Theories With Duality
DRAFT VERSION ONLY

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Abstract

We present a formalism for describing categories equipped with extra structure that involves covariant and contravariant functors related by dinatural transformations. A typical example is the concept of 'cartesian closed category', or more generally 'monoidal closed category'.

While at first this viewpoint may seem overly abstract, it can be very useful to treat a strict monoidal category as a monoid in the category Cat of categories and functors. The idea here is that a strict monoidal category $(\mathscr{M}, \otimes, 1)$ is the same thing as an object $\mathscr{M} \in \mathrm{Cat}$ equipped with morphisms

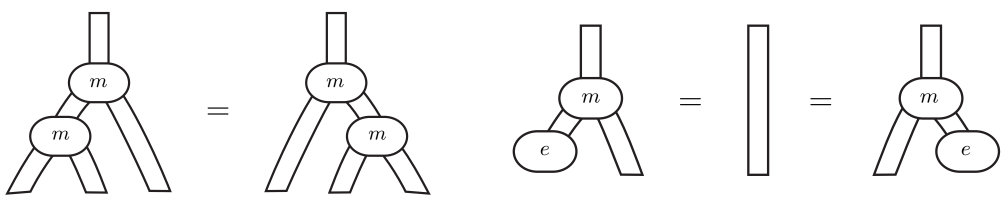
$$\mathscr{M} \times \mathscr{M} \xrightarrow{\otimes} \mathscr{M} \qquad\qquad 1 \xrightarrow{e} \mathscr{M} \tag{1}$$

that make these diagrams commute:

$$
\begin{tikzcd}
\mathscr{M} \times \mathscr{M} \times \mathscr{M} \arrow[r, "\otimes \times \mathscr{M}"] \arrow[d, "\mathscr{M} \times \otimes"'] & \mathscr{M} \times \mathscr{M} \arrow[d, "\otimes"] \\
\mathscr{M} \times \mathscr{M} \arrow[r, "\otimes"'] & \mathscr{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathscr{M} \arrow[dl, bend right, "e \times \mathscr{M}"'] \arrow[dr, bend left, "\mathscr{M} \times e"] \arrow[dd, "\mathrm{id}"] & \\
\mathscr{M} \times \mathscr{M} \arrow[dr, bend right, "\otimes"'] & & \mathscr{M} \times \mathscr{M} \arrow[dl, bend left, "\otimes"] \\
 & \mathscr{M} &
\end{tikzcd}
\tag{2}
$$

The first diagram expresses associativity; the second expresses the left and right unit laws. This formulation lets us treat strict monoidal categories as models of a particular algebraic theory (in this case, the theory of monoids) presented by generators and relations.

The equational nature of the formulation comes out clearly when the notion of monoid is expressed in the graphical language of string diagrams: the generators (1) are depicted in this case as a binary node $\otimes$ and as a nullary node e, with the commutative diagrams (2) formulated as follows:



This model-theoretic point of view on monoidal categories is conceptually neat, and technically useful. Among other benefits, it leads to the definition of a monoidal functor between monoidal categories as a morphism

$$f\colon \mathscr{M} \quad \longrightarrow \quad \mathscr{N}$$

making the diagrams

$$
\begin{tikzcd}
\mathscr{M} \times \mathscr{M} \arrow[r, "f \times f"] \arrow[d, "\otimes"'] & \mathscr{N} \times \mathscr{N} \arrow[d, "\otimes"] \\
\mathscr{M} \arrow[r, "f"] & \mathscr{N}
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\mathrm{id}"] \arrow[d, "e"'] & 1 \arrow[d, "e"] \\
\mathscr{M} \arrow[r, "f"] & \mathscr{N}
\end{tikzcd}
$$

commute in the category Cat. It also follows that there exists a universal construction giving for any category A the free monoidal category TA generated by A, defined as the colimit

$$TA = 1 + A + A^2 + A^3 + \dots$$

This free construction defines a monad T on the category Cat. As expected, the category of strict monoidal categories and strict monoidal functors coincides with the category of algebras for this monad T. Even better, T extends to a 2-monad on the 2-category CAT of categories, functors and natural transformations. This allows us to recover the *2-category* of strict monoidal categories, strict monoidal functors and monoidal natural transformations.

Algebraic theories may be relaxed or 'weakened' in several ways. A nice illustration is provided by the notion of weak monoidal category, where the equations (2) are replaced by natural isomorphisms:

$$
\begin{tikzcd}
\mathscr{M}\times\mathscr{M}\times\mathscr{M} \arrow[r, "\otimes\times\mathscr{M}"] \arrow[d, "M\times\otimes"'] \arrow[dr, phantom, "\Downarrow a"] & \mathscr{M}\times\mathscr{M} \arrow[d, "\otimes"] \\
\mathscr{M}\times\mathscr{M} \arrow[r, "\otimes"'] & \mathscr{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathscr{M} \arrow[dl, bend right, "e\times\mathscr{M}"'] \arrow[dr, bend left, "\mathscr{M}\times e"] \arrow[dd, "\mathrm{id}"] & \\
\mathscr{M}\times\mathscr{M} \arrow[dr, bend right, "\otimes"'] \arrow[r, phantom, "\Rightarrow"] & & \mathscr{M}\times\mathscr{M} \arrow[dl, bend left, "\otimes"] \arrow[l, phantom, "\Leftarrow"] \\
 & \mathscr{M} &
\end{tikzcd}
$$

satisfying coherence laws including Mac Lane's pentagon identity:

$$
\begin{tikzcd}
 & ((w\otimes x)\otimes y)\otimes z \arrow[dl, "a_{w,x,y}\otimes\mathrm{id}"'] \arrow[dr, "a_{w\otimes x,y,z}"] & \\
(w\otimes(x\otimes y))\otimes z \arrow[d, "a_{w,x\otimes y,z}"'] & & (w\otimes x)\otimes(y\otimes z) \arrow[d, "a_{w,x,y\otimes z}"] \\
w\otimes((x\otimes y)\otimes z) \arrow[rr, "\mathrm{id}\otimes a_{x,y,z}"'] & & w\otimes(x\otimes(y\otimes z))
\end{tikzcd}
$$

A weak monoidal category may be seen as a model of a 2-dimensional algebraic theory — in this case, the 2-theory of weak monoids — in the 2-category CAT. Alternatively, it may be seen as a weak algebra, or 'pseudoalgebra', of the 2-monad T [3]. This 2-dimensional extension of algebraic theories is particularly useful when one tries to characterize algebraic structures up to deformation. For example, the coherence theorem for monoidal categories states that a category equivalent to a strict monoidal category is the same thing as a weak monoidal category.

In this article, we would like to develop a similar algebraic point of view on monoidal closed categories. By definition, a monoidal category is closed (on the left) when, for every object a of the category, the functor

$$a \otimes -\colon \mathscr{M} \longrightarrow \mathscr{M}$$

has a right adjoint

$$a \multimap -\colon \mathscr{M} \longrightarrow \mathscr{M}. \tag{3}$$

This adjunction is defined for every object a by a bijection

$$\theta_{a,b,c}\colon \mathscr{M}(a \otimes b, c) \longrightarrow \mathscr{M}(b, a \multimap c) \tag{4}$$

natural in b and c. Since the definition of a monoidal closed category is based on a family of adjunctions, a first step towards an equational reformulation is to remember that the notion of adjunction may be presented by generators and relations in any 2-category [5]. More specifically, the generators of the adjunction $a \otimes - \dashv a \multimap -$ associated to an object a are the natural transformations called **evaluation**:

$$\mathrm{eval}_a\colon a \otimes (a \multimap x) \longrightarrow x \tag{5}$$

and **coevaluation**:

$$\mathrm{coeval}_a\colon x \longrightarrow a \multimap (a \otimes x) \tag{6}$$

while the relations (called triangular laws) satisfied by an adjunction require that the natural transformations obtained by composition

$$a \multimap - \xrightarrow{\mathrm{coeval}_a} a \multimap (a \otimes (a \multimap -)) \xrightarrow{a \multimap \mathrm{eval}_a} a \multimap - \tag{7}$$

$$a \otimes - \xrightarrow{a \otimes \mathrm{coeval}_a} a \otimes (a \multimap (a \otimes -)) \xrightarrow{\mathrm{eval}_a} a \otimes - \tag{8}$$

are equal to the identity for every object a of the category $\mathscr{M}$.

Let us summarize: a monoidal category $\mathscr{M}$ is closed precisely when there exists a functor (3) and two natural transformations (5) and (6) such that the composite natural transformations (7) and (8) are equal to the identity for every object a. This provides a family of generators and relations formulated in the language of categories, functors and natural transformations — apparently just like the weak notion of monoidal category!

There is a big difference, however: this formulation requires that we parametrize the generators and relations by an object a of the category $\mathscr{M}$. This should not be considered satisfactory. In fact, just as the parameter x in (5) and (6) has been eliminated using the hypothesis that eval_a and coeval_a define natural transformations, one should make the parameter a disappear by treating eval and coeval as transformations of some kind. But as we will see, this step involves shifting from natural to *extranatural* transformations.

A good starting point in this direction is provided by the following well-known fact: parameter-theorem Suppose that

$$L\colon \mathscr{A} \times \mathscr{B} \quad \longrightarrow \quad \mathscr{C}$$

is a functor, and suppose that for every object $a \in \mathscr{A}$, the functor $L(a,-)$ has a right adjoint R_a, thus defining a family of bijections

$$\theta_{a,b,c}\colon \mathscr{B}(L_a(b), c) \quad \longrightarrow \quad \mathscr{A}(b, R_a(c))$$

natural in b and c. Then there exists a unique functor

$$R\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \quad \longrightarrow \quad \mathscr{C}$$

such that $R(a,-) = R_a$ for every object $a \in \mathscr{A}$, and such that the bijection $\theta_{a,b,c}$ is natural in a, b and c. proposition The parameter theorem lets us see the family of functors $(a \multimap -)_{a\in\mathscr{M}}$, parametrized by the object a, as a single functor

$$\multimap\colon \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \quad \longrightarrow \quad \mathscr{M} \tag{9}$$

making the bijection (4) natural in a, b and c.

Having seen how $\multimap$ becomes a functor, we can now study the sense in which the evaluation eval_a and coevaluation coeval_a are natural in a. Note that for any morphism $f\colon a \to b$, the following diamonds commute:

$$\begin{tikzcd}
 & a \otimes (a \multimap x) \arrow[dr, "\mathrm{eval}_{a,x}"] & \\
a \otimes (b \multimap x) \arrow[ur, "a\otimes(f\multimap x)"] \arrow[dr, "f\otimes(b\multimap x)"'] & & x \\
 & b \otimes (b \multimap x) \arrow[ur, "\mathrm{eval}_{b,x}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & a \multimap (a \otimes x) \arrow[dr, "a\multimap(f\otimes x)"] & \\
x \arrow[ur, "\mathrm{coeval}_{a,x}"] \arrow[dr, "\mathrm{coeval}_{b,x}"'] & & a \multimap (b \otimes x) \\
 & b \multimap (b \otimes x) \arrow[ur, "f\multimap(b\otimes x)"'] &
\end{tikzcd}$$

This follows from the equations

$$\mathrm{eval}_{a,x} \quad = \quad \theta^{-1}_{a,a\multimap x,x}(1_{a\multimap x}) \qquad\qquad \mathrm{coeval}_{a,x} \quad = \quad \theta_{a,x,a\otimes x}(1_{a\otimes x}).$$

together with the naturality of θ as defined in (4).

Next, recall that a dinatural transformation between two functors

$$F, G\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{C}$$

is defined a family of morphisms

$$\kappa_a\colon F(a,a) \quad \longrightarrow \quad G(a,a)$$

indexed by objects a of the category $\mathscr{A}$, such that hexagons of the form

$$\begin{tikzcd}
 & F(a,a) \arrow[r, "\kappa_a"] & G(a,a) \arrow[dr, "G(a,f)"] & \\
F(b,a) \arrow[ur, "F(f,a)"] \arrow[dr, "F(b,f)"'] & & & G(a,b) \\
 & F(b,b) \arrow[r, "\kappa_b"] & G(b,b) \arrow[ur, "G(f,b)"'] &
\end{tikzcd}$$

commute for every morphism $f\colon a \to b$ of the category $\mathscr{A}$. When the functor F is constant, and thus reduces to a functor $F\colon 1 \to \mathscr{C}$, or in other words just an object of $\mathscr{C}$, the dinaturality hexagon reduces to a diamond:

$$\begin{tikzcd}
 & G(a,a) \arrow[dr, "G(a,f)"] & \\
F \arrow[ur, "\kappa_a"] \arrow[dr, "\kappa_b"'] & & G(a,b) \\
 & G(b,b) \arrow[ur, "G(f,b)"'] &
\end{tikzcd} \tag{10}$$

The dinaturality hexagon also reduces to a diamond when the functor G is constant:

$$\begin{tikzcd}
 & F(a,a) \arrow[dr, "\kappa_a"] & \\
F(b,a) \arrow[ur, "F(f,a)"] \arrow[dr, "F(b,f)"'] & & G \\
 & F(b,b) \arrow[ur, "\kappa_b"'] &
\end{tikzcd} \tag{11}$$

In both cases, the dinatural transformation κ is called an **extranatural** transformation from F to G. So, the diagrams above indicate that the transformations eval_a and coeval_a are extranatural in the parameter a.

In short, we can define a monoidal closed category to be a monoidal category equipped with a functor (3) and two transformations (5) and (6) natural in x and extranatural in a, such that the composites (7) and (8) are equal to the identity. To formalize this definition in terms of some sort of 'bivariant algebraic theory', we need a framework in which extranatural transformations can be defined and composed.

Here it is useful to introduce profunctors, also known as distributors [1]. Given categories $\mathscr{A}$ and $\mathscr{B}$, a **profunctor** $F\colon \mathscr{A} \nrightarrow \mathscr{B}$ is defined to be a functor from $\mathscr{A} \times \mathscr{B}^{\mathrm{op}}$ to Set. We may also think of it as a cocontinuous functor $\hat{F}\colon \hat{\mathscr{A}} \to \hat{\mathscr{B}}$, where $\hat{\mathscr{A}}$ denotes the category of presheaves on $\mathscr{A}$, and similarly for $\hat{\mathscr{B}}$. The latter description makes it easy to compose profunctors in a strictly associative way, obtaining a 2-category PROF where:

- the objects are categories,
- the morphisms are profunctors,
- the 2-morphisms are natural transformations between profunctors.

This description also lets us treat a functor as a special sort of profunctor: namely, one that sends representable presheaves to representable presheaves. As a consequence, we can compose a functor and a profunctor and obtain a profunctor.

Given functors $F\colon \mathscr{A} \times \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$ and $G\colon 1 \to \mathscr{B}$, an extranatural transformation from F to G is the same as a natural transformation

$$\begin{tikzcd}
\mathscr{A} \times \mathscr{A}^{\mathrm{op}} \arrow[r, "F"] & \mathscr{C} \\
1 \arrow[u, "i_{\mathscr{A}}", "\shortmid"{marking}] \arrow[r, "G"'] \arrow[ur, phantom, "\Searrow\kappa"] & \mathscr{C} \arrow[u, "\mathrm{id}"', "\shortmid"{marking}]
\end{tikzcd}$$

Here we are using $i_{\mathscr{A}}$ to denote the functor

$$\begin{array}{rccc}
\hom_{\mathscr{A}}\colon & \mathscr{A}^{\mathrm{op}} \times \mathscr{A} & \to & \mathrm{Set} \\
 & (a,b) & \mapsto & \mathscr{A}(a,b)
\end{array}$$

regarded as a profunctor from 1 to $\mathscr{A} \times \mathscr{A}^{\mathrm{op}}$. Note that in the above square, we are using our ability to compose functors with profunctors. Most importantly, the naturality of κ here is equivalent to the fact that the extranaturality diamond (10) commutes!

Similarly, given functors $F\colon 1 \to \mathscr{C}$ and $G\colon \mathscr{A} \times \mathscr{A}^{\mathrm{op}} \to \mathscr{C}$, an extranatural transformation from F to G is the same as a natural transformation

$$\begin{tikzcd}
1 \arrow[r, "F"] & \mathscr{C} \\
\mathscr{A}^{\mathrm{op}} \times \mathscr{A} \arrow[u, "e_{\mathscr{A}}", "\shortmid"{marking}] \arrow[r, "G"'] \arrow[ur, phantom, "\Searrow\kappa"] & \mathscr{C} \arrow[u, "\mathrm{id}"', "\shortmid"{marking}]
\end{tikzcd}$$

Here e_A denotes the functor $\hom_{\mathscr{A}}$ regarded as a profunctor from $\mathscr{A}^{\mathrm{op}} \times \mathscr{A}$ to 1. Now the naturality of κ is equivalent to the fact that the extranaturality diamond (11) commutes.

In short, extranatural transformations can be seen as certain squares in the double category PROCAT where:

- the objects are categories,
- the horizontal morphisms are functors,
- the vertical morphisms are distributors,
- the squares are natural transformations.

This suggests that we use the formalism of double categories to describe bivariant algebraic theories.

On a technical note: one may wonder if the above structure is a true double category or merely a 'pseudo double category', in which the vertical morphisms compose in a weakly associative manner. The answer depends on how we treat profunctors. If we treat them as cocontinuous functors from $\hat{\mathscr{A}}$ to $\hat{\mathscr{B}}$, they compose in a strictly associative way, and we obtain an honest double category. So, this is the approach we shall take. It is worth noting here that *every* pseudo double category is equivalent to a double category [2].

In our treatment of extranatural transformations, we are implicitly using some special features of the double category PROCAT. First, it is an example of a 'proarrow equipment'. This concept was first developed by Wood [6] as part of 'formal category theory': that is, the project of generalizing tools from CAT to more general 2-categories. Profunctors are part of a network of concepts including the Yoneda embedding, Kan extensions, end, and coends. All these concepts may be generalized from CAT to other 2-categories using Wood's formalism. Wood defined a **proarrow equipment** to be a functor $i\colon \mathrm{X} \longrightarrow \mathrm{Y}$ between 2-categories which is bijective on objects, locally fully faithful, and such that the image of each arrow of X has a right adjoint in Y. The example to keep in mind is where $\mathrm{X} = \mathrm{CAT}$, $\mathrm{Y} = \mathrm{PROF}$, and $i\colon \mathrm{X} \longrightarrow \mathrm{Y}$ is the already discussed method of treating a functor as a special sort of profunctor.

However, as emphasized by Shulman [4], a proarrow equipment may be profitably regarded as double category with a special property. In this approach, a **proarrow equipment** is a double category such that each diagram of this form:

$$\begin{tikzcd} \mathscr{C} \arrow[r, "G"] & \mathscr{D} \\ \mathscr{A} \arrow[r, "F"'] & \mathscr{B} \arrow[u, "H"', "|" marking] \end{tikzcd}$$

extends to a square

$$\begin{tikzcd} \mathscr{C} \arrow[r, "G"] & \mathscr{D} \\ \mathscr{A} \arrow[u, "A", "|" marking] \arrow[r, "F"'] \arrow[ur, phantom, "\Searrow\alpha"] & \mathscr{B} \arrow[u, "H"', "|" marking] \end{tikzcd}$$

with the following universal property: any square of the form

$$\begin{tikzcd} \mathscr{C}' \arrow[r, "GG'"] & \mathscr{D} \\ \mathscr{A}' \arrow[u, "K", "|" marking] \arrow[r, "FF'"'] \arrow[ur, phantom, "\Searrow\kappa"] & \mathscr{B} \arrow[u, "H"', "|" marking] \end{tikzcd}$$

factors uniquely as

$$\begin{tikzcd}
\mathscr{C}' \arrow[r, "G'"] & \mathscr{C} \arrow[r, "G"] & \mathscr{D} \\
\mathscr{A}' \arrow[u, "K", "|"'] \arrow[r, "F'"'] \arrow[ur, phantom, "\Downarrow\beta"] & \mathscr{A} \arrow[u, "A", "|"'] \arrow[r, "F"'] \arrow[ur, phantom, "\Downarrow\alpha"] & \mathscr{B} \arrow[u, "H"', "|"]
\end{tikzcd}$$

We call a square with this universal property a **universal filler** of the original diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "G"] & \mathscr{D} \\
\mathscr{A} \arrow[r, "F"'] & \mathscr{B} \arrow[u, "H"', "|"]
\end{tikzcd}$$

and we call $A\colon \mathscr{A} \nrightarrow \mathscr{C}$ the **universal proarrow**. As Shulman notes, this definition of equipment is equivalent to Wood's original definition. (Check!!!) In the case of PROCAT, the universal proarrow A is defined as follows:

$$A(a, c) = K(Fa, Gc).$$

In addition to being an equipment, PROCAT also has a symmetric monoidal structure, coming from the cartesian product in CAT and the tensor product in PROF... defined how??? We may quickly summarize this as follows. A double category is a category internal to Cat. Similarly, we may define a **symmetric monoidal double category** to be a category internal to SymmMonCat. Here SymmMonCat is the category of (weak) symmetric monoidal categories and (weak) symmetric monoidal functors. One can check that PROCAT is a symmetric monoidal double category. (Do it!!!)

Even better, PROCAT is a symmetric monoidal equipment, meaning that the tensor product of universal fillers of diagrams

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "G"] & \mathscr{D} \\
\mathscr{A} \arrow[r, "F"'] & \mathscr{B} \arrow[u, "H"', "|"]
\end{tikzcd}$$

and

$$
\begin{tikzcd}
\mathscr{C}' \arrow[r, "G'"] & \mathscr{D}' \\
\mathscr{A}' \arrow[r, "F'"'] & \mathscr{B}' \arrow[u, "\shortmid" marking, "H'"']
\end{tikzcd}
$$

is a universal filler for

$$
\begin{tikzcd}
\mathscr{C} \otimes \mathscr{C}' \arrow[r, "G\times G'"] & \mathscr{D} \otimes \mathscr{D}' \\
\mathscr{A} \otimes \mathscr{A}' \arrow[r, "F\times F'"'] & \mathscr{B} \arrow[u, "\shortmid" marking, "H\otimes H'"']
\end{tikzcd}
$$

From Naturality to Extranaturality (old stuff)

In order to understand how naturality produces extranaturality, it is useful to start from the observation that both kinds of transformation may be expressed by end formulas. The set of natural transformations between two functors $F, G\colon \mathscr{A} \longrightarrow \mathscr{B}$ is equal to the end

$$\int_{a\in\mathscr{A}} \mathscr{B}(Fa, Ga)$$

computed in the category of sets and functions. In particular, the set of natural transformations from the functor $\multimap\colon \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{M}$ to itself is equal to the end

$$\int_{a\in\mathscr{M}^{\mathrm{op}},\, x\in\mathscr{M}} \mathscr{M}(a \multimap x, a \multimap x). \tag{12}$$

Applying the natural bijection $\theta_{a,a\multimap x,x}$ componentwise induces a bijection between (12) and the end

$$\int_{a\in\mathscr{M}^{\mathrm{op}}, x\in\mathscr{M}} \mathscr{M}(a \otimes (a \multimap x), x). \tag{13}$$

This end describes the set of transformations natural in x and extranatural in $a = b$ from the functor

$$
\begin{array}{rccc}
1 \otimes (2 \multimap 3)\colon & \mathscr{M} \times \mathscr{M}^{\mathrm{op}} \times \mathscr{M} & \longrightarrow & \mathscr{M} \\
 & (a, b, x) & \mapsto & a \otimes (b \multimap x)
\end{array}
$$

to the identity functor on the category $\mathscr{M}$. (SORT OF AWKWARD!!! Do you say 'extranatural in $a = b$' here, or what???) The extranatural transformation eval is an element of (13), the image of the element of (12) associated to the identity natural transformation. Similarly, one shows that coeval is an element of the end

$$\int_{a\in\mathscr{M},\, x\in\mathscr{M}} \mathscr{M}(x, a \multimap (a \otimes x)).$$

In order to formulate an algebraic theory of monoidal closed categories, it seems necessary to consider there dinatural transformations.

To that purpose, it is tempting to start from the strong relationship observed between natural and dinatural transformations when they are formulated as ends, and to investigate how the 2-category CAT of categories, functors and natural transformations may be extended to incorporate a sufficiently large class of dinatural transformations: namely, the extranatural ones. To that purpose, it appears useful to start from the observation that the end of a functor

$$\varphi\colon \quad \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \quad \longrightarrow \quad \mathbf{Set}$$

may be computed as a right Kan extension in the bicategory of categories, distributors and natural transformations. More specifically, the end is the right Kan extension

$$\begin{tikzcd} \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \arrow[r, "\varphi"] \arrow[d, "\epsilon_{\mathscr{A}}"'] \arrow[dr, phantom, "\Uparrow"] & 1 \\ 1 \arrow[ur, bend right, "\int_{a\in\mathscr{A}} \varphi(a,a)"'] & \end{tikzcd}$$

of the distributor φ along the distributor $e_{\mathscr{A}}$, which is the functor

$$\begin{array}{rccc} \hom_{\mathscr{A}}\colon & \mathscr{A}^{\mathrm{op}} \times \mathscr{A} & \to & \mathrm{Set} \\ & (a,b) & \mapsto & \mathscr{A}(a,b) \end{array}$$

regarded as a profunctor from $\mathscr{A}^{\mathrm{op}} \times \mathscr{A}$ to 1.

From this follows that an element of the end

$$\int_{a\in\mathscr{A}} \varphi(a,a)$$

is the same thing as a natural transformation

$$
\begin{tikzcd}
\mathscr{A}^{\mathrm{op}} \times \mathscr{A} \arrow[r, "\varphi"] \arrow[d, "\hom_{\mathscr{A}}"'] \arrow[dr, phantom, "\Rightarrow"] & 1 \arrow[d, "\mathrm{id}"] \\
1 \arrow[r, "\mathrm{id}"'] & 1
\end{tikzcd}
$$

Eval and Coeval (old stuff)

The above constructions may be applied to eval

$$
\begin{tikzcd}
\mathscr{M} \times \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \arrow[r, "1\otimes(2\multimap 3)"] & \mathscr{M} \\
\mathscr{M} \arrow[u, "\hom \times \mathrm{id}"] \arrow[r, "\mathrm{id}"] \arrow[ur, phantom, "\Downarrow \mathrm{eval}"] & \mathscr{M} \arrow[u, "\mathrm{id}"']
\end{tikzcd}
$$

and to coeval:

$$
\begin{tikzcd}
\mathscr{M} \arrow[r, "\mathrm{id}"] & \mathscr{M} \\
\mathscr{M}^{\mathrm{op}} \times \mathscr{M} \times \mathscr{M} \arrow[u, "\hom \times \mathrm{id}"] \arrow[r, "1\multimap(2\otimes 3)"] \arrow[ur, phantom, "\Downarrow \mathrm{coeval}"] & \mathscr{M} \arrow[u, "\mathrm{id}"']
\end{tikzcd}
$$

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