "target") [1]. In essence, they proved that there's no cloning in classical statistical mechanics. Because their definition of cloning is slightly different than the one I've used, I can't tell whether or not their result implies the one given here.

3.2 No-cloning in category theory

Recently, Samson Abramsky proved an extremely general no-cloning theorem in the setting of category theory. Unfortunately, I don't know enough category theory to understand the statement of the theorem! Fortunately, Abramsky provides a short, non-technical description of his result:

The Cloning Collapse theorem can be read as a No-Go theorem. It says that it is not possible to combine basic structural features of quantum entanglement with a uniform cloning operation without collapsing to degeneracy. [2]

References

[1] A. Plastino A. Daffertshofer, A.R. Plastino. Classical no-cloning theorem. *Physical Review Letters*, 88(21), 2002.

[2] Samson Abramsky. No-cloning in categorical quantum mechanics. `http://arxiv.org/abs/0910.2401v1`, 2009.

[3] Ana Cannas da Silva. Symplectic geometry. `http://www.math.princeton.edu/~acannas/symplectic.pdf`, 2004.

[4] Alex Hoffnung John Baez. Classical mechanics: The Hamiltonian approach. `http://math.ucr.edu/home/baez/classical/#hamiltonian`, 2008.