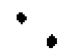
Eugenia Cheng Jan. 2007

$\boxed{\text{I}}$ n-Categories with duals and TQFT

Atiyah: An n-TQFT is a symmetric monoidal functor

$$n\,\mathrm{Cob} \longrightarrow \mathrm{Vect}$$

Want n-categories to deal w/ cobordisms having corners:

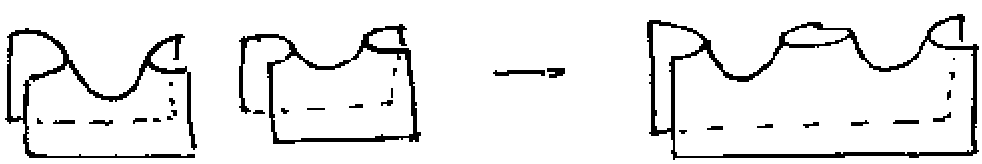
0-cells 

1-cells 

2-cells

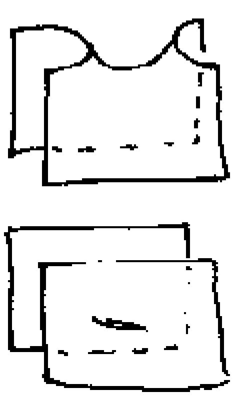
Should form n-category.

k-cells should have k kinds of composition



Attach along 0-cells

To ~~keep track of structure~~



along 1-cells

Actually, want $n\,\mathrm{Cob}_* \longrightarrow n\,\mathrm{Vect}.$

3. Hypotheses of Baez-Dolan

- <u>Extended TQFT hypothesis</u>

The n-category of which extended nTQFTs are reps is the free ~~monoidal~~ stable n-category w/ duals on one object.

- <u>Tangle hypothesis</u>

The n-category of framed n-tangles in $(n+k)$ dimensions is equivalent to the free weak k-tuply monoidal n-category w/ duals on one object.

An n-tangle in $(n+k)$-dims is an n-manifold w/ corners so embeds in $(n+k)$ space if $k \geq n+2$

Informally a k-tuply monoidal n-category is an n-category w/ k-coherent monoidal structures on it. These arise as higher dimensional cats trivial below dim k.

$$\left.\begin{array}{l} 0\text{-cells} \\ 1\text{-cells} \\ \vdots \\ k\text{-}1\text{-cells} \end{array}\right\} \text{trivial (one of each)}$$

k-cells New category

$\vdots \longrightarrow$

k-cells $\rightarrow$ 0-cells
$k+1$ $\rightarrow$ 1-cells
$\vdots$

k kinds of monoidal structure

k \ n	0	1	2	3
0	set	cat	bicat	tricat
1	monoid	mon. cat	mon bicat	mon. tricat
2	comm. monoid	braided mon. cat	brd mon bicat	brd. mon. tricat
3			symm. mon. cat	syleptic mon bicat	syleptic
4			symm. syleptic mon. bicat	symm.

$\swarrow$ degen. case 1object

$s\downarrow$ stable hypothesis

$\uparrow f$ forgetful functor

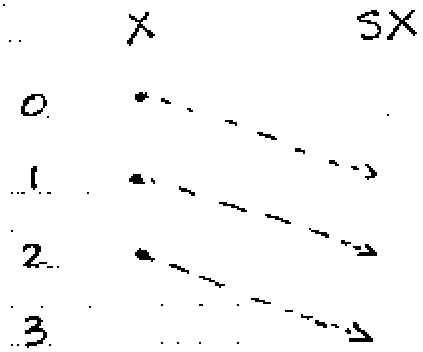
<u>Stabilization hypothesis</u>

$$n\,Cat_K \xrightarrow{F} n\,Cat_{K+1} \quad \text{forgetful}$$

k-tuply mon. n cats

F has some sort of weak higher dimensional left adjoint S and S is an equivalence for $K \geq n+2$.

Suspension



$\pi^n(SX) \simeq S\pi^n(X)$ where on left have suspension of space, on right of n-groupoid.

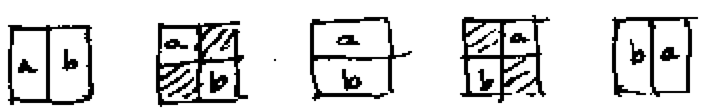
<u>Eckmann-Hilton argument</u>

<u>Given</u> two mults $\times_1$ and $\times_2$ on set that are unital with same unit and distributive

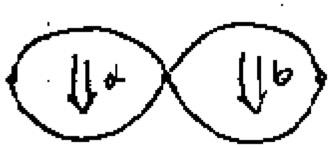
$$(a \times_1 b) \times_2 (c \times_1 d) = (a \times_2 c) \times_1 (b \times_2 d)$$

then $\times_1 = \times_2$ are the same and commutative.

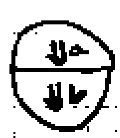
Same proof as π_2 abelian:



Look at bicategory w/ one 0-cell, one 1-cell, and set of 2-cells. Two 2-cell compositions give two mults:

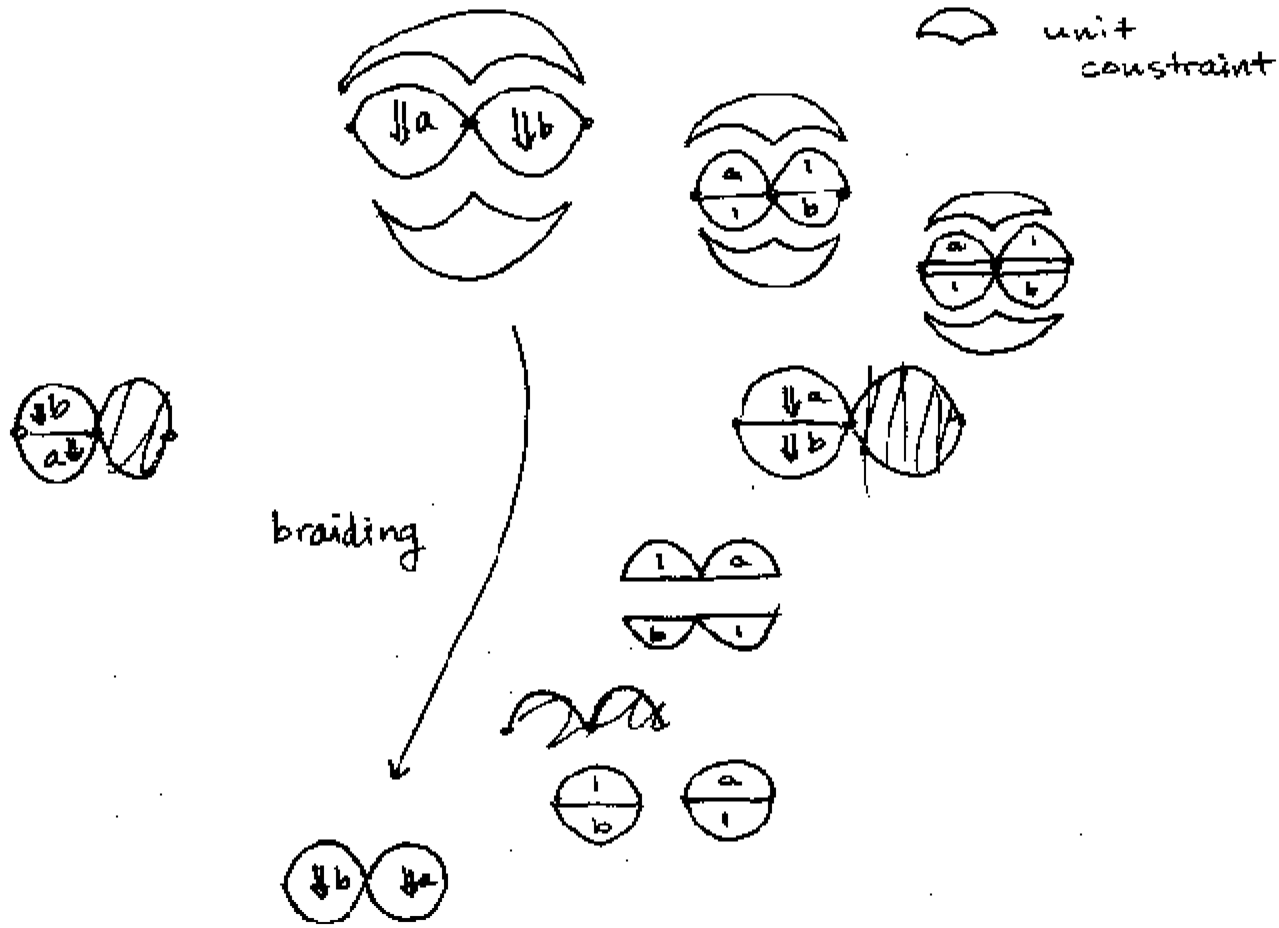


Horizontal



Vertical

Eckmann-Hilton clock



Given tricategory w/ only one 0-cell, 1-cell we get a braided mon cat

The braiding comes from weakness of interchange and units.

SLOGAN: all sorts of interesting structure arises when we allow non-trivial morphisms between identity cells.

II The tangle hypothesis in low dimensions

The n-category of framed n-tangles in (n+k) dimensions is (n+k)-equiv to the free k-tuply monoidal n-category w/ duals on one object.

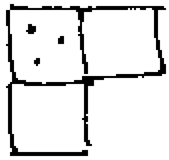
A framed n-tangle in (n+k) dims "is" an n-manifold with corners embedded in $[0,1]^{n+k}$ such that codim j corners are mapped to codim j corners of the cube together with a trivialization of the normal bundle (which in particular gives an orientation)

Should be k-degenerate (n+k)-cat

$$\left.\begin{array}{l} 0\text{-cells} \\ 1\text{-cells} \\ \vdots \\ k{-}1\text{-cells} \end{array}\right\}\text{trivial}$$

k-cells	framed 0 manifolds in k cubes	0
k+1-cells	1 mnflds in (k+1) cubes	1
$\vdots$		$\vdots$
(k+n)-cells	Isotopy classes of n manifolds in (k+n) cells	n

In general, m-dimn'l boxes can be stacked in m ways so we get K compositions on our 0-cells



Why do we expect duals?

A dual is the internal notion of adjunction.

Ex. Adjoint functors are dual 1-cells in the 2-cat. CAT.

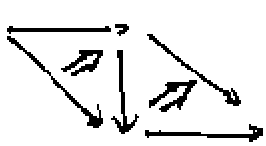
A right dual for

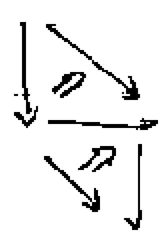
$a \xrightarrow{f} b$ is a $b \xrightarrow{g} a$

and 2-cells

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "1_a"'] & b \arrow[d, "g"] \arrow[dl, phantom, "\Rightarrow \eta"] \\ & a \end{tikzcd} \qquad \begin{tikzcd} b \arrow[d, "g"'] \arrow[dr, "1_b"] & \\ a \arrow[r, "f"'] \arrow[ur, phantom, "e \Rightarrow"] & b \end{tikzcd}$$

Satisfying

 $= 1_f$

 $= 1_g$

~~In strict~~ 2-cat

In strict 2-cat

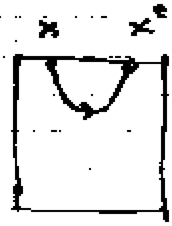
$$\begin{tikzcd} f \arrow[r, "f\eta"] \arrow[dr, "1_f"'] & fgf \arrow[d, "\varepsilon f"] \\ & f \end{tikzcd}$$

In a bicat

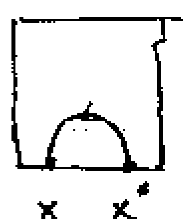
$$\begin{tikzcd} f \arrow[r, "r^{-1}"] \arrow[dddr, "1_f"'] & f \arrow[r, "f\eta"] & f(gf) \arrow[d, "a^{-1}"] \\ & & (fg)f \arrow[d, "\varepsilon f"] \\ & & \bullet \arrow[d] \\ & & f \end{tikzcd}$$

For manifolds

$$\begin{tikzcd} X\otimes X^* \arrow[d, "g"'] \\ \emptyset \end{tikzcd}$$



$$\begin{tikzcd} \emptyset \arrow[d, "f"'] \\ X\otimes X^* \end{tikzcd}$$



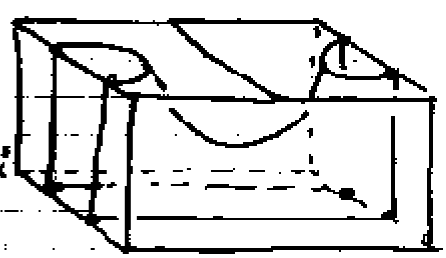
Duality between 1-cells:
$f^* = g$

counit

$$X\otimes X^* \xrightarrow{g} \emptyset \xrightarrow{f} X\otimes X^*$$

$\varepsilon \Downarrow$

$1_{X\otimes X^*}$



$$\emptyset \xrightarrow{f} X\otimes X^* \xrightarrow{g} \emptyset$$

For n-cells in n-category have no dimensions in which to have unit / counit.
A dual for an n-cell

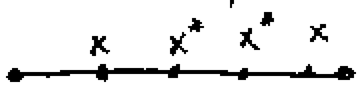
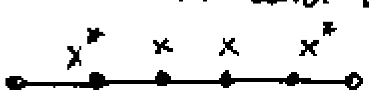
$$a \longrightarrow b$$

is simply an n-cell $b \longrightarrow a$ well-behaved.

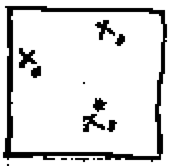
For 0cells we can only have (co)unit if we have some monoidal structure.

Issues:

- interaction of duals of different dimension
- double duals
- duals and composition
- duals and coherence cells for n-cat (which are equivs, adjoint and thus duals)

	manifolds	algebra
$\boxed{n=0}$		
k=0	oriented 0 mnflds in 0 space $\{x, x^*\}$	0-mon 0-cat just a set dual is an involution.
k=1	oriented 0-manifolds in 1-space  dual: reflect and reverse  $(xy)^* = y^* x^*$	1-mon 0-cat - monoid dual: $* : A \longrightarrow A$ such that $(xy)^* = y^* x^*$

$k=2$



Can move them around up to isotopy.

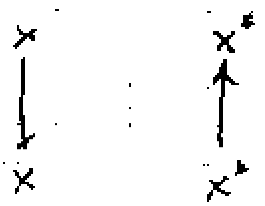
And $k>2$ gives same stabilization.

Commutative monoid w/ duals.

$\boxed{n=1}$ $k=0$

objs - 0 mnflds in 0 space

morphs - 1 mnflds in 1 space



0-mon 1-cat - category

duals - on objects, involution

On morphs

$a \xrightarrow{f} b$

$b \xrightarrow{f^*} a$

s.t. $f^{**} = f$, $(fg)^* = g^* f^*$

$\Rightarrow 1^* = 1$

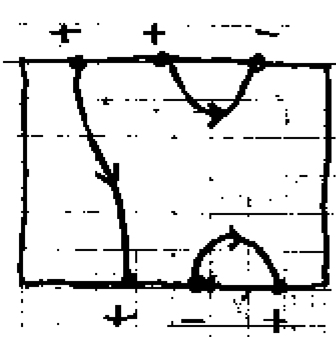
So free on one object

$\leftarrow$

$k=1$

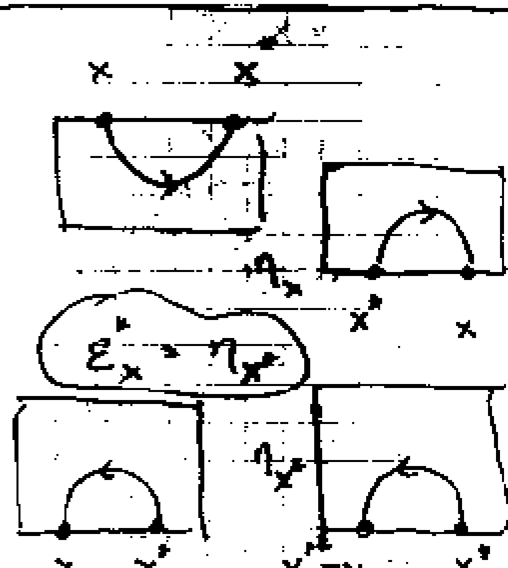
objs:

morphs



$(f\otimes g)^* = f^* \otimes g^*$

$x^{**} = x$ $\quad (x\otimes y)^* = y^* \otimes x^*$

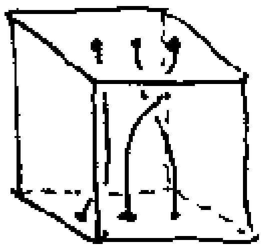


1-mon 1-cat – mon cat w/ duals satisfying above

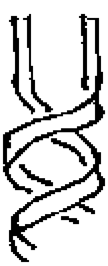
k = 2

objs

morphs



w/ twists, so ribbons



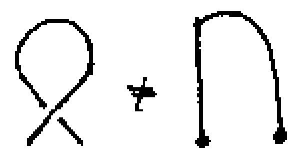
2-mon 1-cat – braided mon. cat

- For just braid mon cat, just take braid cat.

- symm. mon cats = compact w/ duals for objs closed

Kelly-Laplaza

- Shum did braided mon. cat w/ duals for objs

 in general

TTC

Tortile monoidal cats / ribbon cats

BMC w/

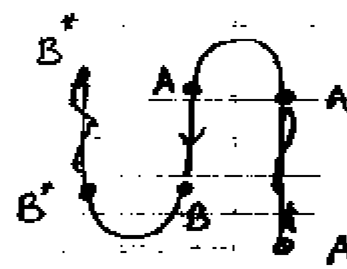
- ∀ obj a, right dual w/ unit / counit
- $a \xrightarrow{\theta_a} a$

s.t. $\theta_I = I$

$$\begin{tikzcd}
A \otimes B \arrow[r, "\gamma_{AB}"] \arrow[d, "\theta_{A\otimes B}"'] & B \otimes A \arrow[d, "\theta_B \otimes \theta_A"] \\
A \otimes B & B \otimes A \arrow[l, "\gamma_{BA}"']
\end{tikzcd}$$

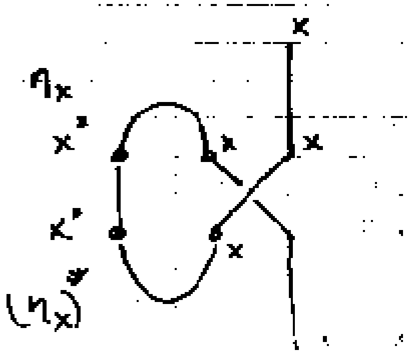
• $\theta_{A^*} = (\theta_A)^*$ NB: here * different

Given any $A \xrightarrow{f} B$
get $B^* \xrightarrow{f^*} A^*$



Coherence

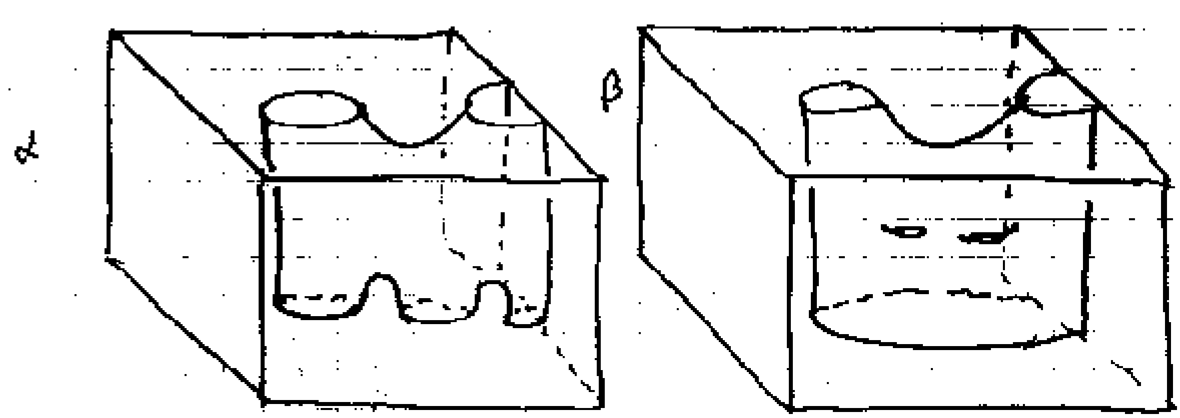
Free TTC on 1 obj is equiv to tangle cat of framed oriented 1-tangle in 3-space.

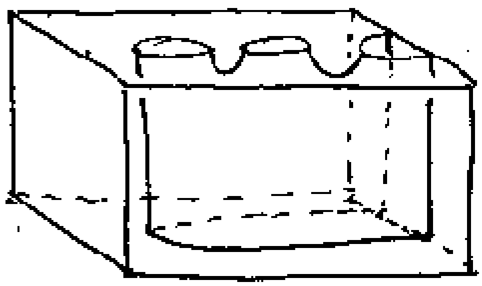
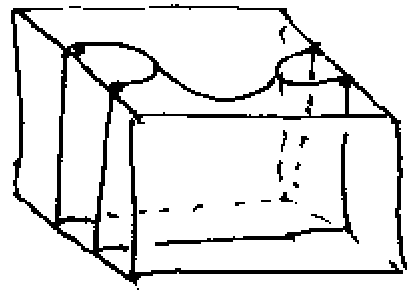


We had $(\eta_x)^* = \varepsilon_{x^*}$
Can only say this since $x^{**} = x$

NB: The ribbon cat simply does have duals for morphisms even though not explicitly forced by free structure. Comes from twist.

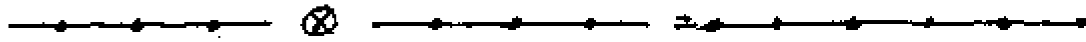
III Higher low dimensions



γ   δ

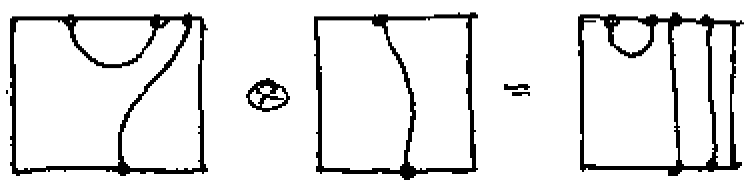
Start w/ $n=2$, $k=1$

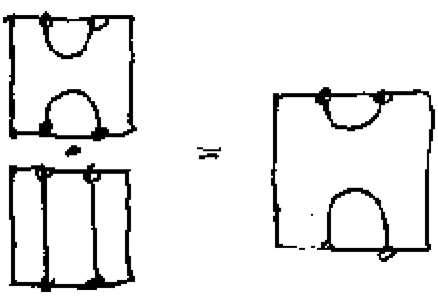
0-cells



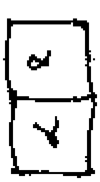
~~ccells~~

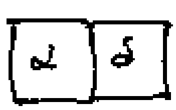
1-cells

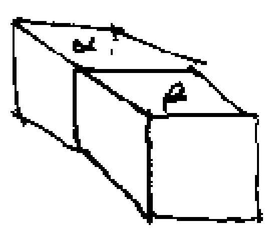




2-cells







say. Tensor product

Baez/Langford HDA 4

$k=2$, $n=2$

1. Only did unframed case.
2. Semi-strict version instead of fully weak.

2. Take equivalence classes at all dimensions. Ignore reparametrization issues. So ignore coherence for composition.

3. Restrict to generic tangles with interesting features spread out.
4. Use "collars" to preserve smoothness in composition.

More precisely, a generic 1 tangle is a 1 mnfld w/ boundary in $[0,1]^3$ w/ boundary in top interior of top and bottom face. Product structure near top and bottom. If project, finitely many crossings. Want critical pts (in 4. slicing along last coordinate) to be local extrema non-degenerate. Critical pts and crossings all occur at different heights.

Subject to equiv relation:

ambient isotopy, level preserving, product structure near top and bottom.

1-cells are equiv classes of generic 1-tangles

Generic 2 tangles:

boundary in right place, collars, intersections with hyperplanes of constant height are all generic 1-tangles except finitely many. Must have certain catastrophe forms classified by Carter, R, Saito.

Use equiv relation: careful isotopy.

NB: Stacking boxes is a strict composition since can always pick reparametrization satisfying necessary conditions.

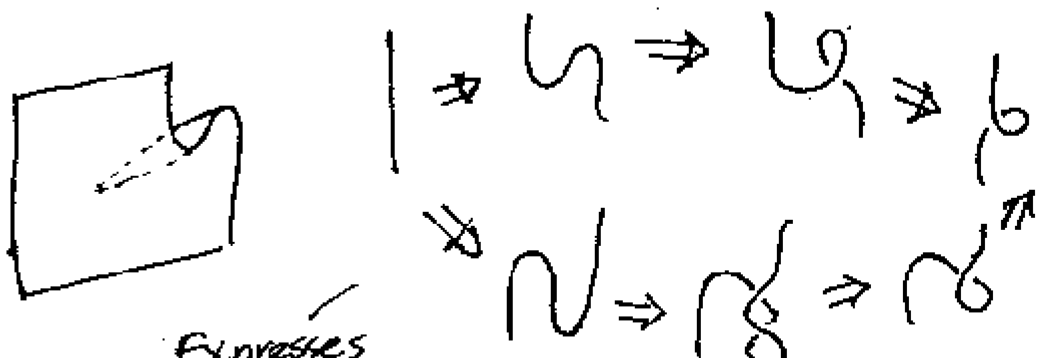
Category side:

Define: Braided monoidal 2 category, strict

Thm Above gives free braided monoidal 2-cat on one unframed self-dual object. ~~Kex~~

$x^* = x$

Specify the writhing, a 2-morphism



Expresses unframedness.

General n

Trimble's definition uses reparametrization of intervals.

Use maps $[1] \longrightarrow [k]$ (length $1 \longrightarrow$ length k) to parameterize composition.

Start with 1-cells

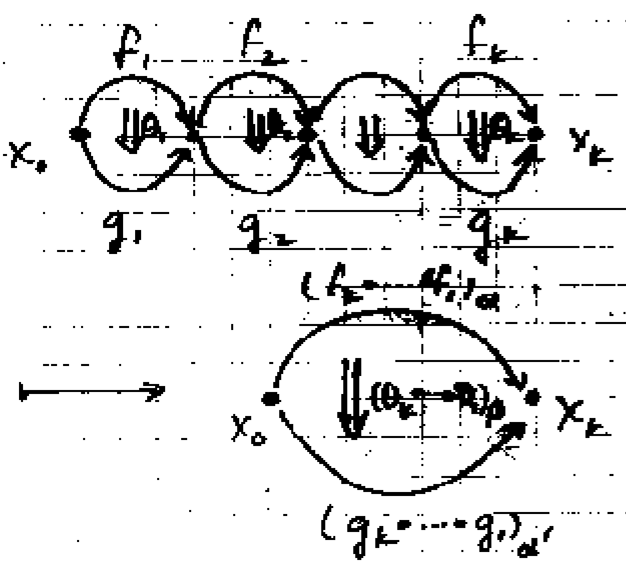
$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \longrightarrow \cdots \xrightarrow{f_k} x_k$$

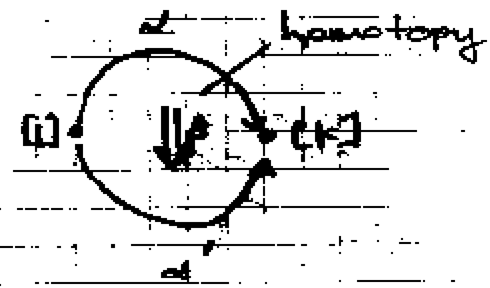
and $[1] \xrightarrow{\alpha} [K]$

Composing gives

$$(f_k \circ \cdots \circ f_1)_\alpha : X_0 \longrightarrow X_k$$

2-cells





Features

- Enriched, so n-cat is given by ~~set of objects~~

* set of objs
* $\forall$ a, b, an $(n-1)$-category $A(a,b)$
* comp $(n-1)$ functor

(There's a topological operad $E(k) = \text{maps } [1] \to [k]$)

$$\Pi_{n-1}(E(k)) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k)$$

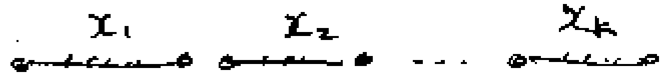
Coherence from functoriality, interaction w/ operad structure, and contractability of $E(k)$.

Warm-up exercise: try on subsets of cubes.

$[1] = I \qquad I^k \longrightarrow 2$, 2-element set. k-cells.

Subsets of I: $I \longrightarrow 2$, in or out.

Compose:



$x_1, \dots, x_k : I \longrightarrow 2$

$x_1 + \dots + x_k : [k] \longrightarrow 2$

Given $\alpha : I \longrightarrow [k]$, compose

$$I \xrightarrow{\alpha} [k] \xrightarrow{x_1 + \dots + x_k} 2$$

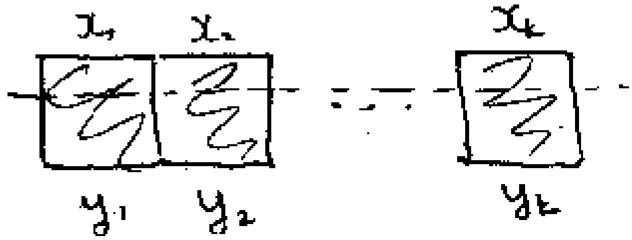
Horizontal compo of 2-boxes:

Given 2-cells $\begin{matrix} x_1 \\ \theta_1 \Downarrow \\ y_1 \end{matrix} \quad \begin{matrix} x_2 \\ \theta_2 \Downarrow \\ y_2 \end{matrix} \quad \cdots \quad \begin{matrix} x_k \\ \theta_k \Downarrow \\ y_k \end{matrix} \qquad I \times I \longrightarrow 2$

$$I \xrightarrow{f,g} [k]$$

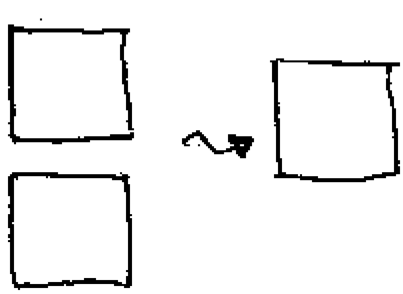
$f \xrightarrow{\alpha} g$ homotopy $\qquad I \times I \xrightarrow{\alpha} [k]$

$\alpha(0,-) = f$, $\alpha(1,-) = g$



at height t reparam by $\alpha(t,-)$

Vertical composition



Can we apply to mnflds?

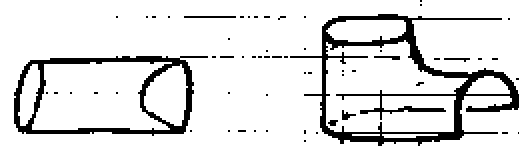
Problem: want to restrict $\Pi_{n-1}(E(K))$ at every dim.

We Can't find subspace $E'(K)$ of $E(K)$ whose fund n-grpd is the one we want.

Ask for an operad in globular sets.

IV Cubical

Aaron's zipper

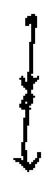


/ put in box

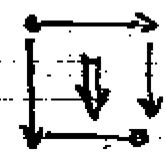
Double category: (as opposed to globular)

horizontal cat

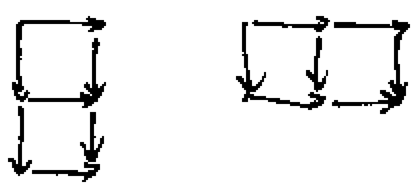
vertical cat



2 cells



A double category is an internal category in Cat. Can demand that all vertical arrows are identities to get ~~two~~ 2 categories.



Two directions of composition.

Have we lost our zipper by ignoring 2-cells w/ non-trivial vertical arrows?

Usually, people take

2 Cat $\longrightarrow$ Double cat

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Double cat arises here only if horizontal = vertical cat.

Need a "connection" on double cat to get from 2 Cat

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, "g'"] \\ \bullet \arrow[r, "f'"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "cc"] & \cdot \arrow[r, "f"] \arrow[d, "g'"] \arrow[dr, phantom, "\Downarrow"] & \cdot \arrow[r, "g"] \arrow[d, "g"'] \arrow[dr, phantom, "cc"] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "g'"'] & \cdot \arrow[r, "f'"'] & \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

cc connection cell.

<u>Can</u> do to get our zipper

ω dimensions: Don't have to quotient. Helps coherence

Version of tangle hypothesis:

There is an ω cat whose n-cells are framed n-manifolds in $(n+k)$-cubes

$\simeq$ free k-tuply monoidal cat w/ duals one one object

<u>Thm</u> (Oo)

An ω cat w/ duals is an ω groupoid

NB: All you need of dual structure the unit and counit. So in part. don't need any comp. of length > 2. Just need some putative identities and binary composites of k-cells along $(k-1)$-cells.

Idea Notion of weakly invertible becomes weaker and weaker as you add dims to your structure

In 1-cell in 1-cat

$$a \xrightarrow{f} b \xrightarrow{g} a \xrightarrow{f} b$$

(weak) Inverse $fg = 1$, $gf = 1$.

In 2Cat need 2cells

$$1 \Rightarrow gf \qquad fg \Rightarrow 1$$

s.t. η and ε weak invertible (in Hom cat).

In n-Cat, need η and ε weakly invertible as 1-cells in Hom $(n-1)$-cat (by induction)

In an ω cat — (Batanin)

Can take n-coskeleton: ignore dim $> n$ and equiv on n-cells $\alpha \sim \beta$ if there is an $(n+1)$-cell

$$\alpha \xRightarrow{\theta} \beta \quad \text{or} \quad \beta \xRightarrow{\theta} \alpha$$

Then weakly invertible in ω-cat iff weakly invertible in every n-skeleton.

OR Demand set of 'witnesses' W to invertibility

- $f \in \omega$
- $\forall \underset{a \to b}{\alpha} \in W$ $\exists$ $\alpha^* : b \to a \in W$

and $\eta : 1 \to \alpha^*\alpha$, $\varepsilon : \alpha\alpha^* \to 1 \in W$

If $\mathcal{C}$ an ω-cat w/ all (pre) duals
Can put all cells (of dim ≥ 1) in W

$\Rightarrow$ all cells weakly invertible.

n-costeleton like taking cobord classes of n-mnflds rather than our careful isotopy classes.

$\Longrightarrow$

Stable version of tangle hypothesis ($k \geq n+2$)

n-cat whose n cells are isotopy classes of framed n-mnflds $\simeq$ free stable n-cat w/ duals on one object

$$\begin{tikzcd}
nCob_k \arrow[rr, "k \text{ large}"] \arrow[d, leftrightarrow, "2"] \arrow[dd, bend right] & & nCob_\omega \arrow[d, leftrightarrow, "2"] \arrow[dl, "\text{don't term}"] \\
\mathcal{F}_{n,k} \arrow[r, "k \text{ large}"] \arrow[d] & \omega Cob_k \arrow[dl] \arrow[d, leftrightarrow, "2"] & \mathcal{F}_{n,\omega} \arrow[dl, "\text{don't term}"] \\
\omega Cob_\omega \arrow[d, leftrightarrow, "2"] & \mathcal{F}_{\omega,k} \arrow[dl] & \\
\mathcal{F}_{\omega,\omega} & &
\end{tikzcd}$$

$nCob_k$ - n mnflds in $(n+k)$-cubes

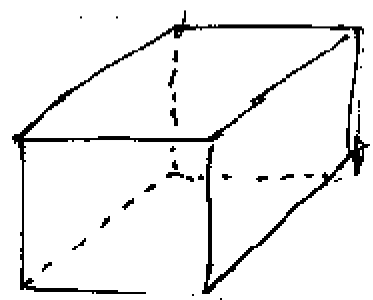
$\mathcal{F}_{n,k}$ = free k-tuply mon. n cat w/ duals on 1 obj.

ωCob_ω - n mnflds at all dimns

ωCob_k

$\mathcal{F}_{\omega,\omega}$ = free stable ω cat w/ duals on one obj?

free stable ω grpd on one object



What's a stable ω-cat?

Can't just say let $K \geq n+2$. Take some sort of limit?

Think about cobordism classes $\rightsquigarrow$ Thom-Pontryagin

a grp / cobordism classes of framed n-mnflds $\simeq$ $\pi_n(\Omega^\infty S^\infty)$

generalize

n grpds / cobord classes of framed n mnflds w/ corners $\xleftrightarrow{?}$ $\Pi_n(\Omega^\infty S^\infty)$

Don't terminate.

ω-cat of framed cobords w/ corners $\xleftrightarrow{?}$ $\Pi_\omega(\Omega^\infty S^\infty)$

$\simeq$ ω cat w/ duals

ω grpd

$n\mathrm{Cob}_\omega \in \omega\,\mathrm{Cat}$

isotopy classes

n cat

n-cosk

free st. ω cat w/ duals $\simeq$ free st. ω grpd

cob classes of n-mnflds $\in n$ grpd

n cosk

free stable n-cat w/ duals on 1 obj

fund grpd of $\Omega^\infty S^\infty$