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On Quantum Fields Satisfying a Given Wave Equation

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Existence and uniqueness questions are treated for quantum fields on $\mathbb{R} \times S^1$ satisfying the nonlinear Klein–Gordon equation $(\square + m^2)\phi + \lambda :P'(\phi):_v = 0$, where P is a given real polynomial, bounded below, and v is the physical vacuum. When $\lambda > 0$ is sufficiently small, there exists a solution, but it need not be unique unless v is required to depend continuously on λ in certain sense. In particular, there exist two unitarily inequivalent solutions of the equation $(\square + m^2)\phi + \lambda :\phi^3:_v = 0$, for $\lambda > 0$ sufficiently small.

1. INTRODUCTION

One can rigorously construct quantum fields on $\mathbb{R} \times S^1$ satisfying nonlinear Klein–Gordon equations. Curiously, however, it is not known under what conditions the solution of a given equation is unique, *nor which equations can be solved.* The problem is that, in contrast to classical nonlinear wave equations, for quantum fields the Hamiltonian is not in general given *a priori*, but depends on the physical vacuum, which is also to be solved for. We begin with a summary of this problem; for more detailed accounts the reader is referred to our earlier work [2, 3].

Let ϕ denote the free scalar field of mass m, which is a distribution on $\mathbb{R} \times S^1$ whose values are self-adjoint operators on a Hilbert space $\mathbf{K}$, and which satisfies the linear Klein–Gordon equation

$$(\square + m^2)\phi = 0$$

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together with the canonical commutation relations

$$[\phi(t,x),\phi(t,y)]=[\dot\phi(t,x),\dot\phi(t,y)]=0,\qquad [\phi(t,x),\dot\phi(t,y)]=i\delta(x-y).$$

Let $P\colon \mathbb{R}\to\mathbb{R}$ be a polynomial, bounded below. At a formal level, if one defines $\phi_{\mathrm{int}}(t,x)$ to be the operator-valued distribution $e^{itH}\phi(0,x)\,e^{-itH}$, where the Hamiltonian H is an operator on $\mathbf{K}$ given by

$$\int_{t=0} \tfrac{1}{2}((\nabla\phi)^2+m^2\phi^2+\dot\phi^2)+P(\phi)\,dx,$$

then ϕ_{int} should satisfy the canonical commutation relations and the nonlinear Klein–Gordon equation

$$(\Box+m^2)\,\phi_{\mathrm{int}}+P'(\phi_{\mathrm{int}})=0.$$

The problem is to interpret the nonlinear functions of operator-valued distributions appearing in the formula for H. These may be defined using the theory of renormalized powers of quantum fields developed by Segal [7], which generalizes and makes precise Wick's "normal-ordered" powers. For any sufficiently regular vector $v\in\mathbf{K}$ there exists a self-adjoint operator on $\mathbf{K}$,

$$H(P,v)=H_0+\int_{t=0} :P(\phi):_v\,dx.$$

Up to a scalar multiple, $H(P,v)$ has a unique ground state, or "vacuum." If the vacuum of $H(P,v)$ is v itself, setting

$$\phi_{\mathrm{int}}(t,x)=e^{itH(P,v)}\phi(0,x)\,e^{-itH(P,v)},$$

one can indeed show that ϕ_{int} satisfies the nonlinear wave equation

$$(\Box+m^2)\,\phi_{\mathrm{int}}+:P'(\phi_{\mathrm{int}}):_v=0$$

on $\mathbb{R}\times S^1$.

The problem is thus reduced to that of finding a suitably regular vector $v\in\mathbf{K}$ such that the Hamiltonian $H(P,v)$ has v as its own vacuum. In physical terms, this condition expresses the fact that while the vacuum is defined in quantum field theory as the state of least energy, the definition of energy (i.e., the Hamiltonian) depends in turn on the vacuum, since the powers of fields involved must be renormalized. Since the condition that Hamiltonian $H(P,v)$ has v as its own vacuum is highly nonlinear, it is difficult to determine for which P there exists such v, or whether v is unique in any sense if it exists. Here we show that for any polynomial P that is bounded below and any sufficiently small $\lambda>0$ there exists $v\in\mathbf{K}$ such that the vacuum of

$H(\lambda P, v)$ is v. We find that uniqueness fails to hold in certain simple cases. Indeed, for $\lambda > 0$ sufficiently small, there exist at least two *unitarily inequivalent* solutions of

$$(\Box + m^2)\,\phi_{\text{int}} + \lambda :\phi_{\text{int}}^3:_v = 0,$$

where v is the vacuum of the Hamiltonian for ϕ_{int}.

These problems, and a plan of attack, were proposed by Segal [8], whom we thank for many useful discussions. This paper continues previous work by the authors [3], which in turn was greatly influenced by the work of Friedman [4]. It is also interesting to compare the work of Jaffe, Lesniewski, and Wieczerkowski [5] on quantum fields on $\mathbb{R} \times S^1$ satisfying *a priori* nonlinear wave equations, for which the problems discussed here do not arise.

2. Preliminaries

We begin by establishing notation; for more details see [2]. Given a complex Hilbert space $\mathbf{H}$, the "free boson field" over $\mathbf{H}$ is a system $(\mathbf{K}, W, \Gamma, v_0)$, unique up to unitary equivalence, such that:

(1) $\mathbf{K}$ is a complex Hilbert space;

(2) W is a strongly continuous map from $\mathbf{H}$ to unitaries on $\mathbf{K}$ satisfying the "Weyl relations":

$$W(x)\, W(y) = e^{-i\,\mathrm{Im}\langle x, y\rangle/2}\, W(x+y);$$

(3) Γ is a strongly continuous unitary representation of $U(\mathbf{H})$ on $\mathbf{K}$ such that

$$\Gamma(T)\, W(x)\, \Gamma(T)^{-1} = W(Tx)$$

for all $T \in U(\mathbf{H})$ and $x \in \mathbf{H}$, and $d\Gamma(A) \geqslant 0$ for all positive self-adjoint A on $\mathbf{H}$;

(4) v_0, the "free vacuum," is a unit vector in $\mathbf{K}$ invariant under Γ and cyclic for W.

We shall be working with the free scalar field of mass $m > 0$ on $\mathbb{R} \times S^1$. In this case the space of Cauchy data is $\mathbf{H} = H^{1/2}(S^1) \oplus H^{-1/2}(S^1)$, where the Sobolev space $H^p(S^1)$ denotes the completion of $C^\infty(S^1)$, the space of smooth *real* functions on S^1, in the norm given by

$$\|f\|^2 = \int_{S^1} (B^p f)^2\, dx,$$

where $B = (-\Delta + m^2)^{1/2}$ and we identify S^1 with $\mathbb{R}/\mathbb{Z}$. $\mathbf{H}$ is a complex Hilbert space with inner product

$$\langle (f_1, f_2), (g_1, g_2) \rangle = \int_{S^1} (B^{1/2} f_1)(B^{1/2} g_1) + (B^{-1/2} f_2)(B^{-1/2} g_2) + i(f_1 g_2 - f_2 g_1),$$

and complex structure $J: \mathbf{H} \to \mathbf{H}$ given by

$$J(f_1, f_2) = (-B^{-1} f_2, B f_1).$$

The "single-particle Hamiltonian" A is the self-adjoint operator on $\mathbf{H}$ given by

$$A(f_1, f_2) = (B f_1, B f_2).$$

The operator A is positive, so the "free field Hamiltonian" $H_0 = d\Gamma(A)$ is a non-negative self-adjoint operator on $\mathbf{K}$.

Let κ be the conjugation on $\mathbf{H}$ given by $\kappa(f_1, f_2) = (f_1, -f_2)$, and let $\mathbf{H}_\kappa$ be the corresponding real part of $\mathbf{H}$, which consists of all vectors of the form $(f, 0)$. Define U and V by

$$U(f) = W(B^{-1} f, 0), \qquad V(f) = W(0, Bf).$$

Then U and V are a "Weyl pair," that is,

$$U(f)\, U(g) = U(f+g), \qquad V(f)\, V(g) = V(f+g),$$

$$V(f)\, U(g) = \exp\left(i \int_{S^1} fg \right) U(g)\, V(f).$$

We shall work with the real wave representation of the free boson field. In this representation, $\mathbf{K}$ is identified with $L^2(\mathbf{H}_\kappa)$, the L^2-space relative to the isonormal Gaussian distribution of variance $\frac{1}{2}$ on $\mathbf{H}_\kappa$. On $L^2(\mathbf{H}_\kappa)$, U acts as multiplication operators, while V acts as unitarized translation operators.

We recall some basic facts about renormalized powers of the free field [7, 8]. Let $\mathbf{K}_p$ denote $L^p(\mathbf{H}_\kappa)$, and let

$$\mathbf{D} = D(H_0) \cap \bigcap_{p \in [2, \infty)} \mathbf{K}_p.$$

Let $u \in \mathbf{D}$ and $g \in C^\infty(S^1)$. Then for all $n \geqslant 0$ there exists a self-adjoint operator $:\Phi^n(g):_u$ on $\mathbf{K}$, which we may also write heuristically as

$$\int_{t=0} :\phi(x)^n:_u\, g(x)\, dx,$$

uniquely determined in a recursive manner by the commutation relations

$$U(f)\,{:}\Phi^n(g){:}_u\,U(-f) = {:}\Phi^n(g){:}_u, \tag{1}$$

$$V(f)\,{:}\Phi^n(g){:}_u\,V(-f) = \text{closure of} \sum_{m=0}^{n} \binom{n}{m} {:}\Phi^m(f^{n-m}g){:}_u, \tag{2}$$

for all $f \in C^\infty(S^1)$, together with the normalization conditions

$${:}\Phi^0(g){:}_u = \int_{S^1} g(x)\,dx \tag{3}$$

and

$$\langle u, {:}\Phi^n(g){:}_u\,u\rangle = 0 \tag{4}$$

for $n \geqslant 1$. For each n, the operator ${:}\Phi^n(g){:}_u$ is essentially self-adjoint on $\mathbf{D}$. We use the abbreviations ${:}\Phi^n{:}_u = {:}\Phi^n(1){:}_u$, ${:}\Phi^n{:} = {:}\Phi^n{:}_{v_0}$, $\Phi(f) = {:}\Phi^1(f){:}_{v_0}$, and $\Phi = \Phi(1)$. Note that $\Phi(f)$ is the self-adjoint generator of the one-parameter group $U(tf)$. There is a useful formula for "change of vacuum,"

$${:}\Phi^n{:}_u = \sum_{m=0}^{n} \binom{n}{m} \langle v, {:}\Phi^m{:}_u\,v\rangle\,{:}\Phi^{n-m}{:}_v \tag{5}$$

as operators on $\mathbf{D}$ for all $u, v \in \mathbf{D}$ with v invariant under $\Gamma(R(t))$, where $R(t)$ is the one-parameter group of unitaries on H induced by rotations of S^1. Such vectors v will be called "translation-invariant."

Let $\mathbf{P}$ denote the vector space of real-valued polynomials of degree $\leqslant 2d$, and let $\mathbf{C} \subset \mathbf{P}$ denote the cone of polynomials that are bounded below. If $P \in \mathbf{C}$ has $P(x) = \sum a_j x^j$, and $u \in \mathbf{D}$, we define $H(P, u)$ to be the closure of the operator $H_0 + \sum a_j\,{:}\Phi^j{:}_u$. Suppose $P \in \mathbf{C}$. Then $H(P, u)$ is self-adjoint, with pure point spectrum, and has a nondegenerate lowest eigenvalue which we denote by $E(P, u)$. There is a unique unit vector $v \in \mathbf{K}$, the "vacuum" of $H(P, u)$, such that $H(P, u)v = E(P, u)v$ and $v \geqslant 0$ as an element of $L^2(\mathbf{H}_\kappa)$. Moreover, $v \in \mathbf{D}$ and v are translation-invariant.

Before we state our main results we wish to make very clear in what sense having $u \in \mathbf{D}$ which is the vacuum of $H(P, u)$ gives rise to a quantum field satisfying a nonlinear Klein–Gordon equation. Suppose that u is the vacuum of $H(P, u)$, and let $f \in C^\infty(S^1)$. Then we define

$$\Phi_{\text{int}}(t, f) = e^{itH(P,u)}\Phi(f)\,e^{-itH(P,u)},$$

and we may define renormalized powers ${:}\Phi^n_{\text{int}}(t, f){:}_u$ by relations analogous to Eqs. (1)–(4), using instead of $U(f)$ and $V(f)$ the time-evolved Weyl pair

$$U_{\text{int}}(t, f) = e^{itH(P,u)}U(f)\,e^{-itH(P,u)},$$

$$V_{\text{int}}(t, f) = e^{itH(P,u)}V(f)\,e^{-itH(P,u)}.$$

Moreover, we have

$$:\Phi^n_{\mathrm{int}}(t, f):_u = e^{itH(P,u)} :\Phi^n(f):_u e^{-itH(P,u)}.$$

The proof of this makes essential use of the fact that the vacuum of $H(P, u)$ is u itself.

The entire vectors for $H(P, u)$ lie in $\mathbf{D}$, so if w is an entire vector, the integral

$$\int_{\mathbb{R}} \Phi_{\mathrm{int}}(t, f(t, \cdot))w \, dt$$

converges absolutely in $\mathbf{K}$ for all $f \in C_0^\infty(\mathbb{R} \times S^1)$, and is norm-continuous as a function of $f \in C_0^\infty(\mathbb{R} \times S^1)$. We may write this integral as

$$\left(\int_{\mathbb{R} \times S^1} \phi_{\mathrm{int}}(t, x) f(t, x) \, dt \, dx\right) w,$$

where ϕ_{int} is an operator-valued distribution, the "interacting field" with Hamiltonian $H(P, u)$. We similarly define the operator-valued distribution $:\phi^n_{\mathrm{int}}:_u$ by

$$\left(\int_{\mathbb{R} \times S^1} :\phi^n_{\mathrm{int}}(t, x):_u f(t, x) \, dt \, dx\right) w = \int_{\mathbb{R}} :\Phi^n_{\mathrm{int}}(t, f(t, \cdot)):_u w \, dt$$

and for any polynomial Q we define $:Q(\Phi_{\mathrm{int}})(t, f):_u$ and $:Q(\phi_{\mathrm{int}}):_u$ by linearity.

LEMMA 1. *Suppose that $u \in \mathbf{D}$ is the vacuum of $H(P, u)$. Then*

$$(\Box + m^2)\, \phi_{\mathrm{int}} + :P'(\phi_{\mathrm{int}}):_u = 0,$$

as distributions having as values operators defined on the entire vectors for $H(P, u)$.

It is physically important to note that the above equation is "local." That is, integrating any term of the left side against a test function $f \in C_0^\infty(\mathbb{R} \times S^1)$, we obtain an operator affiliated to the von Neumann algebra generated by the field operators

$$\int_{\mathbb{R} \times S^1} \phi_{\mathrm{int}}(t, x)\, g(t, x) \, dt \, dx$$

with $\mathrm{supp}(g) \subseteq \mathrm{supp}(f)$. For details, see Corollary 8.8.1 of [2].

Now we consider existence and uniqueness, for a given polynomial P, of a vector $u \in \mathbf{D}$ that is the vacuum of $H(P, u)$.

THEOREM 1. *Suppose that $P \in \mathbf{C}$. Then for some $\lambda_0 > 0$, there exists a unique function u: $[0, \lambda_0] \to \mathbf{D}$ such that:*

1. *$u(\lambda)$ is the vacuum of $H(\lambda P, u(\lambda))$.*
2. *u is norm-continuous from $[0, \lambda_0]$ to $\mathbf{K}$.*
3. *For some $p > 2$, u is bounded from $[0, \lambda_0]$ to $\mathbf{K}_p$.*

In fact, from the proof of this theorem, as $\lambda \downarrow 0$ the vacuum $u(\lambda)$ converges to the free vacuum v_0 in the norm topology on $\mathbf{K}_p$ for all p. Interestingly, if Conditions 2 and 3 in Theorem 1 are removed we have the following example of non-uniqueness:

THEOREM 2. *Let $P(x) = x^4$. There exists $\lambda_0 > 0$ and functions u_1, u_2: $(0, \lambda_0] \to \mathbf{D}$ such that*

1. *$u_i(\lambda)$ is the vacuum of $H(\lambda P, u_i(\lambda))$.*
2. *$u_i(\lambda)$ is even as an element of $L^2(\mathbf{H}_\kappa)$.*
3. *For all $\lambda \in (0, \lambda_0]$, $u_1(\lambda) \neq u_2(\lambda)$.*

Here $v \in L^2(\mathbf{H}_\kappa)$ is said to be "even" if it is invariant under the transformation $\Gamma(-I)$, which corresponds to reflection about the origin of $\mathbf{H}_\kappa$.

By Lemma 1, both of the vacua $u_i(\lambda)$ in Theorem 2 give rise to solutions ϕ_i of the equation

$$(\square + m^2)\,\phi_i + \lambda :\phi_i^3:_{u_i(\lambda)} = 0.$$

We would like to make very precise the sense in which these are physically distinct quantum fields satisfying the same nonlinear wave equation.

THEOREM 3. *For $\lambda \in (0, \lambda_0]$, let P, $u_1 = u_1(\lambda)$, $u_2 = u_2(\lambda)$ be as in Theorem 2. Then there is no unitary operator U on $\mathbf{K}$ such that*

$$H(\lambda P, u_1) = UH(\lambda P, u_2)\, U^{-1}$$

and

$$\Phi(f) = U\Phi(f)\, U^{-1}$$

for all $f \in C^\infty(S^1)$.

3. Proofs

We begin by showing that the interacting field satisfies a wave equation.

Proof of Lemma 1. We need to show that for any $f \in C_0^\infty(\mathbb{R} \times S^1)$,

$$\int_{\mathbb{R} \times S^1} ((\Box + m^2)\, \phi_{\rm int}(t, x) + :P'(\phi_{\rm int})(t, x):_u)\, f(t, x)\, dx\, dt = 0.$$

We will write simply f for the function $f(t, \cdot)$ on S^1, leaving the dependence on t implicit, and similarly write $\dot{f}$ and $\ddot{f}$ for the first and second derivatives of $f(t, \cdot)$ with respect to t. We also write H for $H(P, u)$. Let

$$\dot{\Phi}_{\rm int}(t, f) = i[H, \Phi_{\rm int}(t, f)].$$

Then as operators on entire vectors for H,

$$\frac{d}{dt} \Phi_{\rm int}(t, f) = \dot{\Phi}_{\rm int}(t, f) + \Phi_{\rm int}(t, \dot{f}),$$

$$\begin{aligned} \frac{d^2}{dt^2} \Phi_{\rm int}(t, f) &= i[H, \dot{\Phi}_{\rm int}(t, f)] + 2\dot{\Phi}_{\rm int}(t, \dot{f}) + \Phi_{\rm int}(t, \ddot{f}) \\ &= \Phi_{\rm int}(t, \ddot{f} - B^2 f) - :P'(\Phi_{\rm int})(t, f):_u + 2\dot{\Phi}_{\rm int}(t, \dot{f}), \end{aligned} \tag{6}$$

and

$$\begin{aligned} \frac{d}{dt} \dot{\Phi}_{\rm int}(t, f) &= i[H, \dot{\Phi}_{\rm int}(t, f)] + \dot{\Phi}_{\rm int}(t, \dot{f}) \\ &= -\Phi_{\rm int}(t, B^2 f) - :P'(\Phi_{\rm int})(t, f):_u + \dot{\Phi}_{\rm int}(t, \dot{f}). \end{aligned} \tag{7}$$

Integrating Eq. (6) with respect to t yields

$$\begin{aligned} &\int_{\mathbb{R} \times S^1} ((\partial_t^2 - B^2)\, \phi_{\rm int}(t, x) - :P'(\phi_{\rm int})(t, x):_u)\, f(t, x)\, dt\, dx \\ &\quad = -2 \int_{\mathbb{R}} \dot{\Phi}(t, \dot{f})\, dt. \end{aligned}$$

Integrating Eq. (7) with respect to t gives

$$\int_{\mathbb{R}} \dot{\Phi}(t, \dot{f})\, dt = \int_{\mathbb{R} \times S^1} (B^2 \phi_{\rm int}(t, x) + :P'(\phi_{\rm int})(t, x):_u)\, f(t, x)\, dt\, dx.$$

Combining the last two equations finishes the proof. ∎

Next we consider Hamiltonians renormalized relative to the free vacuum. Recall that $\mathbf{C} \subset \mathbf{P}$ denotes the cone of polynomials of degree $2d$ that are bounded below. Given $P \in \mathbf{C}$, let $H(P)$ denote $H(P, v_0)$, let $v(P)$ denote the vacuum of $H(P)$, and let the "vacuum energy" $E(P)$ be given by

$$H(P)\, v(P) = E(P)\, v(P).$$

Note that $H(0) = H_0$, $v(0) = v_0$, and $E(0) = 0$.

Using Eq. (5), for any $Q \in \mathbf{C}$ there exists a unique $\tilde{Q} \in \mathbf{C}$ such that

$$H(Q) = H(\tilde{Q}, v(Q)).$$

Explicitly, if $Q(x) = \sum_j a_j x^j$, we have

$$\tilde{Q}(x) = \sum_{0 \leqslant k \leqslant j} a_j \binom{j}{k} \langle v(Q), :\Phi^k: v(Q) \rangle\, x^{j-k}. \tag{8}$$

Note that $v(Q)$ is the vacuum of $H(\tilde{Q}, v(Q))$. Conversely, if $v \in \mathbf{D}$ is the vacuum of $H(P, v)$ for some $P \in \mathbf{C}$, we may use Eq. (5) to write $H(P, v) = H(Q)$ for some $Q \in \mathbf{C}$. It follows that $v = v(Q)$ and $P = \tilde{Q}$.

In short, for any $P \in \mathbf{C}$ there exists $v \in \mathbf{D}$ such that v is the vacuum of $H(P, v)$ if and only if $P = \tilde{Q}$ for some $Q \in \mathbf{C}$. To use this fact we must understand the range of the map $T: \mathbf{C} \to \mathbf{C}$ given by

$$T(Q) = \tilde{Q}.$$

We shall use the implicit function theorem to show that "sufficiently small" polynomials are in the range of T, obtaining Theorem 2. To this end we calculate the Jacobian of the map T near the origin in $\mathbf{C}$.

LEMMA 2. *The function $E(P)$ is C^1 on the interior of* $\mathbf{C}$. *The map $P \mapsto v(P)$ is C^1 from the interior of* $\mathbf{C}$ *to* $\mathbf{K}$, *and continuous from the interior of* $\mathbf{C}$ *to* $\mathbf{K}_p$ *for all $p \in [2, \infty)$. Given any convex open cone* $\mathbf{C}_0$ *in the interior of* $\mathbf{C}$, *$E(P) \to 0$ and $v(P) \to v_0$ in* $\mathbf{K}_p$ *for all $p \in [2, \infty)$ as $P \to 0$ in* $\mathbf{C}_0$.

Proof. For P in the interior of $\mathbf{C}$, $H(P)$ is an analytic family of type B in the sense of Kato [6, 9]. Moreover $E(P)$ is a nondegenerate isolated eigenvalue of $H(P)$, so by Kato's perturbation theory $v(P)$ is analytic from the interior of $\mathbf{C}$ to $\mathbf{K}$, and $E(P)$ is an analytic function on the interior of $\mathbf{C}$. (Note that while Kato only considers one-parameter families, the generalization to multi-parameter families is straightforward.) Let S be a fixed subset of $\mathbf{C}_0$ whose closure is compact. By an estimate of Segal [8], for any $p \in [2, \infty)$ there exists $t > 0$ such that $e^{-tH(P)}$ is bounded from $\mathbf{K}$ to $\mathbf{K}_p$, uniformly for all $P \in S$. Since

$$v(P) = e^{tE(P)} e^{-tH(P)} v(P)$$

and $E(P) \leqslant p_0$, where p_0 is the constant term of P, $\|v(P)\|_p$ is uniformly bounded on S. Since $v(P)$ is continuous from S to $\mathbf{K}$, it follows from Hölder's inequality that $v(P)$ is continuous from S to $\mathbf{K}_p$ for all $p \in [2, \infty)$. Also by Hölder's inequality, $\langle v(P), :P(\Phi): v(P) \rangle \to 0$ as $P \to 0$ in $\mathbf{C}_0$. Note that

$$p_0 = \langle v_0, H(P) v_0 \rangle \geqslant E(P) = \langle v(P), H(P)\, v(P) \rangle \geqslant \langle v(P), :P(\Phi): v(P) \rangle,$$

so $E(P) \to 0$ as $P \to 0$ in $\mathbf{C}_0$.

To prove that $v(P) \to v_0$ in $\mathbf{K}_p$ as $P \to 0$ in C_0 it suffices to prove this for $p = 2$. Write $v(P) = a(P)\, v_0 + v_1(P)$, where $a(P) = \langle v_0, v(P) \rangle$, so that $v_1(P)$ is orthogonal to v_0. Since $\langle v(P), H(P)\, v(P) \rangle = E(P)$ and $H_0 v_0 = 0$,

$$\langle v_1(P), H_0 v_1(P) \rangle = E(P) - \langle v(P), :P(\Phi): v(P) \rangle \to 0$$

as $P \to 0$ in $\mathbf{C}_0$, which means that $\lambda_1 \, \|v_1(P)\|^2 \to 0$, where λ_1 is the first positive eigenvalue of H_0. Hence $v_1(P) \to 0$, which completes the proof that $v(P) \to v_0$. ∎

LEMMA 3. *The map T is C^1 in the interior of $\mathbf{C}$. Given any convex open cone $\mathbf{C}_0$ in the interior of $\mathbf{C}$, as $P \to 0$ in $\mathbf{C}_0$, $T(P) \to 0$ and $dT(P) \to I$.*

Proof. By Lemma 2 and Eq. (8), T is C^1 in the interior of $\mathbf{C}$ and $T(P) \to 0$ as $P \to 0$ in $\mathbf{C}_0$.

Differentiating the equation $H(P)\, v(P) = E(P)\, v(P)$, one obtains

$$\frac{d}{d\varepsilon} E(P + \varepsilon Q)\bigg|_{\varepsilon = 0} = \langle v(P), Qv(P) \rangle$$

and

$$(H(P) - E(P)) \frac{d}{d\varepsilon} v(P + \varepsilon Q)\bigg|_{\varepsilon = 0} = (\langle v(P), Qv(P) \rangle - Q)\, v(P)$$

for all $P \in \mathbf{C}_0$ and $Q \in \mathbf{P}$. The former clearly converges to $\langle v_0, Qv_0 \rangle$ as $P \to 0$ in $\mathbf{C}_0$. We claim the latter implies that

$$\frac{d}{d\varepsilon} v(P + \varepsilon Q)\bigg|_{\varepsilon = 0} \to \tilde{H}_0^{-1}(\langle v_0, Qv_0 \rangle - Q)\, v_0, \tag{9}$$

where $\tilde{H}_0^{-1}$ is the inverse of the operator $\tilde{H}_0$ that is H_0 on the space of vectors orthogonal to v_0 and the identity on the space spanned by v_0. To see this, let $\tilde{H}_P$ be the operator that is $H(P) - E(P)$ on the space of vectors orthogonal to $v(P)$ and the identity on the space spanned by $v(P)$, i.e., $\tilde{H}_P = H(P) - E(P) + \pi_{v(P)}$, where $\pi_{v(P)}$ is the orthogonal projection onto the subspace spanned by $v(P)$. Noting that there is an $\varepsilon_0 > 0$ such that $\tilde{H}_P > \varepsilon_0$ for all P near 0 in $\mathbf{C}_0$ and that $(\langle v(P), Qv(P)\rangle - Q)\, v(P) \to (\langle v_0, Qv_0\rangle - Q)\, v_0$ in $\mathbf{K}$, Eq. (9) follows if we can prove that $\tilde{H}_P^{-1}u \to \tilde{H}_0^{-1}u$ as $P \to 0$ in $\mathbf{C}_0$ for any $u \in \mathbf{K}$. It is actually enough to show that

$$\tilde{H}_P^{-1}u \to \tilde{H}_0^{-1}u = E^{-1}u, \tag{10}$$

for every eigenfunction u of $\tilde{H}_0$ associated with the eigenvalue E. To this end, let us consider the identity

$$\begin{aligned}\tilde{H}_P \tilde{H}_0^{-1}u &= (\tilde{H}_0 + :P(\phi):_{v(P)} + \pi_{v(P)} - \pi_{v_0} - E(P))\, \tilde{H}_0^{-1}u \\ &= u + E^{-1}(:P(\phi):_{v(P)} + \pi_{v(P)} - \pi_{v_0} - E(P))u.\end{aligned}$$

It follows that

$$\tilde{H}_P^{-1}u = \tilde{H}_0^{-1}u - E^{-1}\tilde{H}_P^{-1}(:P(\phi):_{v(P)} + \pi_{v(P)} - \pi_{v_0} - E(P))u.$$

Since the functional corresponding to the multiplication operator $:P(\phi):_{v(P)}$ converges to 0 in $\mathbf{K}_p$ for some $p \in (2, \infty)$, and $u \in \mathbf{K}_{p'}$, where $1/p + 1/p' = \frac{1}{2}$, it follows that $(:P(\phi):_{v(P)} + \pi_{v(P)} - \pi_{v_0} - E(P))u \to 0$ in $\mathbf{K}$ as $P \to 0$ in $\mathbf{C}_0$. Consequently

$$E^{-1}\tilde{H}_P^{-1}(:P(\phi):_{v(P)} + \pi_{v(P)} - \pi_{v_0} - E(P))u \to 0$$

which completes the proof of (10).

Since $v(P) \to v_0$ in each $\mathbf{K}_p$ as $P \to 0$ in $\mathbf{C}_0$ by Lemma 2, and the derivative of $v(P)$ converges in $\mathbf{K}$ by (9), it follows from Eq. (8) that the derivative of T converges to the identity as $P \to 0$ in $\mathbf{C}_0$. ∎

Proof of Theorem 1. Let $\mathbf{C}_0$ be a convex open cone in $\mathbf{C}$ containing P. By Lemma 3 and the implicit function theorem for cones [3], for some $\lambda_0 > 0$ there exists a unique continuous function Q: $[0, \lambda_0] \to \mathbf{C}_0 \cup \{0\}$ such that $Q(0) = 0$ and

$$T(Q_\lambda) = \lambda P.$$

Let $u(\lambda) = v(Q_\lambda)$. It follows from the definition of T that

$$H(Q_\lambda) = H(\lambda P, u(\lambda)).$$

Thus $u(\lambda)$ is the vacuum of $H(\lambda P, u(\lambda))$, as desired. Since $u(\lambda)$ is the vacuum of $H(Q_\lambda)$, it follows that $u(\lambda) \in \mathbf{D}$. Since Q_λ is a continuous function from $[0, \lambda_0]$ to $\mathbf{C}_0 \cup \{0\}$, it follows from Lemma 2 that $u(\lambda)$ is continuous from $[0, \lambda_0]$ to $\mathbf{K}_p$ for all $p \in [2, \infty)$. This proves the existence part of the theorem.

For the uniqueness, suppose that $u(\lambda)$ satisfies Conditions 1–3. It follows that $\lambda P = T(Q_\lambda)$ for some $Q_\lambda \in \mathbf{C}$, and $u(\lambda) = v(Q_\lambda)$. By Conditions 2 and 3 it follows that $\langle v(Q_\lambda), :\Phi^n: v(Q_\lambda)\rangle \to 0$ as $\lambda \downarrow 0$, so Eq. (8) implies that $Q_\lambda \to 0$ in $\mathbf{C}_0$ as $\lambda \downarrow 0$. The same reasoning shows that Q_λ depends continuously on λ for $\lambda \geqslant 0$. But as shown above, for some $\lambda_0 > 0$ there is unique continuous function $Q: [0, \lambda_0] \to \mathbf{C}_0$ such that $T(Q_\lambda) = \lambda P$. It follows that Q_λ, and hence $u(\lambda) = v(Q_\lambda)$, is unique. ∎

We conclude with proofs of the nonuniqueness results, Theorems 2 and 3.

Proof of Theorem 2. Given $b \in \mathbb{R}$ and $\lambda > 0$, let $H_{b,\lambda}$ denote the closure of

$$H_0 + \frac{b}{2} :\Phi^2: + \lambda :\Phi^4:.$$

Let $v_{b,\lambda}$ be the vacuum of $H_{b,\lambda}$, and let $E_{b,\lambda}$ be the vacuum energy given by $H_{b,\lambda} v_{b,\lambda} = E_{b,\lambda} v_{b,\lambda}$. Then $H_{b,\lambda} = H(R, v_{b,\lambda})$, where

$$R(x) = \lambda x^4 + \frac{1}{2} a(b, \lambda) x^2 + c(b, \lambda),$$

where the constant $c(b, \lambda)$ does not concern us, but

$$a(b, \lambda) = b + 12\lambda \langle v_{b,\lambda}, :\Phi^2: v_{b,\lambda}\rangle$$

by Eq. (5).

First we estimate $a(b, \lambda)$ as a function of b when $b < -m^2$. We have

$$\begin{aligned} E_{b,\lambda} &\leqslant \langle V(s) v_0, H_{b,\lambda} V(s) v_0 \rangle \\ &= \left\langle v_0, \left(\int_{t=0} :\frac{1}{2}(m^2 + b)(\phi - s)^2 + \lambda(\phi - s)^4: dx \right) v_0 \right\rangle \\ &= \frac{m^2 + b}{2} s^2 + \lambda s^4 \end{aligned}$$

for every $s \in \mathbb{R}$. Minimizing over s, we obtain

$$E_{b,\lambda} \leqslant -\frac{(m^2 + b)^2}{16\lambda}.$$

Note that $H_0 + :\Phi^4: > -c$ for some $c > 0$, so that if $\lambda < 1$ we have $H_0 + \lambda :\Phi^4: > -\lambda c$, hence

$$\frac{b}{2} \langle v_{b,\lambda}, :\Phi^2: v_{b,\lambda} \rangle = E_{b,\lambda} - \langle v_{b,\lambda}, (H_0 + \lambda :\Phi^4:) v_{b,\lambda} \rangle \leqslant -\frac{(m^2+b)^2}{16\lambda} + \lambda c.$$

It follows that for $b < -m^2$, we have

$$a(b, \lambda) = b + 12\lambda \langle v_{b,\lambda}, :\Phi^2: v_{b,\lambda} \rangle \geqslant b - \frac{3(m^2+b)^2}{2b} + \frac{24\lambda^2 c}{b}. \tag{11}$$

Next, for $b = 0$,

$$a(0, \lambda) = 12\lambda \langle v_{0,\lambda}, :\Phi^2: v_{0,\lambda} \rangle,$$

which converges to 0 as $\lambda \downarrow 0$ by Lemma 2. Similarly, for any $b > 0$ we have $a(b, \lambda) \to b$ as $\lambda \downarrow 0$, since $v(b, \lambda)$ converges in $\mathbf{D}$ (while this is not a consequence of Lemma 2, the technique of the proof applies). Fixing $k > 0$, we can choose $\lambda_1 > 0$ such that

$$a(0, \lambda) < k/2, \qquad a(2k, \lambda) > k,$$

for all $\lambda \in (0, \lambda_1]$. By Estimate (11), for some $b < 0$ with $|b|$ sufficiently large there exists $\lambda_2 > 0$ such that

$$a(b, \lambda) > k$$

for all $\lambda \in (0, \lambda_2]$. Let $\lambda_0 = \min(\lambda_1, \lambda_2)$. Note that by Lemma 2, $a(b, \lambda)$ is continuous as a function of $b \in \mathbb{R}$ for $\lambda > 0$. It follows from the intermediate value theorem that for any $\lambda \in (0, \lambda_0]$ there exists $b_1 > 0$ and $b_2 < 0$ with

$$a(b_1, \lambda) = a(b_2, \lambda) = k.$$

Recalling the definition of $a(b, \lambda)$, this implies that $v_{b_1,\lambda} \neq v_{b_2,\lambda}$. Let us write u_i for $v_{b_i,\lambda}$. Since u_i is the vacuum of $H_{b,\lambda}$, it is even. We thus have obtained, for any $\lambda \in (0, \lambda_0]$, two distinct vectors $u_i \in \mathbf{D}$, each of which is the vacuum of $H(Q, u_i)$, where

$$Q(x) = \frac{1}{2} kx^2 + \lambda x^4.$$

Now let $M^2 = m^2 + k$, and let $\mathbf{H}'$ denote the Hilbert space of Cauchy data for the free scalar field of mass M on $\mathbb{R} \times S^1$. Let $(\mathbf{K}', W', \Gamma', v_0')$ denote the free boson field over $\mathbf{H}'$, let $B' = (-\Delta + M^2)^{1/2}$, and similarly for all the other objects associated with the free scalar field. There is a symplectic real-linear transformation $S: \mathbf{H} \to \mathbf{H}'$ given by $S(f, g) = (Tf, T^{-1}g)$, where $T = BB'^{-1}$. Since $SS^* - I$ is Hilbert–Schmidt (where S^* denotes the real adjoint of S when $\mathbf{H}$ and $\mathbf{H}'$ are considered as real Hilbert spaces), the results of Segal and Shale imply that there exists a unitary operator $U: \mathbf{K} \to \mathbf{K}'$ implementing S. That is, for all $z \in \mathbf{H}$,

$$W'(Sz) = UW(z)\, U^{-1}. \tag{12}$$

(An exposition of the theory of unitary implementability of symplectics appears in Chapter 4 of [2].)

Let $u_i' = Uu_i$. We claim that in fact

$$UH(Q, u_i)\, U^{-1} = H'(\lambda P, u_i'), \tag{13}$$

where $P(x) = x^4$. It will follow that u_i' is a lowest eigenvector of $H'(\lambda P, u_i')$; since U can be chosen to be positivity-preserving, we may assume that u_i' is in fact the vacuum of $H'(\lambda P, u_i')$. One can see from $\Gamma'(-I)\, U = U\Gamma(-I)$ that the vacua u_i' are even in $L^2(\mathbf{H}_\kappa')$. Since m and k, hence M, were arbitrary positive numbers, the theorem follows.

Now let us prove (13). Differentiating Eq. (12), we obtain

$$\Phi'(f) = U\Phi(f)\, U^{-1}, \qquad \dot{\Phi}'(f) = U\dot{\Phi}(f)\, U^{-1}.$$

Using the definition of renormalized powers, it follows that

$$:\Phi'^n(f):_{u_i'} = U :\Phi^n(f):_{u_i} U^{-1}$$

for all n. Thus it suffices to show that

$$U^{-1} H_0' U = \text{closure of } H_0 + \frac{k}{2} :\Phi^2: + c.$$

This may be checked by comparing successive commutators with $\Phi(f)$ and $\dot{\Phi}(f)$ (the generator of $V(tf)$) for arbitrary $f \in C^\infty(S^1)$, using the irreducibility of the free boson field. ∎

Proof of Theorem 3. Suppose such a unitary U exists. Since the self-adjoint operators $\Phi(f)$ generate a maximal abelian subalgebra of bounded linear operators on $\mathbf{K}$, the unitary U must lie in this subalgebra. It follows that

$$\langle u_2, :\Phi^2: u_2 \rangle = \langle Uu_2, :\Phi^2: Uu_2 \rangle.$$

Since $H(\lambda P, u_1) = UH(\lambda P, u_2)\, U^{-1}$ and u_i is the vacuum of $H(\lambda P, u_i)$, we must have $u_1 = \alpha U u_2$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. This implies that

$$\langle u_2, :\Phi^2: u_2 \rangle = \langle u_1, :\Phi^2: u_1 \rangle.$$

This was shown not to be the case in the proof of Theorem 2. ∎

We conclude with some open questions concerning Theorem 2. Very similar results hold for the one-dimensional anharmonic oscillator [3]. Here one seeks $u \in L^2(\mathbb{R})$ such that the vacuum of $\frac{1}{2}(p^2+q^2)+\lambda :q^4:_u$ is u itself. In this case, for sufficiently small $\lambda > 0$ there are two solutions, say $u_1(\lambda)$ and $u_2(\lambda)$. The well-behaved solution, $u_1(\lambda)$, converges to the free vacuum as $\lambda \downarrow 0$, while $u_2(\lambda)$ converges weakly to zero. Moreover, numerical calculations indicate that $u_1(\lambda)$ and $u_2(\lambda)$ are well-defined and continuous for λ in some interval $(0, \lambda_0]$, that these are the only solutions for λ in this interval, that they are distinct for $\lambda < \lambda_0$ but equal for $\lambda = \lambda_0$, and that there is no solution for $\lambda > \lambda_0$. It is not known whether the ϕ^4 theory Hamiltonian,

$$H_0 + \lambda \int_{t=0} :\phi^4(t, x):_u \, dx,$$

behaves in the same manner. Theorem 2 suggests that it may.

References

1. J. Baez, Wick products of the free Bose field, *J. Funct. Anal.* **86** (1989), 211–225.
2. J. Baez, I. Segal, and Z. Zhou, "An Introduction to Algebraic and Constructive Quantum Field Theory," Princeton Univ. Press, New Jersey, 1992.
3. J. Baez and Z. Zhou, Renormalized oscillator Hamiltonians, *Adv. Math.*, to appear.
4. C. Friedman, Renormalized oscillator equations, *J. Math. Phys.* **14** (1973), 1378–1380.
5. A. Jaffe, A. Lesniewski, and C. Wieczerkowski, *A priori* quantum field equations, *Ann. Physics* **192** (1989), 2–20.
6. T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, Berlin, 1980.
7. I. Segal, Nonlinear functions of weak processes, I, *J. Funct. Anal.* **4** (1969), 404–456; II, *J. Funct. Anal.* **6** (1970), 29–75.
8. I. Segal, Construction of non-linear local quantum process, I, *Ann. Math.* **92** (1970), 462–481; II, *Invent. Math.* **14** (1971), 211–241.
9. B. Simon and R. Hoegh-Krohn, Hypercontractive semigroups and two dimensional self-coupled Bose fields, *J. Funct. Anal.* **9** (1972), 121–180.