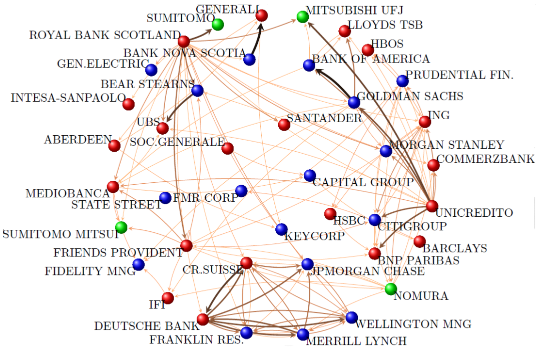
Network Theory

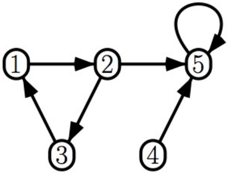
John Baez, for the Azimuth Project



for more, see:

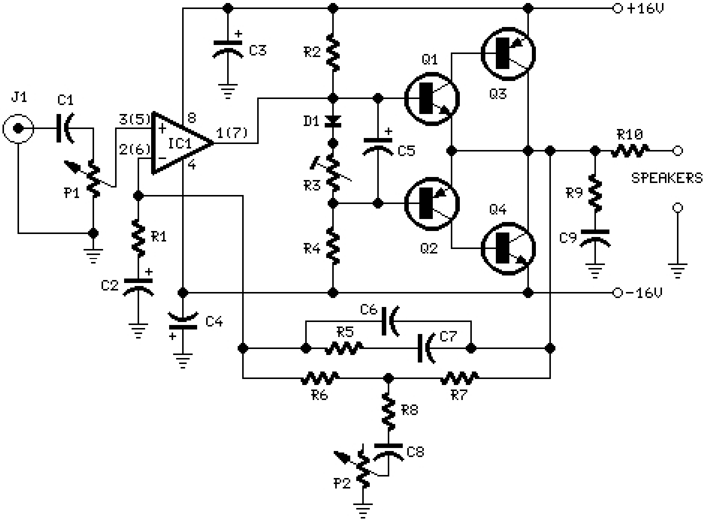
http://math.ucr.edu/home/baez/networks/

Network theory is the study of complex interacting systems that can be represented as graphs equipped with extra structure. A **graph** is a bunch of vertices connected by edges:

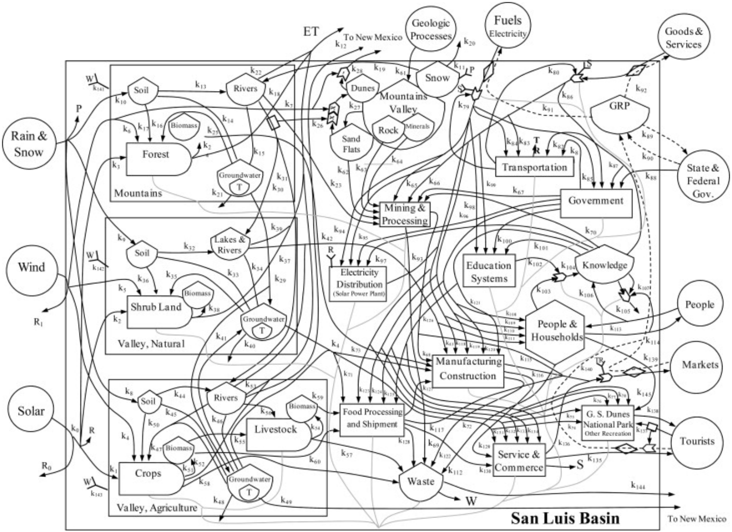


In this example, the 'extra structure' is that the vertices are labelled with numbers and the edges have arrows on them.

Network theory is a vast, sprawling subject. For example, it includes the study of electrical circuits:



In the 1950's, Howard Odum introduced networks to model the flow of resources like energy through ecosystems:



This is from a recent paper on the San Luis Basin in Colorado.

Starting in 2008, biologists have introduced Systems Biology Graphical Notation to describe networks. This is actually 3 different languages. For example, the Entity Relationship Language lets you talk about how entities affect each other:

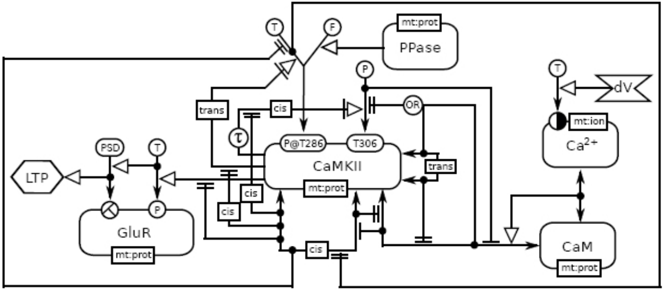
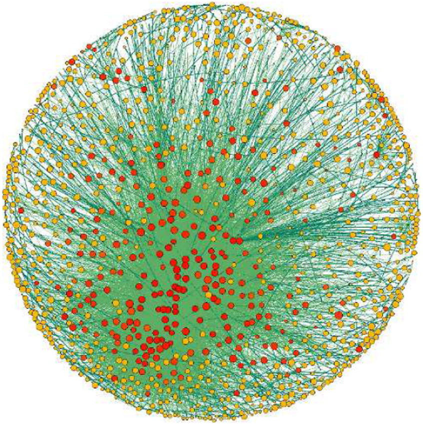


Figure A.2: *Regulation of calcium/calmoduline kinase II effect on synaptic plasticity.*

Many people use ‘network theory’ to mean the study of large graphs, and how they change with time.



This is from a paper on “the network of global corporate control”, which analyzed ownership links between 600,000 companies.

I've been working on 'reaction networks' and their applications to evolutionary game theory—a topic closely connected to economics.

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Reaction networks were born in chemistry. Here's an example:

$$\begin{tikzcd}
\mathrm{C} + \mathrm{O}_2 \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & \mathrm{CO}_2 \arrow[dl, bend left, "\delta"] \\
& \mathrm{CO} + \mathrm{O} \arrow[ur, bend left, "\gamma"] &
\end{tikzcd}$$

Here $\alpha, \beta, \gamma, \delta > 0$ are 'rate constants' for the reactions shown.

Reaction networks are also implicit in evolutionary game theory, a topic important in biology and economics.

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For example, suppose we have a population of agents of two kinds: 'aggressive' (A) and 'cooperative' (C). Their dynamics might be described by this reaction network:

$$A + A \xrightarrow{\alpha} A$$

$$A + C \xrightarrow{\beta} A$$

$$C + C \xrightarrow{\gamma} C + C + C$$

for some constants $\alpha, \beta, \gamma > 0$. The idea is that aggressive agents sometimes destroy the agents they meet, while cooperative ones sometimes reproduce.

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We could elaborate this example indefinitely by introducing more kinds of agents: for example, agents with different strategies, locations, or resources.

More formally, to give a reaction network we start with any finite collection of **species** $A_1, A_2, \ldots, A_k$.

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We define a **complex** to be a linear combination of species with natural number coefficients, e.g.

$$2A_1 + A_3 + A_4$$

We define a **reaction network** to be a graph with:

- vertices labelled by complexes
- edges labelled with arrows and also positive **rate constants**.

For example, if we have species A, B, C, D, E, here is an example of a reaction network:

$$
\begin{tikzcd}
A + 2B \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & C \\
& B + D + E \arrow[ur, "\gamma"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A + B \arrow[rr, bend left, "\delta"] & & 2C \arrow[ll, bend left, "\epsilon"]
\end{tikzcd}
$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are any positive numbers.

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$$\frac{d\psi}{dt} = H\psi$$

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$$\frac{d\psi}{dt} = H\psi$$

Here H is a matrix whose entries describe the probabilistic rate at which one complex turns into another. So, in detail:

$$\frac{d\psi_\ell}{dt} = \sum_{\ell'} H_{\ell\ell'}\psi_{\ell'}$$

where $H_{\ell\ell'}$ is the probabilistic rate at which ℓ' becomes ℓ.

We can write down the matrix entries $H_{\ell\ell'}$ starting from the reaction network by following some simple rules. For example, suppose we have this reaction network:

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Suppose $\ell' = 5A + 3B + C$ and $\ell = 4A + B + 2C$.

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Suppose $\ell' = 5A + 3B + C$ and $\ell = 4A + B + 2C$. Then

$$H_{\ell\ell'} = 5 \times 3 \times 2 \times \alpha$$

since the reaction on top turns $A + 2B$ into C and there are 5 ways to pick an A and 3×2 ways to pick two B's.

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The stability of these equilibria is proved by finding a 'Lyapunov function'. Roughly, this means showing that a certain quantity always decreases, and takes a minimum value at the equilibrium.

In applications to chemistry, this quantity is ‘free energy’. Free energy always decreases, and takes its minimum value in equilibrium. This is a way of saying that entropy approaches a maximum *subject to certain constraints*.

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In economic applications, it is not genomes but *strategies* that are being selected for.

An example: the **1-2-3 coordination game.** In ordinary game theory, this is a 2-player game where each player has 3 strategies, and *both* players win the following payoffs depending on their choice of strategy:

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It's called a **coordination game** since Nash equilibria with pure strategies arise when both players choose the same strategy. There are also Nash equilibria with mixed strategies.

But let's see this as an evolutionary game given by this reaction network:

$$A + A \xrightarrow{\alpha} A + A + A$$

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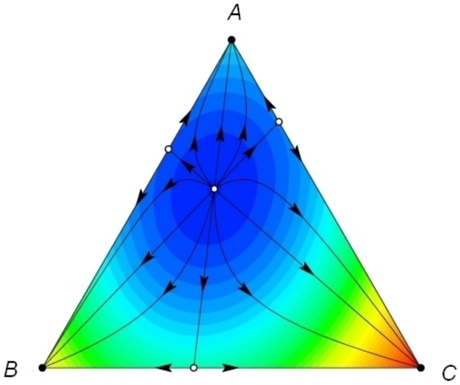
What does the master equation predict in the limit of large numbers?

The fraction of the population playing each strategy evolves as in this picture from Sandolm's *Evolutionary Game Theory*:



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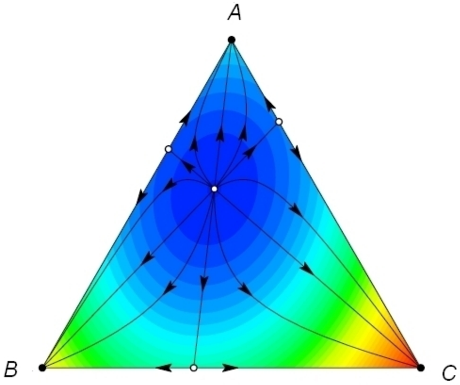

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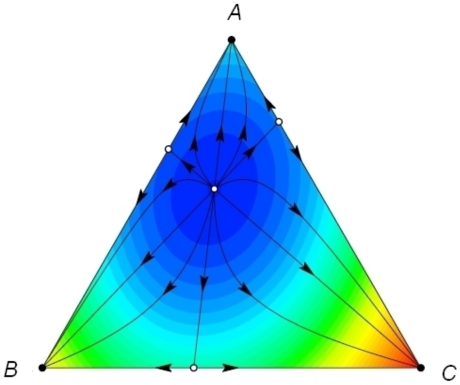

The equilibria here are the Nash equilibria. This example does *not* obey the conditions of the Deficiency Zero Theorem: that's how we can have nonunique and unstable equilibria.

Some references:

- Ernesto Estrada, *The Structure of Complex Networks: Theory and Applications*, Oxford U. Press, 2011.
- William Sandholm, *Evolutionary Game Theory*, 2007.
- Jonathan Guberman, *Mass Action Reaction Networks and the Deficiency Zero Theorem*, B.A. thesis, Department of Mathematics, Harvard University, 2003.
- John Baez, Jacob Biamonte and Brendan Fong, *Network Theory*, 2012.