Quantum 2-States: Sections of 2-vector bundles

Urs Schreiber

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**Abstract**

Quantization of point particles is a process that sends a vector bundle to its space of sections (“states”), and a connection on the vector bundle to an action on this space of states.

This situation can be categorified. Suitable sections of line-2-bundles (≃ line bundle gerbes) describe states of open strings.

Over the endpoints of the string, such a 2-section amounts to a choice of gerbe module. This is known as a “D-brane”.

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3 **Sections of 2-vector bundles**
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1. **sections and states**

(a) for applications in physics, we want to take a vector bundle and form its space of sections (= space of states $H$)

(b) a connection on the vector bundle allows to do parallel transport of sections, its differential is the covariant derivative

(c) from the covariant derivative one obtains a Laplace operator and from that a functor $\Sigma(\mathbb{R}) \to \text{Aut}(H)$

(d) moral: quantization sends a parallel transport functor on a bundle to a propagation functor on the space of states

   if we want to categorify this, we need to formulate the above in suitable terms

(e) a dictionary for arrow-theoretic quantum mechanics; main idea: to get a section of an $n$-bundle with connection, pull it back to configuration space and pick a morphism from the trivial bundle to this pullback (essentially: section=trivialization)

2. **2-vector bundles**

(a) hence we need to know what a 2-vector bundle with connection is – notice how an associated bundle with connection is particularly easy in anafunctor language: simply postcompose with representation

(b) so all we need is to understand reps of 2-groups: should be 2-functor from $\Sigma(G_2)$ to $2\text{Vect}$

(c) what is "2Vect"? as we will later see, for us a useful definition is “module categories for abelian monoidal categories”

(d) three examples: Baez-Crans, Kapranov-Voevodsky, Bimodules

(e) the canonical rep of any strict 2-group on bimodules

(f) plug this canonical rep into 2-anafunctor description of $\Sigma(U(1))$-2-transport to obtain a line 2-bundle with connection

(g) this defines a structure with transition line bundles with connection: a line bundle gerbe

3. **sections of 2-vector bundles**

(a) now put all this together to see what sections of line-2-bundles are and how their parallel transport looks like:

   parameter space is the category $\{a \to b\}$ (the open string) configuration space are maps from this into the cover 2-category of the parallel transport 2-anafunctor

(b) use the dictionary from above to write down the diagram that defines a section of a line-2-bundle
(c) over $a$ and $b$ this defines a gerbe module, over $a \rightarrow b$ a morphism between these gerbe modules

(d) terminology: physicists call a gerbe module a "D-brane"; it is something that the ends of open strings may sit on

(e) to understand the parallel transport of these sections it helps to make the global picture of our 2-anafunctor manifest: our line 2-bundle 2-anafunctor is locally equivalent to a 2-functor with values in bimodules of compact operators

(f) remark: this generalizes to higher rank vector 2-bundles: Stolz-Teichner’s string connections are of this form - just replace $G_2 = \Sigma(U(1))$ by $G_2 = \text{String}(G)$ and its canonical 2-rep

(g) finally the disk diagram: a diagram depicting a 2-section coming in, propagating along a strip, and a 2-section coming out – this yields the formula for gerbe disk holonomy
1 Sections and States

Given a notion of parallel transport, there are three different things one may want to do to it:

- **Local trivialization** expresses a globally defined functor in terms of an ana-functor.

- **Categorification** passes from a 1-functor that sends paths to fiber morphisms to a 2-functor that sends 2-paths to morphisms of fibers that are objects in a 2-category.

These two steps are discussed in my other talk. Here I shall try to indicate, from the point of view of functorial parallel transport, a tiny aspect of a third operation that is of interest: quantization.

<table>
<thead>
<tr>
<th>kinematics</th>
<th>dynamics</th>
</tr>
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<tbody>
<tr>
<td>vector bundle $V \to X$</td>
<td>connection $\nabla$</td>
</tr>
<tr>
<td>space of states</td>
<td>evolution operator</td>
</tr>
<tr>
<td>$H$</td>
<td>$U(t) : H \to H$</td>
</tr>
<tr>
<td>objects</td>
<td>morphisms</td>
</tr>
<tr>
<td>space of sections</td>
<td>path integral</td>
</tr>
<tr>
<td>straightforward</td>
<td>subtle</td>
</tr>
</tbody>
</table>

Table 1: **Quantization** involves a kinematical and a dynamical aspect.

Given an $n$-bundle with connection, the kinematical aspect of quantization should involve finding the space of sections of the $n$-bundle and finding the action of parallel transport on these sections.

This I discuss for line-2-bundles (≃ line bundle gerbes) with connection.

1.1 Ordinary quantization

The usual setup is this:

On a Riemannian space $X$ we have a hermitean vector bundle $V \to X$ with connection $\nabla$. The space of smooth sections

$$\Gamma(V)$$

of this vector bundle models the “space of states” of a particle “charged under” this vector bundle.

The Riemannian structure on $X$ together with the hermitean structure on $V$ induce a scalar product on $\Gamma^2(V) \subset \Gamma(V)$: the space of square integrable sections. Completing with respect to this scalar product yields the Hilbert space

$$H = \bar{\Gamma}^2(V).$$

The connection $\nabla$ on $V$ gives rise to a differential operator

$$\nabla : H \to H$$
Figure 1: **Quantization, categorification and local trivialization** are the three procedures relating $n$-vector $n$-transport that play a role in the local description of $n$-dimensional quantum field theory. Categorification sends $n$-transport to $(n+1)$-transport. Quantization sends $n$-transport on $n$-paths in configuration space to $n$-transport on abstract $n$-paths (parameter space). Local trivialization sends $n$-transport on globally defined $n$-paths to $n$-transport on local $n$-paths glued by $n$-transitions.


on this space: the covariant derivative.

Composed with its Hilbert space adjoint

$$\nabla^\dagger$$

we obtain the covariant Laplace operator

$$\Delta = \nabla^\dagger \circ \nabla.$$  

This induces a representation of the additive group of real numbers

$$U : t \mapsto \exp(it\Delta)$$

on $H$. This models the “time evolution” of a state in $H$ over a period of time $t$. 

5
If we regard the vector bundle with connection as a parallel transport functor

$$\text{tra}_V : \mathcal{P}_1(X) \to \text{Vect}$$

then it pays to regard time evolution as a functor

$$U : \text{1Cob}_{\text{Riem}} \to \text{Hilb}$$

from 1-dimensional Riemannian cobordisms to Hilbert spaces. In fact, this is a morphism of symmetric monoidal categories with duals.

**Moral.** *Quantization is a process that takes a parallel transport functor and sends it to a transport functor acting on its space of sections.*

<table>
<thead>
<tr>
<th>parallel transport on $V$</th>
<th>quantization</th>
<th>time evolution on $\Gamma^2(V)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{tra}_V : \mathcal{P}_1(X) \to \text{Vect}$</td>
<td>$U : \text{1Cob}_{\text{Riem}} \to \text{Aut}(\Gamma^2(V))$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Quantization is an operation that sends parallel transport functors to propagation functors (modelling “time evolution”) acting on the space of sections of that bundle.

### 1.2 Arrow-theoretic reformulation

I want to describe the analog of the above quantization procedure for the case where the vector bundle with connection is replaced by a 2-vector bundle with connection.

In order to do so, it is helpful to reformulate everything in an arrow-theoretic way that lends itself to categorification.

<table>
<thead>
<tr>
<th>target space</th>
<th>tar = $\mathcal{P}_1(X)$</th>
<th>paths in base space</th>
</tr>
</thead>
<tbody>
<tr>
<td>background field</td>
<td>$\text{tra}_V : \mathcal{P}_1(X) \to \text{Vect}$</td>
<td>parallel transport in a vector bundle with connection</td>
</tr>
<tr>
<td>parameter space</td>
<td>par = ${\bullet}$</td>
<td>the discrete category on a single object</td>
</tr>
<tr>
<td>configuration space</td>
<td>$\text{conf} = \text{Disc}(X) \subset [\text{par}, \mathcal{P}_1(X)]$</td>
<td>the space of inequivalent configurations of the particle in target space</td>
</tr>
<tr>
<td>space of phases</td>
<td>$\text{phas} = [\text{par}, \text{Vect}]$</td>
<td>parallel transport of sections takes values here</td>
</tr>
<tr>
<td>abstract space of states</td>
<td>$\text{tra}_* : \text{conf} \to \text{phas}$</td>
<td>the bundle transgressed to configuration space</td>
</tr>
<tr>
<td>concrete space of states</td>
<td>$\text{Hom}(\mathbb{I}, \text{tra}_*)$</td>
<td>a section is a generalized object of the transgressed bundle</td>
</tr>
</tbody>
</table>

Table 3: *The arrow theory of quantum mechanics* of a particle coupled to a vector bundle with connection.
With “parameter space” $\text{par}$ the trivial category on a single object, as in the above table, we simply have

$$[\text{par}, \mathcal{P}_1(X)] \simeq \mathcal{P}_1(X).$$

But the idea is that varying the specification of $\text{par}$ allows us to seamlessly model the propagation of entities richer than than the single point particle.

**Parameter space and configuration space.** For instance, the simplest more interesting example is that where

$$\text{par} = \{a, b\}$$

is the discrete category on two elements. This models a system consisting not of one, but of two point particles propagating on $X$.

In this case configuration space $\text{conf} \subset [\text{par}, \mathcal{P}_1(X)]$ would be

$$\text{conf} \simeq \text{Disc}(X \times X).$$

In general, we will take $\text{conf}$ to be that subcategory of $[\text{par}, \mathcal{P}_1(X)]$ that contains all morphisms that relate configurations which we want to regard as equivalent.

**Configuration space for orbifolds.** For example, the particle might be propagating on an orbifold $O$. This is best thought of as the corresponding action Lie groupoid $\mathcal{G}_O$. In this case we would take “target space”

$$\text{tar} = \mathcal{P}_1(\mathcal{G}_O)$$

to be the category of paths in the groupoid. This is generated from paths in $\text{Obj}(\mathcal{G}_O)$ together with morphisms of $\mathcal{G}_O$ modulo some natural relations.

In this case, we would want to consider the configuration where the particle sits at $x \in \text{Obj}(\mathcal{G}_O)$ to be equivalent to that where it sits at $y \in \text{Obj}(\mathcal{G}_O)$ if $x$ and $y$ are connected by a morphism of $\mathcal{G}_O$. So we would set

$$\text{conf} = \mathcal{G}_O \subset [\text{par}, \mathcal{P}_1(\mathcal{G}_O)].$$

**Configuration space for covers.** Of general interest is the special case of the above situation, where the Lie groupoid in question is that coming from a cover

$$U \rightarrow X$$

of base space by open contractible sets. Paths in the corresponding groupoid

$$U^{[2]} \rightarrowtail U$$

form a category $\mathcal{P}_1(U^{[2]})$ that covers $\mathcal{P}_1(X)$ in such a way that every path has at most one lift with given source and target.

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Figure 2: If \textbf{target space is an orbifold}, we can model it by the corresponding action Lie groupoid. A path \textit{in} this groupoid is a combination of paths in the object space combined with jumps between points related by the group action. As configurations, $\gamma(0)$ and $g\gamma(0)$ are regarded as equivalent.

This is the situation encountered when the parallel transport functor is conceived as an anafunctor

$$P_1(U^{[2]}) \xrightarrow{(\text{tra}, g)} \Sigma(G) \xrightarrow{\rho} \text{Vect}.$$  

The \textbf{concrete space of sections}. We know beforehand what the sections of an ordinary vector bundle are. The above arrow-theory should reproduce that.

So let configuration space be

$$\text{conf} = \text{Disc}(X).$$

Then

$$\text{tra}_*: \text{conf} \rightarrow \text{Vect}$$

is simply the restriction of $\text{tra}$ to constant paths. This functor simply assigns to each point in target space the fiber above it:

$$\text{tra}_*: \text{Id}_x \mapsto \text{Id}_{V_x}.$$  

The tensor unit transport functor on $\text{conf}$ is

$$\mathbb{1}: \text{Id}_x \mapsto \text{Id}_C.$$  

Therefore a natural transformation

$$e: \mathbb{1} \rightarrow \text{tra}$$
is nothing but a morphism
\[ e_x : \mathbb{C} \to V_x \]
for each \( x \in X \). This is indeed nothing but a section of \( V \).

We should get an equivalent result when we pass from the functor \( \text{tra} : P_1(X) \to \text{Vect} \) to that on the cover \((\text{tra}_U, g) : P_1(U[1^2]) \to \Sigma(G)\).

By the above, the transgressed functor
\[ \text{tra}_* : U[1^2] \xrightarrow{g} \Sigma(G) \xrightarrow{\rho} \text{Vect} \]
sends every transition morphism to the corresponding transition between the trivialized fibers:
\[ \text{tra}_* : ( (x, i) \to (x, j) ) \mapsto ( \mathbb{C} \xrightarrow{\rho(g_{ij}(x))} \mathbb{C}^n ) . \]

Now, a section
\[ e : \mathbb{1} \to \text{tra}_* \]
is a morphism
\[ e_i(x) : \mathbb{C} \to \mathbb{C}^n \]
for each \( x \) in each patch \( U_i \) of the cover such that all these diagrams commute:
\[ \begin{array}{c}
\mathbb{C} \\
\downarrow \text{id} \\
\mathbb{C}
\end{array} 
\begin{array}{c}
\mathbb{C} \\
\downarrow e_i(x) \\
\mathbb{C}^n
\end{array} 
\begin{array}{c}
\mathbb{C} \\
\downarrow \text{id} \\
\mathbb{C} \\
\downarrow e_j(x) \\
\mathbb{C}^n \\
\downarrow \rho(g_{ij}(x)) \\
\mathbb{C}^n
\end{array} \]

This is nothing but a global section, as in the previous example, expressed with respect to the chosen local trivialization.

2 2-Vector bundles

The payoff of our efforts is that now it is easy to take the arrow theory discussed so far and consider it internal to 2-categories.

Principal 2-bundles with connection I have already discussed in my other talk. Now we need associated 2-bundles. This means we need to understand some basics of

- 2-vector spaces;
- representations of 2-groups on 2-vector spaces;
- 2-vector bundles associated to principal 2-bundles.
global transport on $X$ \hspace{1cm} local transport on cover $U \to X$

<table>
<thead>
<tr>
<th>transport</th>
<th>functor</th>
<th>configuration space</th>
<th>transgressed transport</th>
<th>section of vector bundle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{tra} : \mathcal{P}_1(X) \to \text{Vect}$</td>
<td>$(\text{tra}_U, g) : \mathcal{P}_1(U[2]) \to \Sigma(G) \to \text{Vect}$</td>
<td>$\text{conf} = \text{Disc}(X)$</td>
<td>$(x, i) \mapsto (x, j)$</td>
<td>$e(x) : \mathbb{C} \to V_x$</td>
</tr>
<tr>
<td>$\text{tra}_* : \text{Id}<em>x \mapsto \text{Id}</em>{V_x}$</td>
<td>$\rho(g_{ij}(x)) : \mathbb{C}_n \to \mathbb{C}_n$</td>
<td>$(x, i) \mapsto (x, j)$</td>
<td>$(\mathbb{C}<em>n \rho(g</em>{ij}(x))) : \mathbb{C}_n \to \mathbb{C}_n$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Summary of the way the arrow-theory reproduces the correct notion of section of a vector bundle with connection for the two cases where the vector bundle is encoded in a global transport functor on target space and as a local functor on a cover, respectively.

2.1 2-Vector spaces.

A vector space is a module for a field. The concept of field is hard to categorify. But fields are rings, and the categorification of a ring is an abelian monoidal category.

2.2 Representations of 2-groups

For $G$ an ordinary group and $\Sigma(G)$ its suspension, i.e. the category with a single object and $G$ worth of morphisms, a representation $\rho$ of $G$ is a functor

$$\rho : \Sigma(G) \to \text{Vect}.$$ 

Accordingly, we say that for $G_2$ a 2-group and $\Sigma(G_2)$ the corresponding 2-groupoid with a single object that a 2-functor

$$\tilde{\rho} : \Sigma(G_2) \to \text{cMod}$$

is a $\mathbb{C}$-linear 2-representation.

A useful example is the 2-representation of a strict 2-group

$$G_2 = (t : H \to G)$$

induced from an ordinary representation $\rho$ of $H$.

$$\tilde{\rho} : \Sigma(G_2) \to \text{Bim(Vect)}.$$
<table>
<thead>
<tr>
<th>1-transport</th>
<th>2-transport</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain: path groupoid $\mathcal{P}_1(X)$</td>
<td>domain: 2-path 2-groupoid $\mathcal{P}_2(X)$</td>
</tr>
<tr>
<td>codomain: vector spaces $\text{Vect}$</td>
<td>codomain: 2-vector spaces $\text{2Vect}$</td>
</tr>
<tr>
<td>structure group: $G$</td>
<td>structure 2-group: $G_2$</td>
</tr>
<tr>
<td>representation $\rho : \Sigma(G) \to \text{Vect}$</td>
<td>2-representation $\rho : \Sigma(G_2) \to \text{2Vect}$</td>
</tr>
<tr>
<td>trivial vector bundles with connection</td>
<td>trivial 2-vector bundle with connection</td>
</tr>
<tr>
<td>smooth functors: $\mathcal{P}_1(X) \to \Sigma(G) \to \text{Vect}$</td>
<td>smooth 2-functor: $\mathcal{P}_2(X) \to \Sigma(G_2) \to \text{2Vect}$</td>
</tr>
<tr>
<td>groupoid covering target space</td>
<td>2-groupoid covering target space</td>
</tr>
<tr>
<td>$U^\bullet = {(x,j) \to (x,k) \parallel (x,i)}$</td>
<td>$U^\bullet = {(x,j) \to (x,k) \parallel (x,i)}$</td>
</tr>
<tr>
<td>vector bundles with connection</td>
<td>2-vector bundle with connection</td>
</tr>
<tr>
<td>smooth anafunctors: $\mathcal{P}_1(U^\bullet) \to \Sigma(G) \to \text{Vect}$</td>
<td>smooth 2-anafunctor: $\mathcal{P}_2(U^\bullet) \to \Sigma(G_2) \to \text{2Vect}$</td>
</tr>
<tr>
<td>$\mathcal{P}_1(X) \xrightarrow{p} \mathcal{P}_1(U^\bullet)$</td>
<td>$\mathcal{P}_2(X) \xrightarrow{p} \mathcal{P}_2(U^\bullet)$</td>
</tr>
</tbody>
</table>

Table 5: On the left, our description of bundles with connection in terms of parallel transport functors. On the right our categorification of this situation.

Let $\langle \rho(H) \rangle$ be the algebra generated by the image of $\rho$ and let $\langle \rho(H) \rangle_g$ for $g \in G$ be the $\langle \rho(H) \rangle$ bimodule which is $\langle \rho(H) \rangle$ itself as an object, with the right action twisted by $g$. Then $\tilde{\rho}$ is given by

$$\tilde{\rho} : \bullet \xrightarrow{\rho(h)} \bullet \to \langle \rho(H) \rangle \xrightarrow{\rho(h)} \langle \rho(H) \rangle \cdot \langle \rho(H) \rangle_g.$$

We shall only need a very simple special case of this, namely the representation of the 2-group

$$\Sigma(U(1)) = (U(1) \to 1)$$
### Table 6: Categorified linear algebra

<table>
<thead>
<tr>
<th>abelian monoidal category $\mathcal{C}$</th>
<th>examples for $\mathcal{C}$-module categories</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Disc}(K)$</td>
<td>Baez-Crans 2-vector spaces: categories internal to $\text{Vect}_K$</td>
</tr>
<tr>
<td>$\text{Vect}_K$</td>
<td>Kapranov-Voevodsky 2-vector spaces: categories of the form $(\text{Vect}<em>K)^n \simeq \text{Mod}</em>{K^n}$</td>
</tr>
<tr>
<td>$\mathcal{C}$</td>
<td>categories of $\mathcal{C}$-internal algebra modules in the image of the canonical embedding $\text{Bim}(\mathcal{C}) \xrightarrow{\subset} \text{Vect}_K \text{Mod}$</td>
</tr>
</tbody>
</table>

### Table 7: Various flavors of 2-vector spaces.

induced from the standard representation of $U(1)$:

$$\tilde{\rho} : \bullet \xrightarrow{\iota} \mathbb{C} \xrightarrow{\subset} \mathbb{C}$$

#### 2.3 Associated 2-bundles

Given a principal $G$-bundle $P$ and a representation $G$ on a vector space $V$, one can form the total space of the associated vector bundle by forming the quotient $P \times_G V$.

Categorifying this description is subtle, since it involves 2-coequalizers.

What is much simpler is the description of associated bundles in terms of local transitions. There it simply amounts to postcomposing the respective
anafunctor with the representation:

\[
\begin{array}{ccc}
(x, j) & \rightarrow & (x, k) \\
\| \quad \| \quad \| \\
(x, i) & \rightarrow & (x, k) \\
\end{array}
\]

This categorifies easily. We find that a \textbf{line-2-bundle}, coming from the canonical 2-rep of \(\Sigma(U(1))\), is given by a 2-anafunctor that labels transition tetrahedra like this:

\[
\begin{array}{ccc}
\mathbb{C} & \rightarrow & \mathbb{C} \\
\downarrow & \downarrow & \downarrow \\
\mathbb{C} & \rightarrow & \mathbb{C} \\
\end{array}
\quad =
\begin{array}{ccc}
\mathbb{C} & \rightarrow & \mathbb{C} \\
\downarrow & \downarrow & \downarrow \\
\mathbb{C} & \rightarrow & \mathbb{C} \\
\end{array}
\]

\textbf{Bundle gerbes with connection.} The above 2-anafunctor of the form

\[
\begin{array}{ccc}
P_2(U^\bullet) & \rightarrow & \Sigma(\Sigma(U(1))) \rightarrow \tilde{\rho} \rightarrow \text{Bim}(\text{Vect}) \\
\downarrow \rho & & \downarrow \tilde{\rho} & \downarrow \tilde{\rho} \\
P_2(X) & & \Sigma(\Sigma(U(1))) & \rightarrow & \text{Bim}_{\text{FinRnk}}(\text{Vect}_\mathbb{C}) \\
\end{array}
\]

can be regarded as the end result of a local trivialization process of several steps, descending along this chain of injections:

\[
\begin{array}{cc}
\Sigma(\Sigma(U(1))) & \rightarrow \Sigma(1d\text{Vect}_\mathbb{C}) \rightarrow \text{Bim}_{\text{FinRnk}}(\text{Vect}_\mathbb{C}) \\
\end{array}
\]

Notice that \(\mathbb{C}\), regarded as an algebra over itself, is (Morita-)equivalent to any algebra of finite-rank operators on a complex vector space.

\textbf{Side remark.} Therefore, in the world of 2-vector bundles with respect to the flavor of 2-vector spaces given by \(\text{Bim}(\text{Vect}_\mathbb{C})\), the most general line-2-bundle with connection is a 2-functor with values in \(\text{Bim}_{\text{FinRnk}}(\text{Vect}_\mathbb{C})\).

Given any principal \(\text{PU}(H)\)-bundle with connection on \(X\), the fact that \(\text{PU}(H)\) is the automorphism group of the algebra of finite-rank operators on \(H\) gives us canonically an associated algebra bundle with connection. This is encoded in a 1-functor \(P_1(X) \rightarrow \text{Bim}_{\text{FinRnk}}(\text{Vect}_\mathbb{C})\). A smooth choice of lift of the \(\text{Lie}(\text{PU}(H))\)-valued curvature at each point to \(\text{Lie}(U(H))\) gives an extension to a 2-functor

\[
\text{tra} : P_2(X) \rightarrow \text{Bim}_{\text{FinRnk}}(\text{Vect}_\mathbb{C})
\]
3 Sections of 2-vector bundles

Now that we know what a line-2-bundle is like, we can plug this into our arrow theory of quantum mechanics and find out what sections into this bundle are like.

3.1 Open string coupled to a line-2-bundle

For this, we have to specify a suitable parameter space. Of particular interest is the parameter space given by the category

$$\text{par} = \{a \to b\}$$

that consists of two objects with a single nontrivial morphism between these.

This choice allows us to consider parallel transport over surfaces with the topology of disk:

![Diagram](image)

Figure 3: The parameter space \(\{a \to b\}\) models “open strings” that trace out disk-like surfaces as they propagate through target space.

Of all maps

$$\text{par} \to \mathcal{P}_2(U^\bullet)$$

we take those to be gauge equivalent that project to the same path in \(X\).

Configuration space

$$\text{conf} \subset [\text{par}, \mathcal{P}_2(U^\bullet)]$$

contains all morphisms describing such gauge equivalences.
### P

<table>
<thead>
<tr>
<th>parameter space</th>
<th>interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>{•}</td>
<td>single point particle</td>
</tr>
<tr>
<td>{a, b}</td>
<td>two point particles</td>
</tr>
<tr>
<td>{a → b}</td>
<td>open string</td>
</tr>
<tr>
<td>(\Sigma(N) = {a → a})</td>
<td>closed string</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c}
\begin{array}{c}
\text{Table 8: Different choices of parameter space categories and the corresponding physics interpretation.}
\end{array}
\end{array}
\]

Notice that for \(\text{par} = \{a → b\}\) this configuration space is a lot like that describing two 1-particles \(\{a, b\}\), but now encoding the information that these two points are connected by a string.

The inclusion

\[
\begin{array}{c}
\begin{array}{c}
\text{The inclusion }
\end{array}
\end{array}
\]

identifies the two endpoints of the open strings as two point particles.

In order to understand sections of the line-2-bundle over the open string it therefore helps to first study their behaviours over these endpoints only. This amounts to pulling back our 2-bundle with connection, encoded in the 2-anafunctor \((\text{tra}_U, g, f)\), all the way to the parameter space of the 2 point particles

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\begin{array}{c}
\text{But the configuration space of two point particles } a \text{ and } b \text{ propagating on } \mathcal{P}_2(U^\bullet)
\end{array}
\end{array}
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\text{Table 9: A morphism in the configuration space of the open string relating two string configurations that differ only by a gauge transformation.}
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\begin{array}{c}
\text{Notice that for } \text{par} = \{a → b\} \text{ this configuration space is a lot like that describing two 1-particles } \{a, b\}, \text{ but now encoding the information that these two points are connected by a string.}
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\]
is simply $U^* \times U^*$. A section $e$ of a line-2-bundle over the endpoint of the string is therefore a morphism

$$e : \mathbb{I}_{|U^*} \to \text{tr}_{\ast}|_{U^*}.$$ 

### 3.2 Gerbe modules and D-branes

What is such a morphism like? Being a morphism of 2-functors, it is a pseudo-natural transformation. This means $e$ is determined by an assignment

$$e : ( (x, i) \mapsto (x, j) ) \mapsto e_i(x) \sim e_j(x) ,$$

for each point $x$ in a double intersection of the cover, where $e_i(x)$ and $e_j(x)$ are $\mathbb{C}$-bimodules, hence vector spaces, and where $e_{ij}(x) : e_j(x) \to e_i(x)$ is a morphism of $\mathbb{C}$-bimodules, hence a linear map.

The consistency condition this assignment has to satisfy is

$$e_{ij} \circ e_{jk} = f_{ijk} e_{ik} .$$

for all $x$ in triple overlaps of the cover.

If you like formulas better, think of this equivalently as saying that

$$e_{ij} \circ e_{jk} = f_{ijk} e_{ik} .$$

It follows that the section $e$ of our line-2-bundle is, over the endpoints of the open string, much like an ordinary vector bundle, but one whose transition cocycle involves a certain “twist” which is measured by the cocycle data of the line-2-bundle.

Such structures are equivalently known as

- twisted vector bundles

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• gerbe modules
• twisted representations of $U^{[2]}$
• D-branes with Chan-Paton bundles.

In conclusion, we find that

**Proposition 1** A section of a line-2-bundle (≈ line bundle gerbe) with respect to the open string $\{a \to b\}$ is a D-brane over $a$, another D-brane over $b$ together with a morphism of D-branes over $a \to b$. 