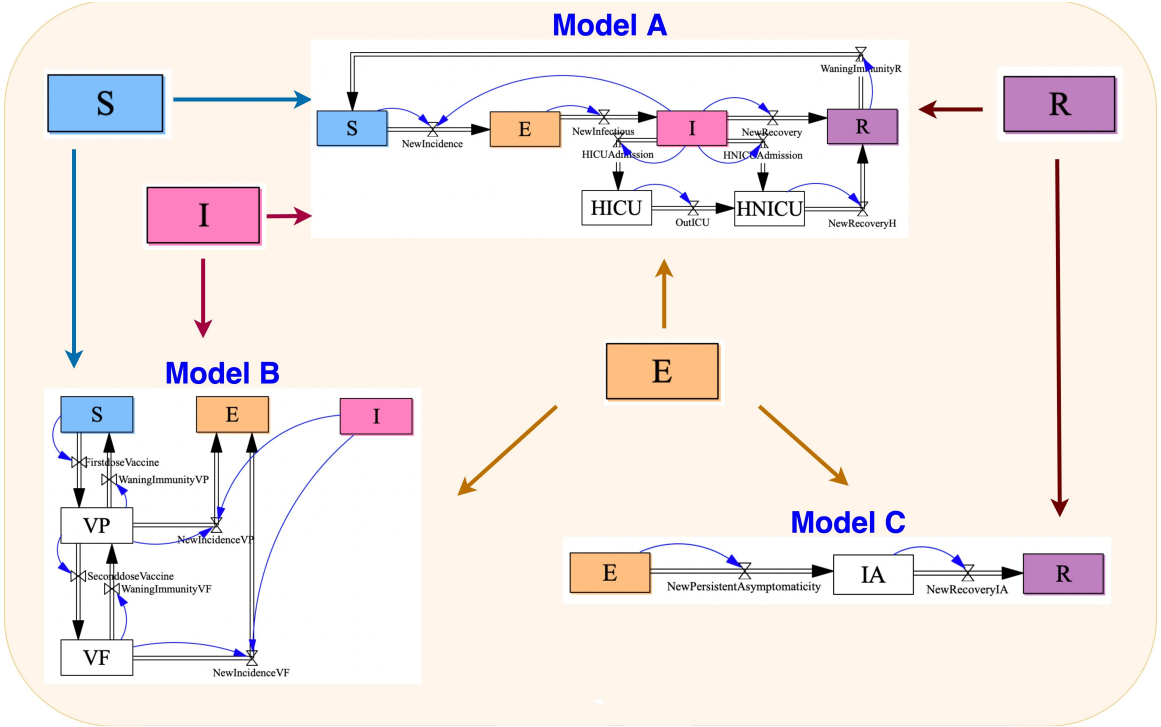
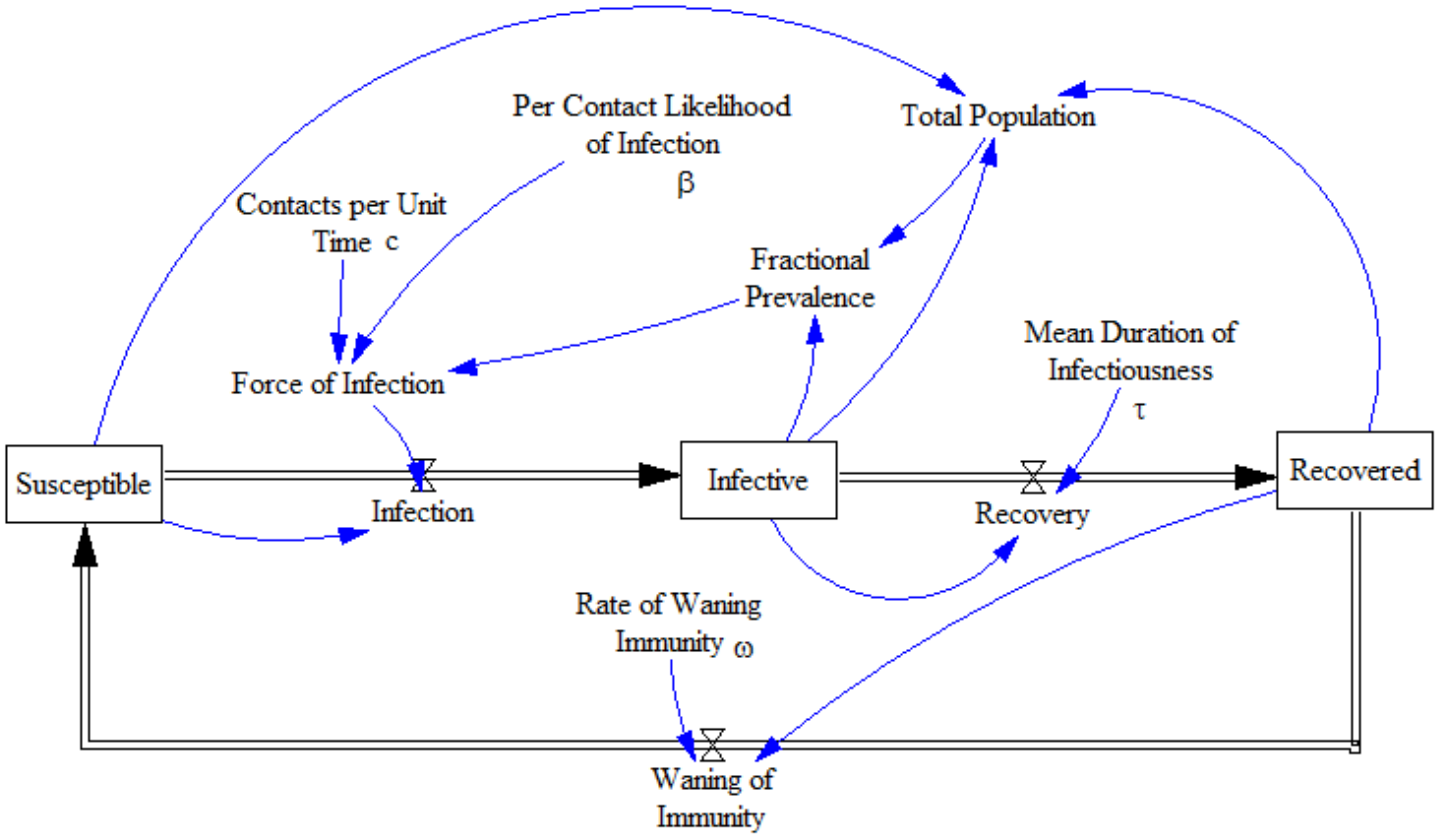
CATEGORY THEORY IN EPIDEMIOLOGY



John Baez

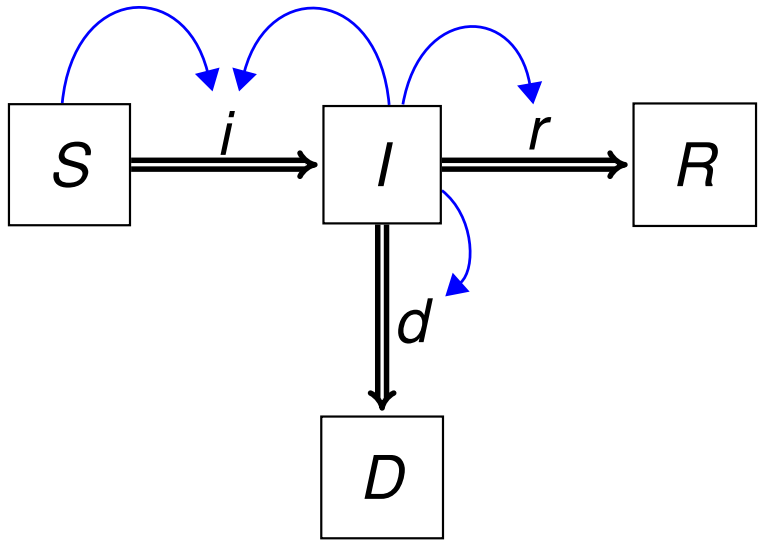
Edinburgh Mathematical Society
2023 December 8

In "System Dynamics", dynamical systems are modeled using "stock-flow diagrams":



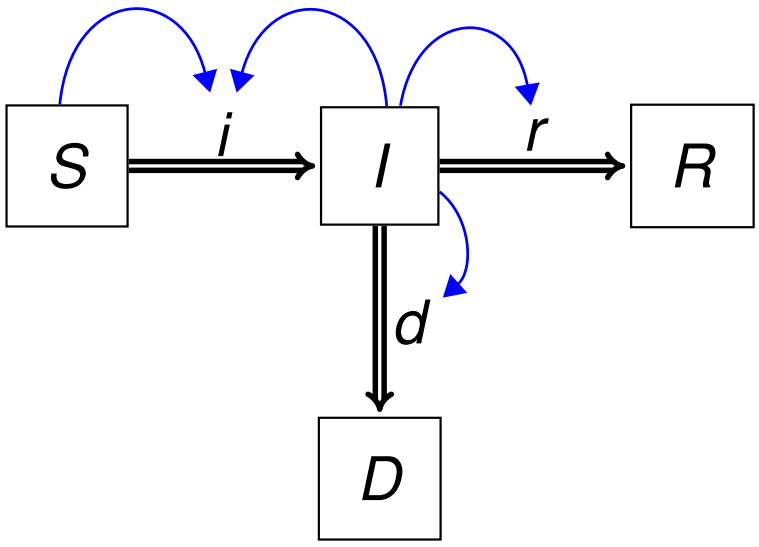
These diagrams are now widely used in economics, population biology, epidemiology, etc.

Here is a simple example of a stock-flow diagram:



The boxes are **stocks**, the double arrows are **flows**, and the blue arrows are **links** from stocks to flows.

If we equip each flow with a **flow function**, a stock-flow diagram gives differential equations describing how the stocks change with time.



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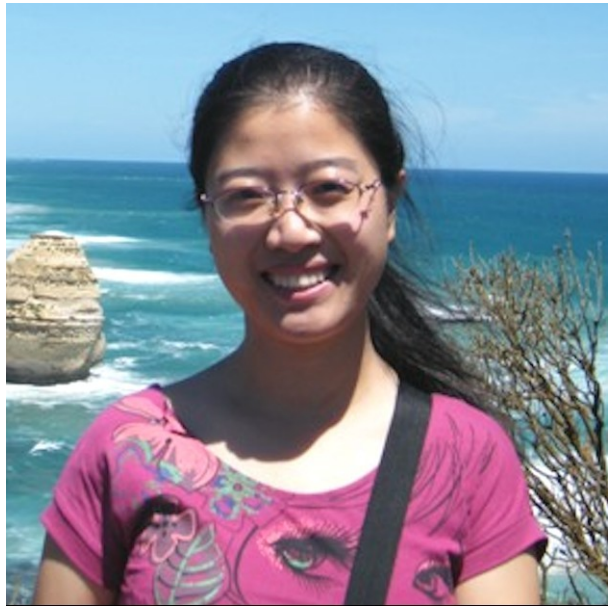
Shockingly, most people find it easier to understand diagrams than differential equations!

In "community-based modeling", diagrams help experts work with community members to build models of the problems they face.

Sociologists would say the power of diagrams is that they're "boundary objects":

> *A **boundary object** is any object that is part of multiple social worlds and facilitates communication between them; it has a different identity in each social world that it inhabits.*

There is a community of epidemiologists who use stock-flow diagrams to model the spread of disease. This includes my coauthors Nate Osgood and Xiaoyan Li, who did COVID modeling for the government of Canada.



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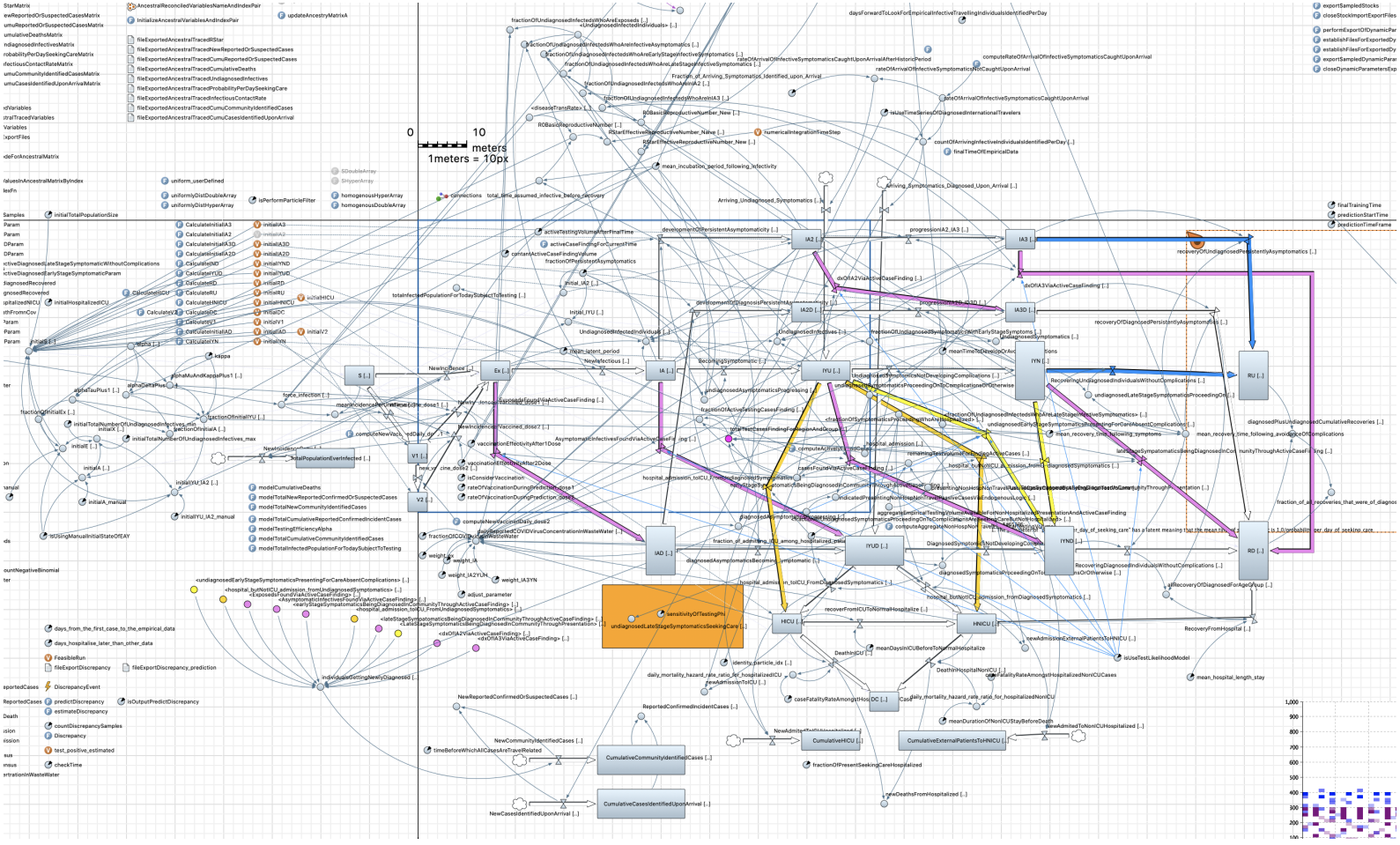
Our new work aims to fix these problems.

The ability to *compose* models is crucial because realistic models are complicated and built out of many smaller parts. Here is Osgood and Li's COVID model used by the government of Canada:



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In traditional applications to computer science, this says roughly:

- There is a category C whose objects are data types and morphisms are programs.
- There is a category D whose objects are sets and whose morphisms are (partially defined) functions.
- There is a functor $F : \mathrm{C} \to \mathrm{D}$ sending each data type to the set of data of that type, and sending each program to the function it computes.

We say F maps **syntax** to **semantics**.

In compositional modeling, functorial semantics works a bit differently:

- There is a category C where morphisms are models and *composing* morphisms lets us putting together smaller models to form bigger ones.

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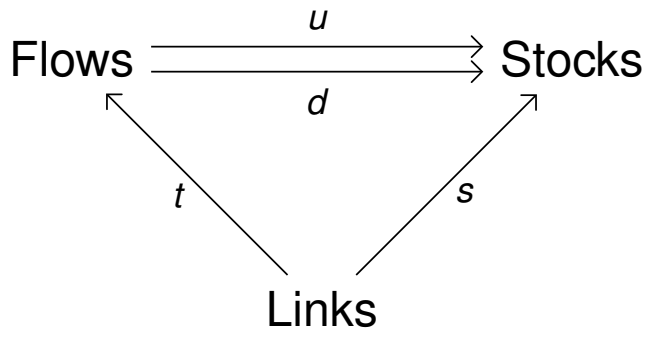
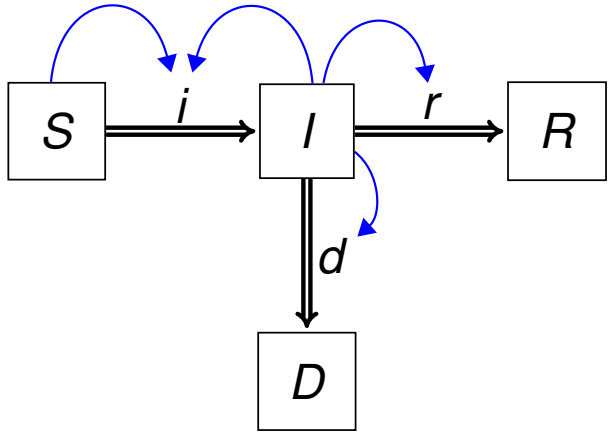
- There is a functor D where morphisms are collections of first-order ordinary differential equations.
- There is a functor $F : \mathrm{C} \to \mathrm{D}$ sending each model to the differential equations it describes.

This needs more explanation!

A **stock-flow diagram** consists of finite sets and functions:



together with, for each $f \in$ Flows, a **flow function** $\phi_f : \mathbb{R}^{L(f)} \to \mathbb{R}$ where $L(f)$ is the set of all $\ell \in$ Links with $t(\ell) = f$.

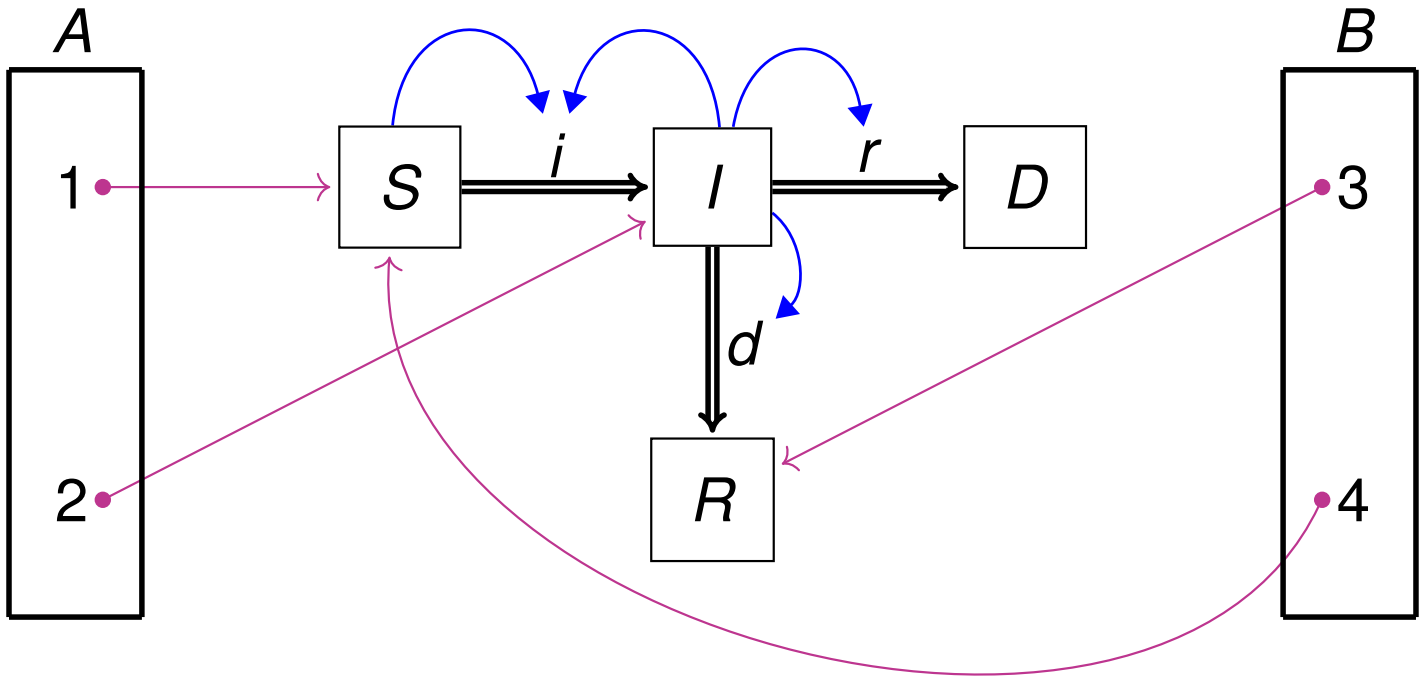


$\phi_i : \mathbb{R}^2 \to \mathbb{R}$ gives $\phi_i(S, I)$

$\phi_r : \mathbb{R} \to \mathbb{R}$ gives $\phi_r(I)$

$\phi_d : \mathbb{R} \to \mathbb{R}$ gives $\phi_d(I)$

An **open stock-flow diagram** is a stock-flow diagram equipped with maps $i\colon A \to \text{Stocks}$, $o\colon B \to \text{Stocks}$ for some finite sets A, B.

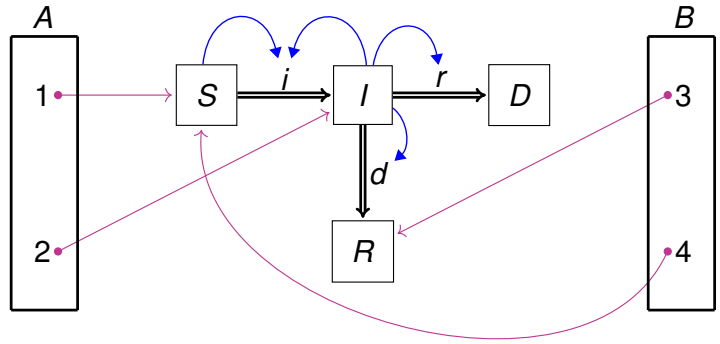


We call this an open stock-flow diagram **from *A* to *B*** and write it as $A \xrightarrow{F} B$.

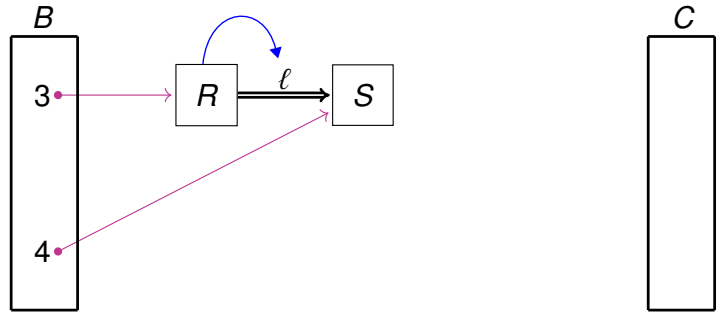
We can **compose** open stock-flow diagrams $A \xrightarrow{F} B$ and $B \xrightarrow{G} C$ by "gluing them together along B".

We get an open stock-flow diagram called $A \xrightarrow{G \circ F} C$.

$A \xrightarrow{F} B$



$B \xrightarrow{G} C$



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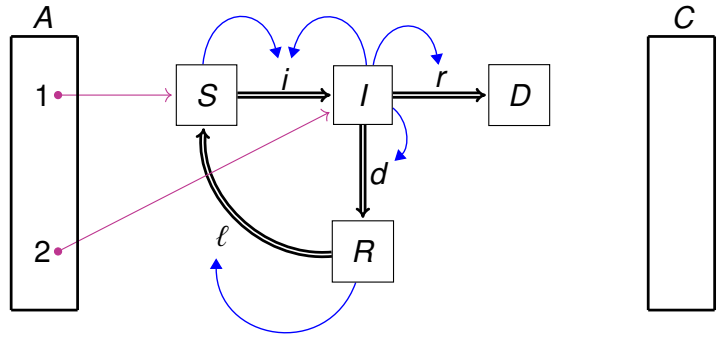


Thus, we get a category Open(StockFlow) with:

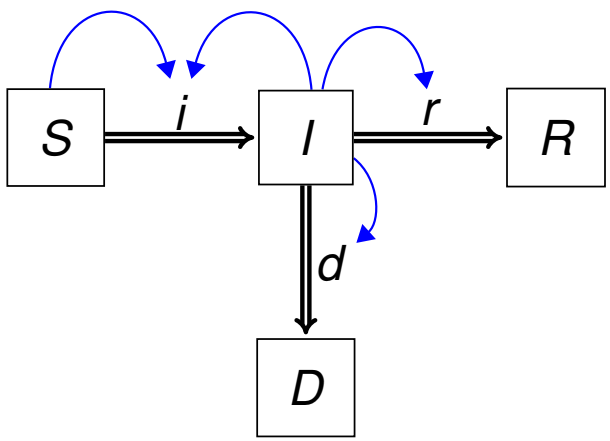
- finite sets as objects,
- open stock-flow diagrams as morphisms.

Next we can construct a category Open(Dynam) of "open dynamical systems", and a functor

$$\Phi : \mathrm{Open}(\mathrm{StockFlow}) \to \mathrm{Open}(\mathrm{Dynam})$$

This turns any open stock-flow diagram into an open dynamical system.

We have already seen how this works without the "openness":



$$\frac{dS}{dt} = -\phi_i(S, I)$$

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A **dynamical system** on some finite set of variables X is a vector field v on $\mathbb{R}^X$. This lets us write down a system of first-order ordinary differential equations.

For example, if $X = \{S, I, D, R\}$ and v is the 4-component vector field (v_S, v_I, v_D, v_R) on $\mathbb{R}^X \cong \mathbb{R}^4$, we get

$$\begin{aligned}
\frac{d}{dt}S(t) &= v_S(S(t), I(t), D(t), R(t)) \\
\frac{d}{dt}I(t) &= v_I(S(t), I(t), D(t), R(t)) \\
\frac{d}{dt}D(t) &= v_D(S(t), I(t), D(t), R(t)) \\
\frac{d}{dt}R(t) &= v_R(S(t), I(t), D(t), R(t))
\end{aligned}$$

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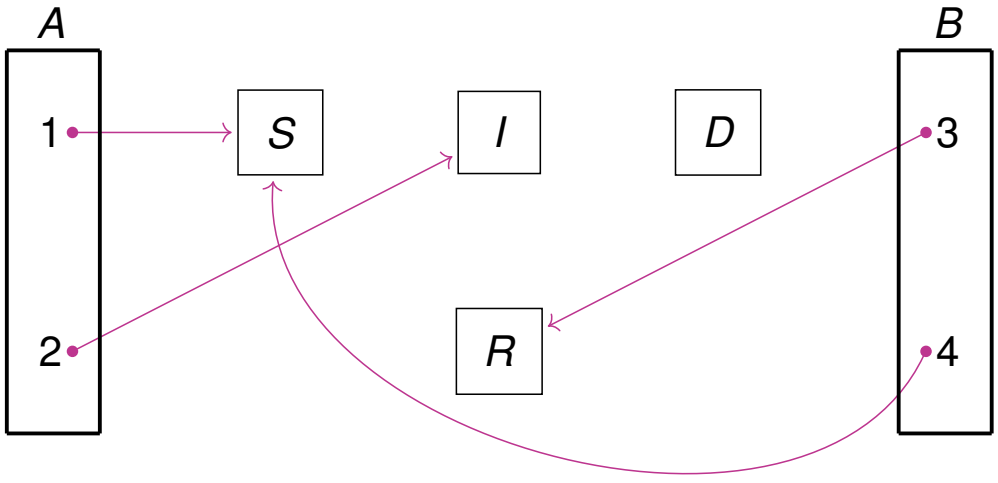
$$\begin{aligned} \frac{dS}{dt} &= v_S \\ \frac{dI}{dt} &= v_I \\ \frac{dD}{dt} &= v_D \\ \frac{dR}{dt} &= v_R \end{aligned}$$

An **open dynamical system** $A \xrightarrow{v} B$ is a dynamical system v on some finite set X equipped with maps $i\colon A \to X$, $o\colon B \to X$ for some finite sets A, B.

For example:



$$\frac{dS}{dt} = v_S$$

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Here $X = \{S, I, D, R\}$.

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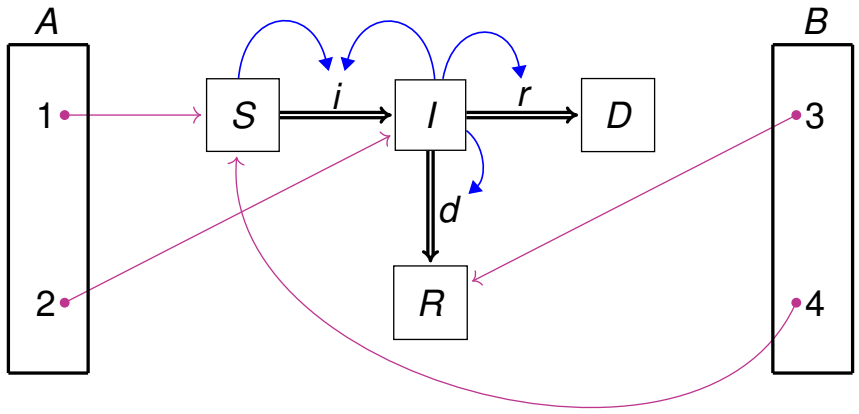
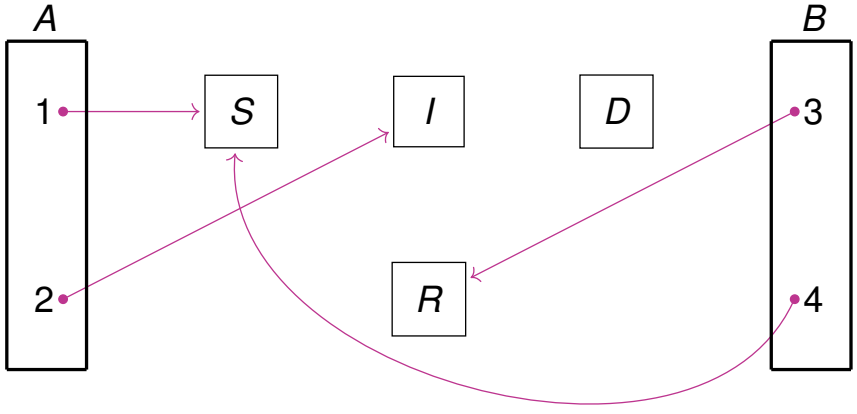
The process we've already seen for converting stock flow diagrams into dynamical systems then gives a functor

$$\Phi\colon \mathrm{Open}(\mathrm{StockFlow}) \to \mathrm{Open}(\mathrm{Dynam})$$

For example, Φ maps this open stock flow diagram:



to this *open* dynamical system:



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What does it mean that

$$\Phi : \mathrm{Open}(\mathrm{StockFlow}) \to \mathrm{Open}(\mathrm{Dynam})$$

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Most importantly, it means that for any open stock-flow diagrams $A \xrightarrow{F} B$ and $B \xrightarrow{G} C$ we have

$$\Phi(G \circ F) = \Phi(G) \circ \Phi(F)$$

We created software to implement these ideas with Evan Patterson and Sophie Libkind at the Topos Institute, who are experts on categories for computing:


- JB, Xiaoyan Li, Sophie Libkind, Nathaniel D. Osgood and Evan Patterson, Compositional modeling with stock and flow diagrams.

We used AlgebraicJulia: a framework for high-performance scientific computing using categories. This was developed by James Fairbanks, Evan, Sophie, and many others.

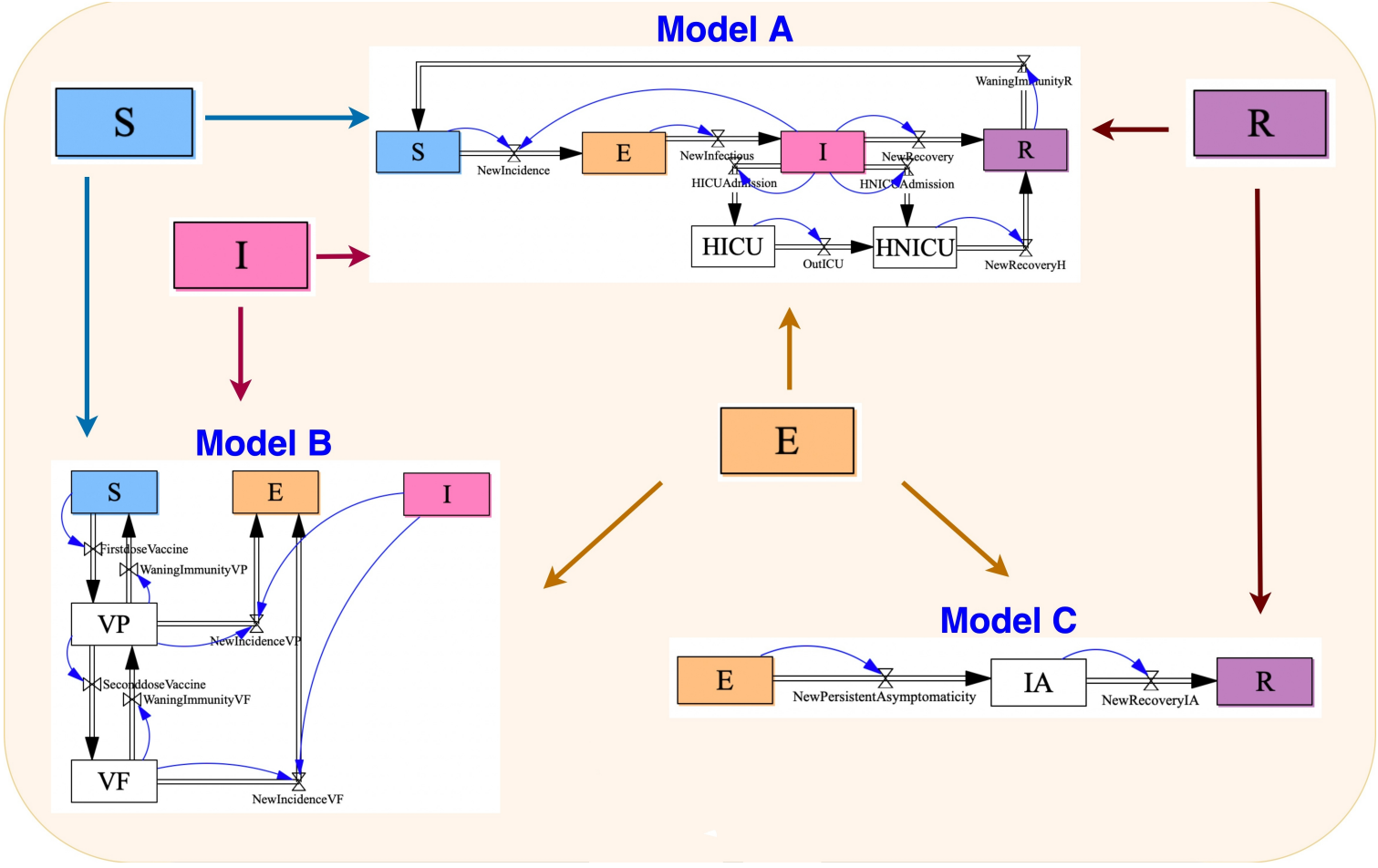
Composition of dynamical systems had already been implemented in AlgebraicJulia:

- Sophie Libkind, Andrew Baas, Evan Patterson and James Fairbanks, Operadic modeling of dynamical systems: mathematics and computation.

Using AlgebraicJulia we created a software package called **StockFlow**, now available on GitHub. This lets you:

- build stock-flow diagrams
- view them
- make them “open”
- apply various functors to interpret stock-flow diagrams, e.g. as systems of differential equations but also other things
- numerically solve the resulting differential equations.

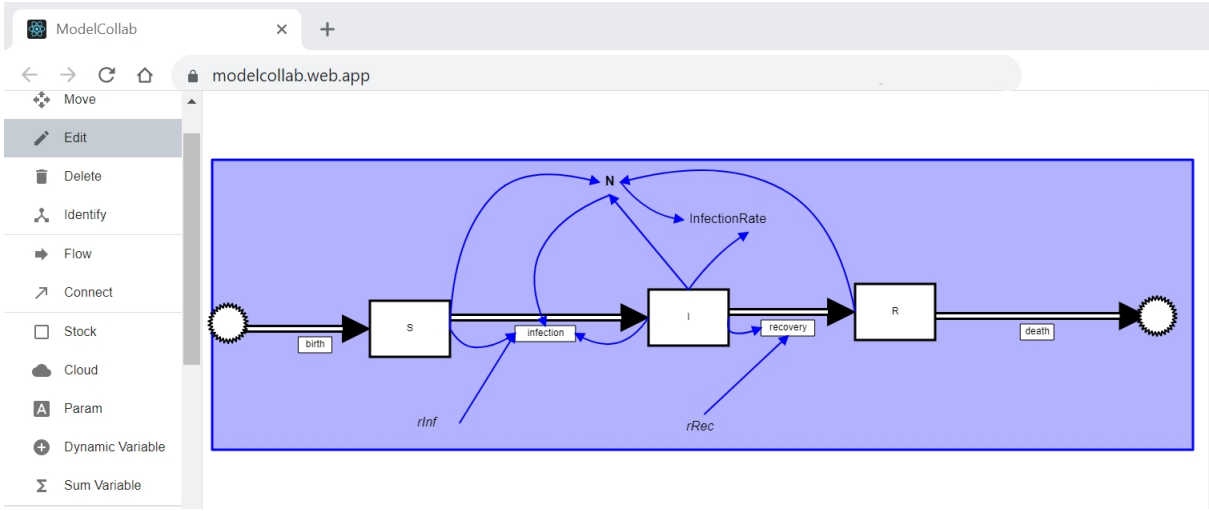
Besides composing open stock-flow diagrams end-to-end, our software also let you compose them in more complex ways, like this:



With Eric Redekopp we have made a graphical user interface for StockFlow, called **ModelCollab** — also available on GitHub.

- John C. Baez, Xiaoyan Li, Sophie Libkind, Nathaniel D. Osgood, Long Pham and Eric Redekopp, A categorical framework for modeling with stock and flow diagrams.

ModelCollab runs in your browser, so teams can collaborate to build stock-flow diagrams.



And you *don't need to know anything about category theory or AlgebraicJulia to use it!*

The fancy stuff is black-boxed.

Next steps:

In the spring of 2024, Nate, Xiaoyan, Evan, Kris Brown (Topos), Sean Wu (Merck) and I will spend 6 weeks at the ICMS here in Edinburgh.

We'll extend our work to "agent-based models". These simulate individual agents rather than treating them *en masse* as mere "stocks".

Together with Patricia Mabry at the HealthPartners Institute for Education and Research, Nate Osgood and I will apply compositional modeling to health problems related to substance abuse.

Finally, a little announcement.

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Nate Osgood and I have been called in to lead a program called Mathematics for Climate Change at the Fields Institute in Toronto.

It's not mainly about the geophysics of climate change — but rather, the human response to it: that is, *figuring out what we should do!*

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I hope we can form working groups where mathematicians learn from experts trying to solve the problems of climate change — and help them with subjects like these:

- Parameter estimation
- Causal discovery and attribution
- Optimization
- Uncertainty quantification
- Agent-based models
- Tipping point theory
- Game theory
- Decision theory