A Survey of Higher Lie Theory

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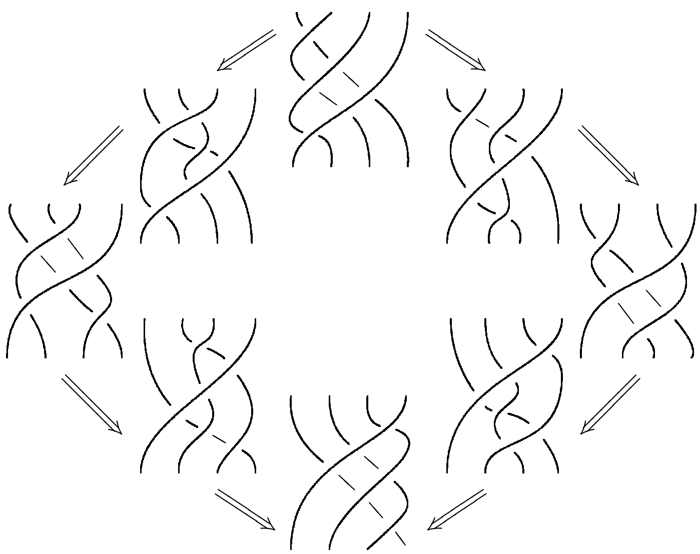

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Higher Gauge Theory

It is natural to assign a group element to each path:

$$\bullet \xrightarrow{g} \bullet$$

since composition of paths then corresponds to multiplication:

$$\bullet \xrightarrow{g} \bullet \xrightarrow{g'} \bullet$$

while reversing the direction of a path corresponds to taking inverses:

$$\bullet \xleftarrow{g^{-1}} \bullet$$

and the associative law makes this composite unambiguous:

$$\bullet \xrightarrow{g} \bullet \xrightarrow{g'} \bullet \xrightarrow{g''} \bullet$$

Internalization

Often a useful first step in the categorification process involves using a technique developed by Ehresmann called 'internalization.'

How do we do this?

- Given some concept, express its definition completely in terms of commutative diagrams.
- Now interpret these diagrams in some ambient category K.

We will consider the notion of a 'category in K' for various categories K.

A **strict 2-group** is a category in Grp, the category of groups.

Categorified Lie Theory, strictly speaking...

A **strict Lie 2-group** G is a category in LieGrp, the category of Lie groups.

A **strict Lie 2-algebra** L is a category in LieAlg, the category of Lie algebras.

We can also define **strict homomorphisms** between each of these and **strict 2-homomorphisms** between them, in the obvious way. Thus, we have two strict 2-categories: SLie2Grp and SLie2Alg.

The picture here is very pretty: Just as Lie groups have Lie algebras, strict Lie 2-groups have strict Lie 2-algebras.

Proposition. There exists a unique 2-functor

$$d\colon \mathrm{SLie2Grp} \to \mathrm{SLie2Alg}$$

Examples of Strict Lie 2-Groups

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

- **Automorphism 2-Group**

$$\begin{aligned} \text{Objects}: &= \mathrm{Aut}(G) \\ \text{Morphisms}: &= G \rtimes \mathrm{Aut}(G) \end{aligned}$$

- **Shifted** $U(1)$

$$\begin{aligned} \text{Objects}: &= * \\ \text{Morphisms}: &= U(1) \end{aligned}$$

- **Tangent 2-Group**

$$\begin{aligned} \text{Objects}: &= G \\ \text{Morphisms}: &= \mathfrak{g} \rtimes G \cong TG \end{aligned}$$

- **Poincaré 2-Group**

$$\begin{aligned} \text{Objects}: &= SO(n,1) \\ \text{Morphisms}: &= \mathbb{R}^n \rtimes SO(n,1) \cong ISO(n,1) \end{aligned}$$

Categorified vector spaces

Kapranov and Voevodsky defined a finite-dimensional 2-vector space to be a category of the form Vect^n.

Instead, we define a **2-vector space** to be a category in Vect, the category of vector spaces.

Thus, a 2-vector space is a category where everything in sight is *linear*!

A **2-vector space**, V, consists of:

- a **vector space** of objects, $Ob(V)$

- a **vector space** of morphisms, $Mor(V)$

together with:

- **linear** source and target maps
$$s, t \colon Mor(V) \to Ob(V),$$
- a **linear** identity-assigning map
$$i \colon Ob(V) \to Mor(V),$$
- a **linear** composition map
$$\circ \colon Mor(V) \times_{Ob(V)} Mor(V) \to Mor(V)$$

such that the following diagrams commute, expressing the usual category laws:

- laws specifying the source and target of identity morphisms:

$$\begin{tikzcd} Ob(V) \arrow[r, "i"] \arrow[dr, "1_{Ob(V)}"'] & Mor(V) \arrow[d, "s"] \\ & Ob(V) \end{tikzcd} \qquad \begin{tikzcd} Ob(V) \arrow[r, "i"] \arrow[dr, "1_{Ob(V)}"'] & Mor(V) \arrow[d, "t"] \\ & Ob(V) \end{tikzcd}$$

- laws specifying the source and target of composite morphisms:

$$\begin{tikzcd} Mor(V) \times_{Ob(V)} Mor(V) \arrow[r, "\circ"] \arrow[d, "p_1"'] & Mor(V) \arrow[d, "s"] \\ Mor(V) \arrow[r, "s"] & Ob(V) \end{tikzcd}$$

$$\begin{tikzcd} Mor(V) \times_{Ob(V)} Mor(V) \arrow[r, "\circ"] \arrow[d, "p_2"'] & Mor(V) \arrow[d, "t"] \\ Mor(V) \arrow[r, "t"] & Ob(V) \end{tikzcd}$$

- the associative law for composition of morphisms:

$$
\begin{tikzcd}
Mor(V) \times_{Ob(V)} Mor(V) \times_{Ob(V)} Mor(V) \arrow[r, "\circ \times_{Ob(V)} 1"] \arrow[d, "1 \times_{Ob(V)} \circ"'] & Mor(V) \times_{Ob(V)} Mor(V) \arrow[d, "\circ"] \\
Mor(V) \times_{Ob(V)} Mor(V) \arrow[r, "\circ"] & Mor(V)
\end{tikzcd}
$$

- the left and right unit laws for composition of morphisms:

$$
\begin{tikzcd}
Ob(V) \times_{Ob(V)} Mor(V) \arrow[r, "i \times 1"] \arrow[dr, "p_2"'] & Mor(V) \times_{Ob(V)} Mor(V) \arrow[d, "\circ"] & Mor(V) \times_{Ob(V)} Ob(V) \arrow[l, "1 \times i"'] \arrow[dl, "p_1"] \\
& Mor(V) &
\end{tikzcd}
$$

2-Vector Spaces

We can also define **linear functors** between 2-vector spaces, and **linear natural transformations** between these, in the obvious way.

Theorem. The 2-category of 2-vector spaces, linear functors and linear natural transformations is equivalent to the 2-category of:

- 2-term chain complexes $C_1 \xrightarrow{d} C_0$,
- chain maps between these,
- chain homotopies between these.

2-Vector Spaces

Proposition. Given 2-vector spaces V and V' there is a 2-vector space $V \oplus V'$ having:

- $\mathrm{Ob}(V) \oplus \mathrm{Ob}(V')$ as its vector space of objects,
- $\mathrm{Mor}(V) \oplus \mathrm{Mor}(V')$ as its vector space of morphisms,

Proposition. Given 2-vector spaces V and V' there is a 2-vector space $V \otimes V'$ having:

- $\mathrm{Ob}(V) \otimes \mathrm{Ob}(V')$ as its vector space of objects,
- $\mathrm{Mor}(V) \otimes \mathrm{Mor}(V')$ as its vector space of morphisms,

Moreover, we have an 'identity object', K, for the tensor product of 2-vector spaces, just as the ground field k acts as the identity for the tensor product of usual vector spaces:

Proposition. There exists a unique 2-vector space K, the **categorified ground field**, with

$$\mathrm{Ob}(K) = \mathrm{Mor}(K) = k \;\text{ and}$$
$$s, t, i = 1_k.$$

Semistrict Lie 2-Algebras

A **semistrict Lie 2-algebra** consists of:

- a 2-vector space L

equipped with:

- a functor called the **bracket**:
$$[\cdot,\cdot]\colon L \times L \to L$$
bilinear and skew-symmetric as a function of objects and morphisms,
- a natural isomorphism called the **Jacobiator**:
$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$
trilinear and antisymmetric as a function of the objects x, y, z,

such that:

- the **Jacobiator identity** holds, meaning the following diagram commutes:

$$\begin{tikzcd}
 & {[[[w,x],y],z]} \arrow[dl, "{[J_{w,x,y},z]}"'] \arrow[dr, "1"] & \\
{[[[w,y],x],z]+[[w,[x,y]],z]} \arrow[d, "{J_{[w,y],x,z}+J_{w,[x,y],z}}"'] & & {[[[w,x],y],z]} \arrow[d, "{J_{[w,x],y,z}}"] \\
\begin{array}{c}[[[w,y],z],x]+[[w,y],[x,z]] \\ +[w,[[x,y],z]]+[[w,z],[x,y]]\end{array} \arrow[d, "{[J_{w,y,z},x]}"'] & & {[[[w,x],z],y]+[[w,x],[y,z]]} \arrow[d, "{[J_{w,x,z},y]}"] \\
\begin{array}{c}[[[w,z],y],x]+[[w,[y,z]],x] \\ +[[w,y],[x,z]]+[w,[[x,y],z]]+[[w,z],[x,y]]\end{array} \arrow[dr, "{[w,J_{x,y,z}]}"'] & & \begin{array}{c}[[w,[x,z]],y] \\ +[[w,x],[y,z]]+[[[w,z],x],y]\end{array} \arrow[dl, "{\begin{array}{c}J_{w,[x,z],y} \\ +J_{[w,z],x,y}+J_{w,x,[y,z]}\end{array}}"] \\
 & \begin{array}{c}[[[w,z],y],x]+[[w,z],[x,y]]+[[w,y],[x,z]] \\ +[w,[[x,z],y]]+[[w,[y,z]],x]+[w,[x,[y,z]]]\end{array} &
\end{tikzcd}$$

Given a vector space V and an isomorphism

$$B\colon V \otimes V \to V \otimes V,$$

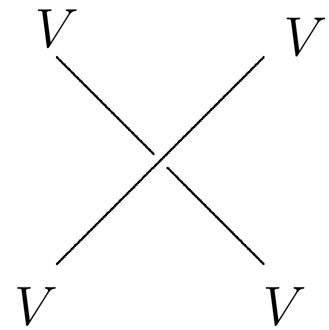
we say B is a **Yang–Baxter operator** if it satisfies the **Yang–Baxter equation**, which says that:

$$(B \otimes 1)(1 \otimes B)(B \otimes 1) = (1 \otimes B)(B \otimes 1)(1 \otimes B),$$

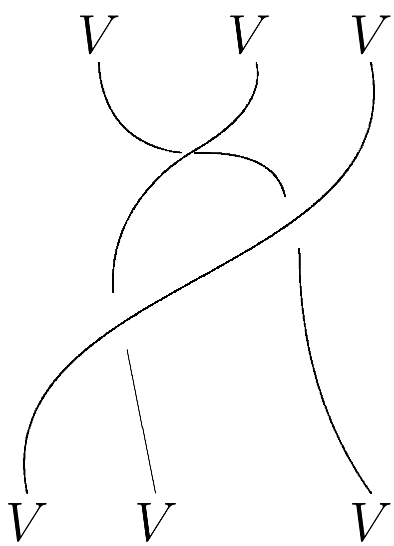
or in other words, that this diagram commutes:

$$\begin{tikzcd}
 & V \otimes V \otimes V \arrow[dl, "1\otimes B"'] \arrow[dr, "B\otimes 1"] & \\
V \otimes V \otimes V \arrow[d, "B\otimes 1"'] & & V \otimes V \otimes V \arrow[d, "1\otimes B"] \\
V \otimes V \otimes V \arrow[dr, "1\otimes B"'] & & V \otimes V \otimes V \arrow[dl, "B\otimes 1"] \\
 & V \otimes V \otimes V &
\end{tikzcd}$$

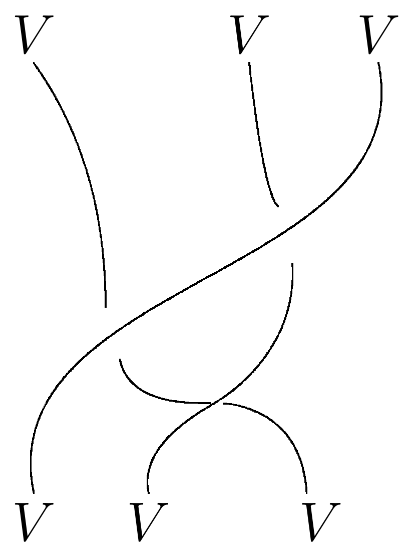
If we draw $B\colon V \otimes V \to V \otimes V$ as a braiding:

$B =$ 

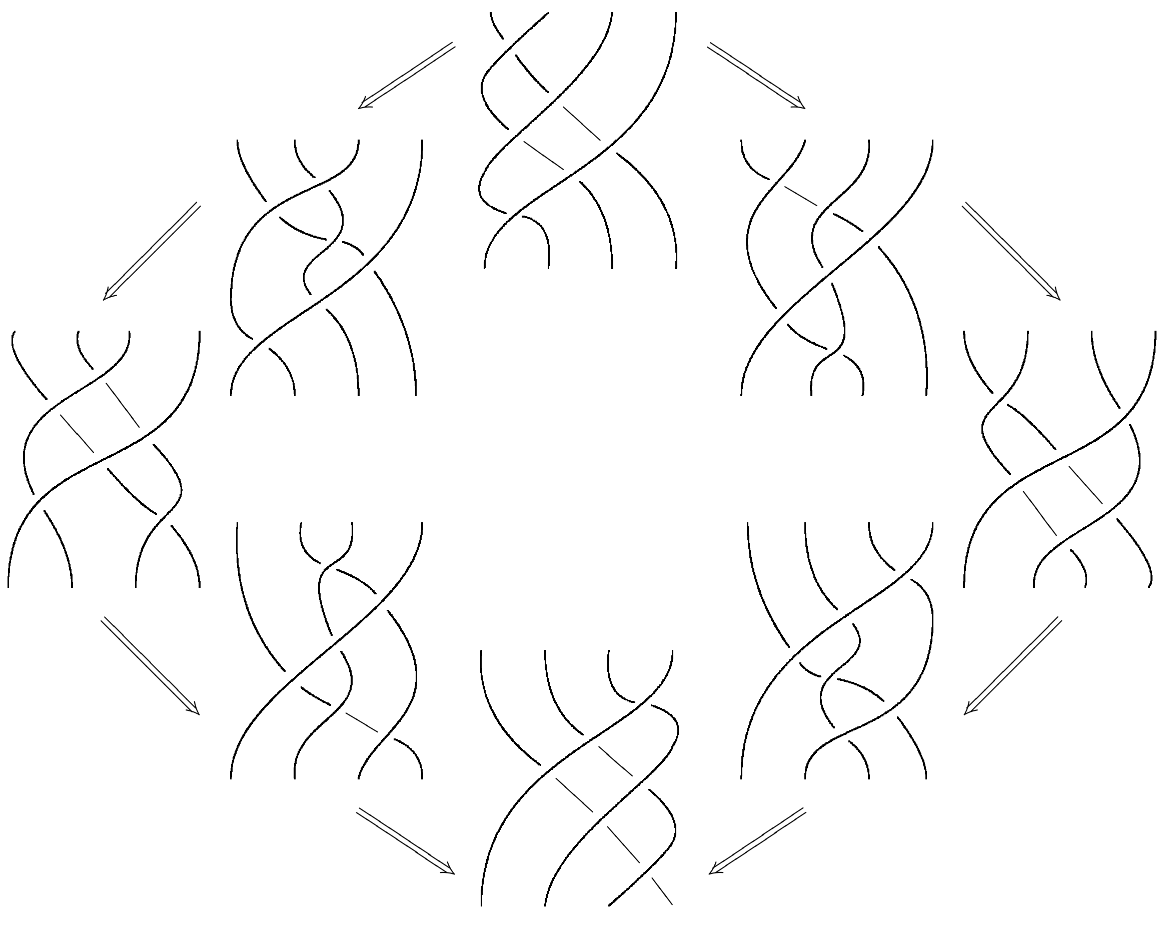
the Yang–Baxter equation says that:



$=$



The Zamolodchikov tetrahedron equation is a formalization of this commutative octagon:



We can define **homomorphisms** between Lie 2-algebras, and **2-homomorphisms** between these.

Given Lie 2-algebras L and L', a **homomorphism** $F\colon L \to L'$ consists of:

- a functor F from the underlying 2-vector space of L to that of L', linear on objects and morphisms,

- a natural isomorphism

$$F_2(x,y)\colon [F(x),F(y)] \to F[x,y],$$

  bilinear and skew-symmetric as a function of the objects $x,y \in L$,

such that:

- the following diagram commutes for all objects $x,y,z \in L$:

$$\begin{tikzcd}
[F(x),[F(y),F(z)]] \arrow[r, "J_{F(x),F(y),F(z)}"] \arrow[d, "{[1,F_2]}"'] & {[[F(x),F(y)],F(z)] + [F(y),[F(x),F(z)]]} \arrow[d, "{[F_2,1]+[1,F_2]}"] \\
{[F(x),F[y,z]]} \arrow[d, "F_2"'] & {[F[x,y],F(z)] + [F(y),F[x,z]]} \arrow[d, "F_2+F_2"] \\
{F[x,[y,z]]} \arrow[r, "F(J_{x,y,z})"] & {F[[x,y],z] + F[y,[x,z]]}
\end{tikzcd}$$

Theorem. The 2-category of Lie 2-algebras, homomorphisms and 2-homomorphisms is equivalent to the 2-category of:

- 2-term L_∞-algebras,
- L_∞-homomorphisms between these,
- L_∞-2-homomorphisms between these.

The Lie 2-algebras L and L' are **equivalent** if there are homomorphisms

$$f\colon L \to L' \qquad \bar{f}\colon L' \to L$$

that are inverses up to 2-isomorphism:

$$f\bar{f} \cong 1, \qquad \bar{f}f \cong 1.$$

Theorem. Lie 2-algebras are classified up to equivalence by quadruples consisting of:

- a Lie algebra $\mathfrak{g}$,
- an abelian Lie algebra (= vector space) $\mathfrak{h}$,
- a representation ρ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

The Lie 2-Algebra $\mathfrak{g}_k$

Suppose $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{R}$. To get a Lie 2-algebra having $\mathfrak{g}$ as objects we need:

- a vector space $\mathfrak{h}$,
- a representation ρ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

Assume without loss of generality that ρ is irreducible. To get Lie 2-algebras with nontrivial Jacobiator, we need $H^3(\mathfrak{g}, \mathfrak{h}) \neq 0$. By Whitehead's lemma, this only happens when $\mathfrak{h} = \mathbb{R}$ is the trivial representation. Then we have

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

with a nontrivial 3-cocycle given by:

$$\nu(x, y, z) = \langle [x, y], z \rangle .$$

The Lie algebra $\mathfrak{g}$ together with the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$ and k times the above 3-cocycle give the Lie 2-algebra $\mathfrak{g}_k$.

In summary: *every simple Lie algebra $\mathfrak{g}$ gives a one-parameter family of Lie 2-algebras, $\mathfrak{g}_k$, which reduces to $\mathfrak{g}$ when $k = 0$!*

Puzzle: Does $\mathfrak{g}_k$ come from a Lie 2-group?

Coherent 2-Groups

A **coherent 2-group** is a weak monoidal category in which every morphism is invertible and every object is equipped with an adjoint equivalence.

A **homomorphism** between coherent 2-groups is a weak monoidal functor. A **2-homomorphism** is a monoidal natural transformation. The coherent 2-groups X and X' are **equivalent** if there are homomorphisms

$$f \colon X \to X' \qquad \bar{f} \colon X' \to X$$

that are inverses up to 2-isomorphism:

$$f\bar{f} \cong 1, \qquad \bar{f}f \cong 1.$$

Theorem. Coherent 2-groups are classified up to equivalence by quadruples consisting of:

- a group G,
- an abelian group H,
- an action α of G as automorphisms of H,
- an element $[a] \in H^3(G, H)$.

Suppose we try to copy the construction of $\mathfrak{g}_k$ for a particularly nice kind of Lie group. Let G be a simply-connected compact simple Lie group whose Lie algebra is $\mathfrak{g}$. We have

$$H^3(G, \mathrm{U}(1)) \xhookrightarrow{\iota} \mathbb{Z} \hookrightarrow \mathbb{R} \cong H^3(\mathfrak{g}, \mathbb{R})$$

Using the classification of 2-groups, we can build a skeletal 2-group G_k for $k \in \mathbb{Z}$:

- G as its group of objects,
- $\mathrm{U}(1)$ as the group of automorphisms of any object,
- the trivial action of G on $\mathrm{U}(1)$,
- $[a] \in H^3(G, \mathrm{U}(1))$ given by $k\, \iota[\nu]$, which is nontrivial when $k \neq 0$.

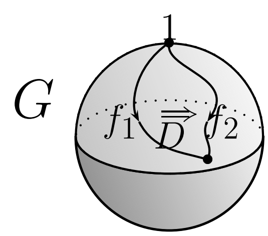
Question: Can G_k be made into a Lie 2-group?

Here's the bad news:

(Bad News) Theorem. Unless $k = 0$, there is no way to give the 2-group G_k the structure of a Lie 2-group for which the group G of objects and the group $\mathrm{U}(1)$ of endomorphisms of any object are given their usual topology.

(Good News) Theorem. For any $k \in \mathbb{Z}$, there is a Fréchet Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra $\mathcal{P}_k \mathfrak{g}$ is equivalent to $\mathfrak{g}_k$.

An object of $\mathcal{P}_k G$ is a smooth path $f\colon [0, 2\pi] \to G$ starting at the identity. A morphism from f_1 to f_2 is an equivalence class of pairs (D, α) consisting of a disk D going from f_1 to f_2 together with $\alpha \in \mathrm{U}(1)$:



For any two such pairs (D_1, α_1) and (D_2, α_2) there is a 3-ball B whose boundary is $D_1 \cup D_2$, and the pairs are equivalent when

$$\exp\left(2\pi i k \int_B \nu\right) = \alpha_2/\alpha_1$$

where ν is the left-invariant closed 3-form on G with

$$\nu(x, y, z) = \langle [x, y], z \rangle$$

and $\langle \cdot, \cdot \rangle$ is the smallest invariant inner product on $\mathfrak{g}$ such that ν gives an integral cohomology class.

$\mathcal{P}_k G$ and Loop Groups

We can also describe the 2-group $\mathcal{P}_k G$ as follows:

- An object of $\mathcal{P}_k G$ is a smooth path in G starting at the identity.
- Given objects $f_1, f_2 \in \mathcal{P}_k G$, a morphism

  $$\widehat{\ell}\colon f_1 \to f_2$$

  is an element $\widehat{\ell} \in \widehat{\Omega_k G}$ with

  $$p(\widehat{\ell}) = f_2/f_1$$

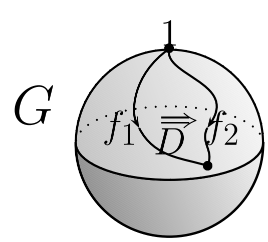
  where $\widehat{\Omega_k G}$ is the level-k Kac–Moody central extension of the loop group ΩG:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \xrightarrow{\;p\;} \Omega G \longrightarrow 1$$

Remark: $p(\widehat{\ell})$ is a loop in G. We can get such a loop with

$$p(\widehat{\ell}) = f_2/f_1$$

from a disk D like this:



The Lie 2-Group $\mathcal{P}_k G$

Thus, $\mathcal{P}_k G$ is described by the following where $p \in P_0 G$ and $\hat{\gamma} \in \widehat{\Omega_k G}$:

- A Fréchet Lie group of **objects**:

$$\mathrm{Ob}(\mathcal{P}_k G) = P_0 G$$

- A Fréchet Lie group of **morphisms**:

$$\mathrm{Mor}(\mathcal{P}_k G) = P_0 G \ltimes \widehat{\Omega_k G}$$

- **source map**: $s(p, \hat{\gamma}) = p$

- **target map**: $t(p, \hat{\gamma}) = p\partial(\hat{\gamma})$ where ∂ is defined as the composite

$$\widehat{\Omega_k G} \xrightarrow{p} \Omega G \overset{i}{\hookrightarrow} P_0 G$$

- **composition**: $(p_1, \hat{\gamma}_1) \circ (p_2, \hat{\gamma}_2) = (p_1, \hat{\gamma}_1 \hat{\gamma}_2)$ when $t(p_1, \hat{\gamma}_1) = s(p_2, \hat{\gamma}_2)$, or $p_2 = p_1 \partial(\hat{\gamma}_1)$

- **identities**: $i(p) = (p, 1)$

Topology of $\mathcal{P}_k G$

The **nerve** of any topological 2-group is a **simplicial** topological group and therefore when we take the **geometric realization** we obtain a topological group:

Theorem. For any $k \in \mathbb{Z}$, the geometric realization of the nerve of $\mathcal{P}_k G$ is a topological group $|\mathcal{P}_k G|$. We have

$$\pi_3(|\mathcal{P}_k G|) \cong \mathbb{Z}/k\mathbb{Z}$$

When $k = \pm 1$,

$$|\mathcal{P}_k G| \simeq \widehat{G},$$

which is the topological group obtained by killing the third homotopy group of G.

When $G = \mathrm{Spin}(n)$, $\widehat{G}$ is called $\mathrm{String}(n)$. When $k = \pm 1$, $|\mathcal{P}_k G| \simeq \widehat{G}$.

The Lie 2-Algebra $\mathcal{P}_k\mathfrak{g}$

$\mathcal{P}_k G$ is a particularly nice kind of Lie 2-group: a *strict* one! Thus, its Lie 2-algebra is easy to compute.

The 2-term L_∞-algebra V corresponding to the Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$ is given by:

- $V_0 = P_0\mathfrak{g}$
- $V_1 = \widehat{\Omega_k\mathfrak{g}} \cong \Omega\mathfrak{g} \oplus \mathbb{R}$,
- $d\colon V_1 \to V_0$ equal to the composite
$$\widehat{\Omega_k\mathfrak{g}} \to \Omega\mathfrak{g} \hookrightarrow P_0\mathfrak{g}\,,$$
- $l_2\colon V_0 \times V_0 \to V_0$ given by the bracket in $P_0\mathfrak{g}$:
$$l_2(p_1, p_2) = [p_1, p_2],$$
and $l_2\colon V_0 \times V_1 \to V_1$ given by the action $d\alpha$ of $P_0\mathfrak{g}$ on $\widehat{\Omega_k\mathfrak{g}}$, or explicitly:
$$l_2(p, (\ell, c)) = \Big([p, \ell],\ 2k \int_0^{2\pi} \langle p(\theta), \ell'(\theta)\rangle \, d\theta\ \Big)$$
for all $p \in P_0\mathfrak{g}$, $\ell \in \Omega G$ and $c \in \mathbb{R}$,
- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ equal to zero.

The 2-term L_∞-algebra V corresponding to the Lie 2-algebra $\mathfrak{g}_k$ is given by:

- $V_0 =$ the Lie algebra $\mathfrak{g}$,
- $V_1 = \mathbb{R}$,
- $d\colon V_1 \to V_0$ is the zero map,
- $l_2\colon V_0 \times V_0 \to V_0$ given by the bracket in $\mathfrak{g}$:

  $$l_2(x, y) = [x, y],$$

  and $l_2\colon V_0 \times V_1 \to V_1$ given by the trivial representation ρ of $\mathfrak{g}$ on $\mathbb{R}$,
- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ given by:

  $$l_3(x, y, z) = k\langle [x, y], z\rangle$$

  for all $x, y, z \in \mathfrak{g}$.

The Equivalence $\mathcal{P}_k\mathfrak{g} \simeq \mathfrak{g}_k$

We describe the two Lie 2-algebra homomorphisms forming our equivalence in terms of their corresponding L_∞-algebra homomorphisms:

- $\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$ has:

  $$\begin{aligned} \phi_0(p) &= p(2\pi) \\ \phi_1(\ell, c) &= c \end{aligned}$$

  where $p \in P_0\mathfrak{g}$, $\ell \in \Omega\mathfrak{g}$, and $c \in \mathbb{R}$.

- $\psi\colon \mathfrak{g}_k \to \mathcal{P}_k\mathfrak{g}$ has:

  $$\begin{aligned} \psi_0(x) &= xf \\ \psi_1(c) &= (0, c) \end{aligned}$$

  where $x \in \mathfrak{g}$, $c \in \mathbb{R}$, and $f\colon [0, 2\pi] \to \mathbb{R}$ is a smooth function with $f(0) = 0$ and $f(2\pi) = 1$.

Theorem. With the above definitions we have:

- $\phi \circ \psi$ is the identity Lie 2-algebra homomorphism on $\mathfrak{g}_k$, and
- $\psi \circ \phi$ is isomorphic, as a Lie 2-algebra homomorphism, to the identity on $\mathcal{P}_k\mathfrak{g}$.

What's Next?

We know how to get Lie n-algebras from Lie algebra cohomology! We should:

- Classify their representations
- Find their corresponding Lie n-groups
- Understand their relation to higher braid theory

Moreover, many other questions remain:

- Weak n-categories in Vect?
- Weakening laws governing addition and scalar multiplication?
- Weakening the antisymmetry of the bracket in the definition of Lie 2-algebra?
- What's a free Lie 2-algebra on a 2-vector space?
- Lie 2-algebra cohomology? L_∞-algebra cohomology?
- Deformations of Lie 2-algebras?