Eugenia Cheng Jan. 2007

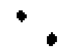
$\boxed{\text{I}}$ n-Categories with duals and TQFT

Atiyah: An n-TQFT is a symmetric monoidal functor

$$n\,\mathrm{Cob} \longrightarrow \mathrm{Vect}$$

Want n-categories to deal w/ cobordisms having corners:

0-cells

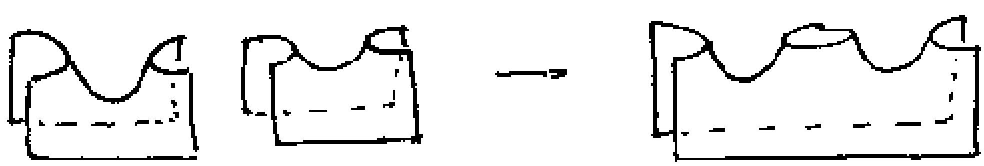


1-cells

2-cells

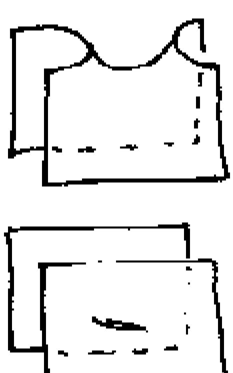
Should form n-category.

k-cells should have k kinds of composition



Attach along 0-cells

To ~~keep track of structure~~



along 1-cells $\longrightarrow$

Actually, want $n\,\mathrm{Cob}_* \longrightarrow n\,\mathrm{Vect}$.

3. Hypotheses of Baez-Dolan

- Extended TQFT hypothesis

The n-category of which extended nTQFTs are reps is the free ~~monoidal~~ stable n-category w/ duals on one object.

- Tangle hypothesis

The n-category of framed n-tangles in (n+k) dimensions is equivalent to the free weak k-tuply monoidal n-category w/ duals on one object.

An n-tangle in (n+k)-dims is an n-manifold w/ corners so embeds in (n+k) space if $k \geq n+2$

Informally a k-tuply monoidal n-category is an n-category w/ k-coherent monoidal structures on it. These arise as higher dimensional cats trivial below dim k.

$$\left.\begin{array}{l} 0\text{-cells} \\ 1\text{-cells} \\ \vdots \\ k\text{-}1\text{-cells} \end{array}\right\} \text{trivial (one of each)}$$

k-cells   New category

$\vdots \longrightarrow$

k-cells $\to$ 0-cells

k+1 $\to$ 1-cells

$\vdots$

k kinds of monoidal structure

k \ n	0	1	2	3
0	set	cat	bicat	tricat
1	monoid	mon. cat	mon bicat	mon. tricat
2	comm. monoid	braided mon. cat	brd mon bicat	brd. mon. tricat
3		symm. mon. cat	syleptic mon bicat	syleptic
4			symm. syleptic mon. bicat	symm.

$\swarrow$ degen. case / 1 object

$s\downarrow$ stable hypothesis

$\uparrow f$ forgetful functor

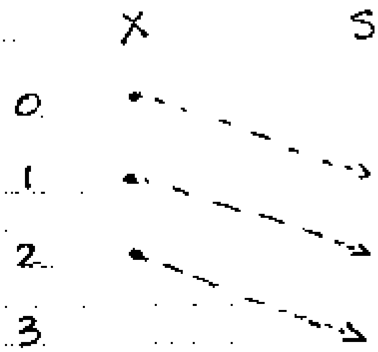
<u>Stabilization hypothesis</u>

$$n\,Cat_k \xrightarrow{F} n\,Cat_{k+1} \quad \text{forgetful}$$

k-tuply mon.
n cats

F has some sort of weak higher dimensional left adjoint S and S is an equivalence for $k \geq n+2$.

Suspension



$\Pi^n(SX) \simeq S\Pi^n(X)$

where on left have suspension of space, on right of n-groupoid.

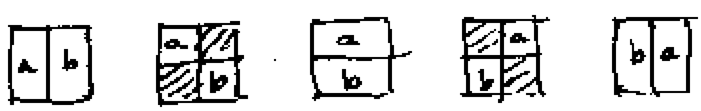
<u>Eckmann-Hilton argument</u>

<u>Given</u> two mults $\times_1$ and $\times_2$ on set that are unital with same unit and distributive

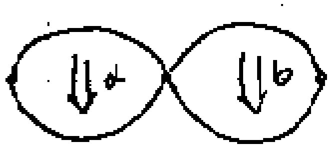
$$(a \times_1 b) \times_2 (c \times_1 d) = (a \times_2 c) \times_1 (b \times_2 d)$$

then $\times_1 = \times_2$ are the same and commutative.

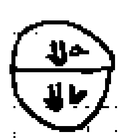
Same proof as π_2 abelian:



Look at bicategory w/ one 0-cell, one 1-cell, and set of 2-cells. Two 2-cell compositions give two mults:

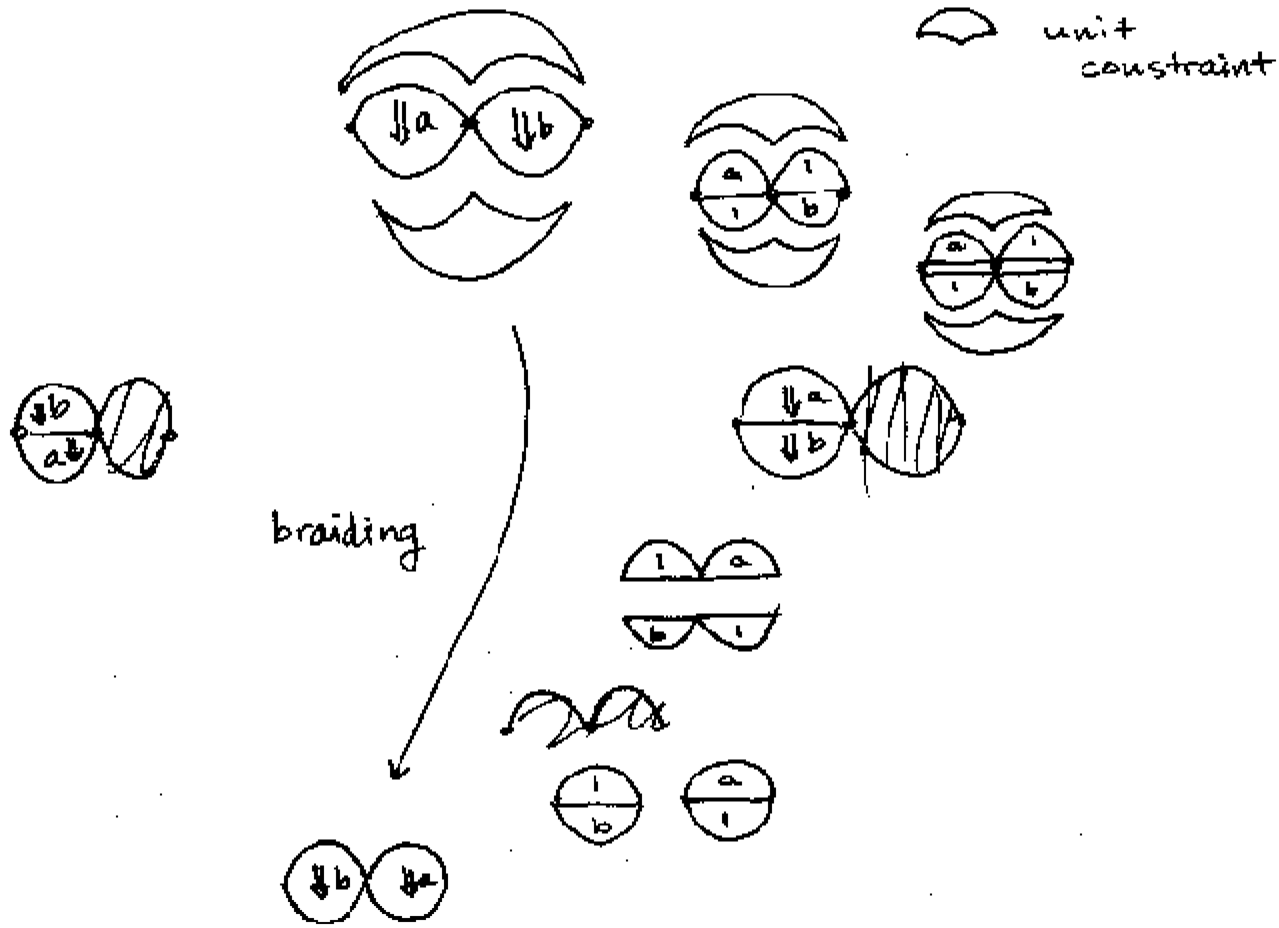


Horizontal



Vertical

Eckmann-Hilton clock



Given tricategory w/ only one 0-cell, 1-cell we get a braided mon cat

The braiding comes from weakness of interchange and units.

SLOGAN: all sorts of interesting structure arises when we allow non-trivial morphisms between identity cells.

II The tangle hypothesis in low dimensions

The n-category of framed n-tangles in (n+k) dimensions is (n+k)-equiv to the free k-tuply monoidal n-category w/ duals on one object.

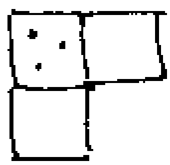
A framed n-tangle in (n+k) dims "is" an n-manifold with corners embedded in $[0,1]^{n+k}$ such that codim j corners are mapped to codim j corners of the cube together with a trivialization of the normal bundle (which in particular gives an orientation)

Should be k-degenerate (n+k)-cat

$$\left.\begin{array}{l} 0\text{-cells} \\ 1\text{-cells} \\ \vdots \\ k\text{-}1\text{-cells} \end{array}\right\} \text{trivial}$$

k-cells	framed 0 manifolds in k cubes	0
k+1-cells	1 mnflds in (k+1) cubes	1
$\vdots$		$\vdots$
(k+n)-cells	Isotopy classes of n manifolds in (k+n) cells	n

In general, m-dimn'l boxes can be stacked in m ways so we get k compositions on our 0-cells



Why do we expect duals?

A dual is the internal notion of adjunction.

<u>Ex</u>. Adjoint functors are dual 1-cells in the 2-cat. CAT.

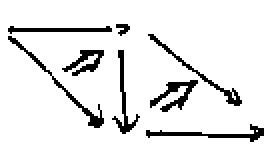
A right dual for

$$a \xrightarrow{f} b \quad \text{is a} \quad b \xrightarrow{g} a$$

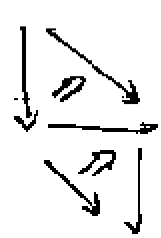
and 2-cells

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "1_a"'] & b \arrow[d, "g"] \arrow[dl, phantom, "\Uparrow\eta"] \\ & a \end{tikzcd} \qquad \begin{tikzcd} b \arrow[d, "g"'] \arrow[dr, "1_b"] & \\ a \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow e"] & b \end{tikzcd}$$

Satisfying



$= 1_f$



$= 1_g$

~~In strict~~ 2-cat

In strict 2-cat

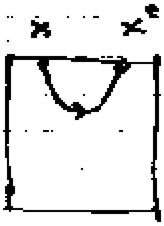
$$\begin{tikzcd} f \arrow[r, "f\eta"] \arrow[dr, "1_f"'] & fgf \arrow[d, "\varepsilon f"] \\ & f \end{tikzcd}$$

In a bicat

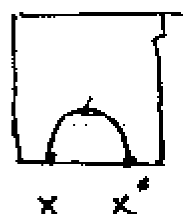
$$\begin{tikzcd} f \arrow[r, "r^{-1}"] \arrow[dddrr, "1_f"'] & f \arrow[r, "f\eta"] & f(gf) \arrow[d, "a^{-1}"] \\ & & (fg)f \arrow[d, "\varepsilon f"] \\ & & \bullet \arrow[d] \\ & & f \end{tikzcd}$$

For manifolds

$$\begin{tikzcd} X \otimes X^* \arrow[d, "g"'] \\ \emptyset \end{tikzcd}$$



$$\begin{tikzcd} \emptyset \arrow[d, "f"'] \\ X \otimes X^* \end{tikzcd}$$



<u>Duality between</u> 1-cells:

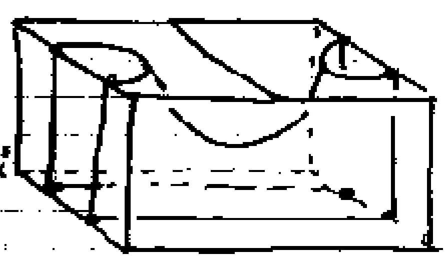
$f^* = g$

counit

$$X \otimes X^* \xrightarrow{g} \emptyset \xrightarrow{f} X \otimes X^*$$

$\varepsilon \Downarrow$

$1_{X \otimes X^*}$



$$\emptyset \xrightarrow{f} X \otimes X^* \xrightarrow{g} \emptyset$$

For n-cells in n-category have no dimensions in which to have unit / counit.
A dual for an n-cell

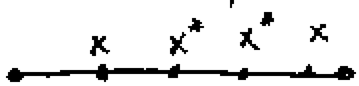
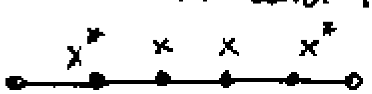
$$a \longrightarrow b$$

is simply an n-cell $b \longrightarrow a$ well-behaved.

For 0cells we can only have (co)unit if we have some monoidal structure.

Issues:

- interaction of duals of different dimension
- double duals
- duals and composition
- duals and coherence cells for n-cat (which are equivs, adjoint and thus duals)

	manifolds	algebra
$\boxed{n=0}$		
$k=0$	oriented 0 mnflds in 0 space $\{x, x^*\}$	0-mon 0-cat just a set dual is an involution.
$k=1$	Oriented 0-manifolds in 1-space  dual: reflect and reverse  $(xy)^* = y^* x^*$	1-mon 0-cat -- monoid dual: $*: A \longrightarrow A$ such that $(xy)^* = y^* x^*$

$k=2$

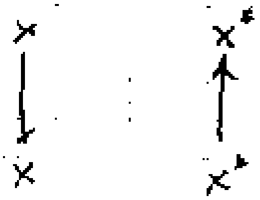
Can move them around up to isotopy.

And $k>2$ gives same stabilization.

Commutative monoid w/ duals.

$\boxed{n=1}$ objs - 0 mnflds in 0 space

$k=0$ morphs - 1 mnflds in 1 space



0-mon 1-cat - category

duals - on objects, involution

On morphs

$a \xrightarrow{f} b$

$b \xrightarrow{f^*} a$

st. $f^{**}=f$, $(fg)^*=g^*f^*$

$\Rightarrow 1^*=1$

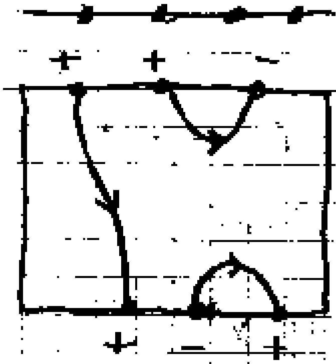
So free on one object

$\leftarrow$

$k=1$

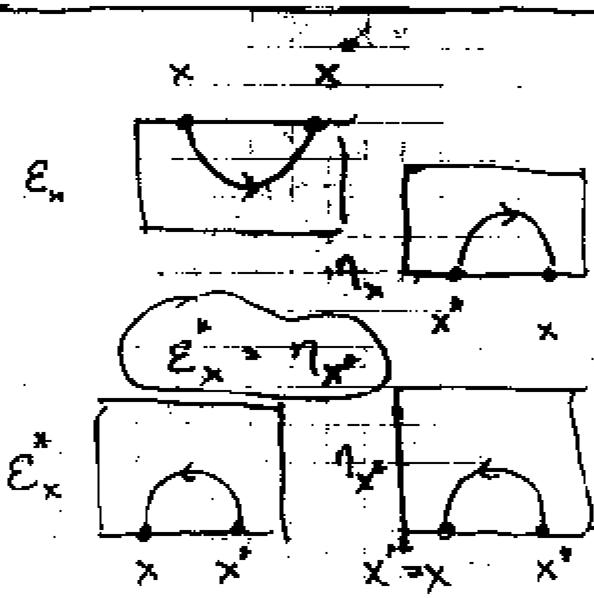
objs:

morphs



$(f\otimes g)^* = f^*\otimes g^*$

$x^{**}=x$ $(x\otimes y)^*=y^*\otimes x^*$



k = 2

ob

mor

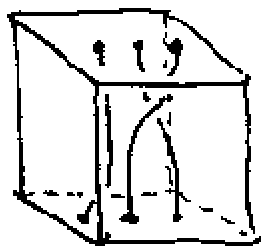
w/

1-mon 1-cat – mon cat w/ duals satisfying above

k = 2

objs

morphs



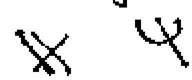
w/ twists, so ribbons

2-mon 1-cat – braided mon. cat

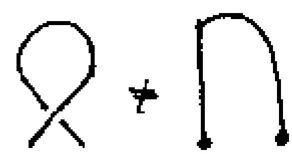
- For just braid mon cat, just take braid cat.

- symm. mon cats = compact w/ duals for objs closed

Kelly-Laplaza



- Shum did braided mon. cat w/ duals for objs

 in general

TTC

Tortile monoidal cats / ribbon cats

BMC w/

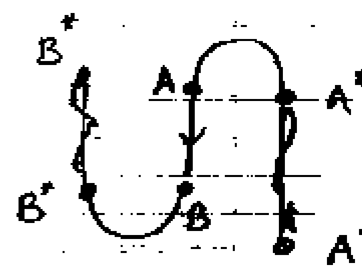
- $\forall$ obj a, right dual w/ unit / counit
- $a \xrightarrow{\theta_a} a$

s.t. $\theta_I = I$

$$
\begin{tikzcd}
A \otimes B \arrow[r, "\gamma_{AB}"] \arrow[d, "\theta_{A\otimes B}"'] & B \otimes A \arrow[d, "\theta_B \otimes \theta_A"] \\
A \otimes B \arrow[r, "\gamma_{BA}"] & B \otimes A
\end{tikzcd}
$$

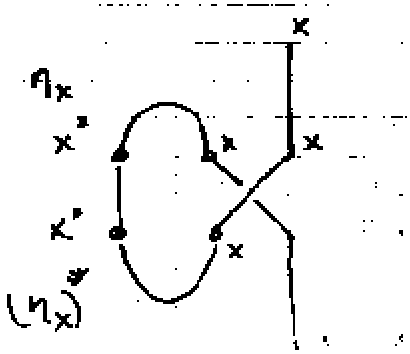
• $\theta_{A^*} = (\theta_A)^*$ NB: here * different

Given any $A \xrightarrow{f} B$
get $B^* \xrightarrow{f^*} A^*$



Coherence

Free TTC on 1 obj is equiv to tangle cat of framed oriented 1-tangle in 3-space.



We had $(\eta_x)^* = \varepsilon_{x^*}$
Can only say this since $x^{**} = x$

NB: The ribbon cat simply does have duals for morphisms even though not explicitly forced by free structure. Comes from twist.

III Higher low dimensions