Combinatorial Quasi-Categories (Thesis Chapters 1 and 2)

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Contents

Chapter 1

Quasi-Categories

1.1 Simplicial Sets and Quasi-Categories

1.1.1 Initial Definitions

In the following, Δ will denote as usual the category of nonempty finite ordinals, with the element $[n] \in \Delta$ representing the ordinal $\{0, 1, \ldots, n\}$.

Definition 1.1.1. A *simplicial set* is a presheaf on Δ, i.e. a functor $\Delta^{\mathrm{op}} \to \mathrm{Sets}$. The image of $[n]$ under such a functor is called the set of *n-simplices.* The function from n-simplices to $(n-1)$-simplices corresponding to the morphism $[n-1] \to [n]$ in Δ whose image is all of $[n]$ except k is called the k^{th} *boundary map* ∂_k; the image of an n-simplex x under ∂_k is called its k^{th} *face.* The function from n-simplices to $(n+1)$-simplices corresponding to the surjection $[n+1] \to [n]$ in Δ which contracts k and $k+1$ onto $k \in [n]$ is called the k^{th} *degeneracy map* σ_k.

For future use, we name a few simplicial sets. For a nonnegative integer n, let Δ^n be the presheaf represented by $[n]$. This simplicial set has a single nondegenerate n-simplex and the $n+1$ faces of this n-simplex are its only nondegenerate $n-1$-simplices. If we omit the nondegenerate n-simplex, we obtain a simplicial set $\partial\Delta^n$, the *boundary* of Δ^n, also called an *n-shell.* If we omit in addition its k^{th} face we obtain a simplicial set which we will call the k^{th} *n-horn* Λ^n_k. If $k < n$, then Λ^n_k is called a *left n-horn*; if $k > 0$, then it is called a *right n-horn*; and if a horn is both a left and right horn, it will be called *inner.* We are now prepared to define a quasi-category (or weak Kan complex as in [BV73]; our terminology follows [Joy02]).

Definition 1.1.2. Let C be a simplicial set. We say that C is a *quasi-category* if for every $m > 1$ and every k with $0 < k < m$, every morphism $\Lambda^m_k \to C$ can be extended to a full m-simplex $\Delta^m \to C$ (in short, every inner horn can be filled). If in addition $0 < n \leq \infty$ and whenever $k > n$ there is only one filler of inner k-horns as above, we say that C is an *n-quasi-category* (so that an ∞-quasi-category is just a quasi-category). If all horns (not just inner ones) have fillers, we say that C is a *quasi-groupoid*, and if all horns of dimension $m > n$ have unique fillers, we say that C is an *n-quasi-groupoid.*

In the sequel, we define the ∞-skeleton of a simplicial set X to just be X itself.

The 0-simplices of a quasi-category are often denoted *objects*, and the 1-simplices 1-*morphisms*, or simply *morphisms* if there is no ambiguity. Given an m-simplex, its 0^{th} and m^{th} faces (the faces which always are present in an inner horn) are denoted the *target* and *source* faces, respectively. More generally, the nondegenerate m-simplex whose vertices are minimal will be referred to as the *source* m-simplex, and the nondegenerate face whose vertices are maximal will be referred to as the *target* m-simplex. We note that the automorphism $\mathrm{op} : \Delta \to \Delta$ which takes a finite ordinal to its opposite order type induces an automorphism $\mathrm{op} : \mathrm{SSets} \to \mathrm{SSets}$ which clearly fixes the classes of n-quasi-categories and n-quasi-groupoids. In the sequel, given a quasi-category X, its image under op will be denoted X^{op} and referred to as the *opposite quasi-category of X*.

1.1.2 Characterizations From Less Data

We provide a criterion for producing an n-quasi-category or n-quasi-groupoid from a finite chunk of data, when n is finite. Recall that for a simplicial set X, the n-skeleton X_n of X is the smallest subfunctor of X which contains all n-simplices of X (including degenerate ones, so it also contains all m-simplices with $m < n$).

Proposition 1.1.3. *Let $n < \infty$ be a positive integer. Every n-quasi-category X is determined by its $(n+1)$-skeleton. An n-skeleton Y_{n+1} is the $(n+1)$-skeleton of an n-quasi-groupoid (n-quasi-category) if and only if it satisfies the (resp. inner) horn-filling conditions up to dimension $n+1$ and also has the property that every (resp. inner) $(n+2)$-horn can be completed to an $(n+2)$-shell.*

Proof. For the first part, we prove that given two n-quasi-categories X and Y and an isomorphism $X_{n+1} \to Y_{n+1}$, the isomorphism can be extended to $X_{n+2} \to Y_{n+2}$; this will establish the result by induction, since n-quasi-categories are m-quasi-categories for each $m \geq n$. To wit, given an $(n+2)$-simplex x in X, consider the Λ_1^{n+2} part h of the $(n+2)$-simplex x. The inner $(n+2)$-horn h (because it is comprised of $(n+1)$-simplices in X) maps to an inner $(n+2)$-horn in Y under the given isomorphism; this horn has a unique filler y in Y, and we declare this y to be the image of x.

To see that this respects the simplicial set structure, we must check that the remaining face $\partial_1 x$ maps to $\partial_1 y$ under the given isomorphism. But the boundary of $\partial_1 x$ is contained in h, and so the boundary of $\partial_1 y$ is the image of the boundary of $\partial_1 x$. Since $(n+1)$-shells can be filled in at most one way in both X and Y, this establishes that $\partial_1 x$ must map to $\partial_1 y$.

Since this map is in fact canonically defined, we may also produce a map $Y_{n+2} \to X_{n+2}$, and these two maps must be inverse to one another.

We first do the case of an n-quasi-category. The only if direction is immediate, so let X_{n+1} be an $(n+1)$-skeleton satisfying the conditions in the proposition. We show by induction on m that there is an extension of X_{n+1} to an m-skeleton X_m ($m > n$) with the inner horn-filling conditions holding uniquely in dimensions $n+1$ through m, and also such that every inner $(m+1)$-horn can be completed to an $(m+1)$-shell.

The base is clear, so assume we have the result for $m-1$. We adjoin to X_{m-1} one m-simplex for every m-shell (with the obvious boundary maps); call this m-skeleton X_m. I claim that X_m has the property that each inner m-horn has a unique filler. Indeed, given an inner m-horn h, it can be completed to a shell in X_{m-1}, whence has at least one filler in X_m. But any two such fillers would agree on the boundary of the face omitted in h (since that boundary is contained in h) and so by the unique horn-filling condition in dimension $m-1$, those two fillers must have the same shell, whence must be the same. Finally, given an inner $(m+1)$-horn in X_m, the boundary of the omitted face is an m-shell, and thus is fillable by the tautological m-simplex, so that the inner $(m+1)$-horn we started with can in fact be completed to an $(m+1)$-shell as we wanted.

The n-quasi-groupoid case follows by the same argument; just omit all occurrences of the word "inner." □

This proposition amounts to saying that to define an n-quasi-category one need only give data up to dimension $n+1$ (objects, morphisms, and "weak composition laws") and have this data satisfy an axiom in dimension $n+2$ ("associativity"). That this characterization is not more complicated is an advantage of this system as compared with other frameworks for higher category theory; we of course are taking advantage of the fact that all our higher morphisms will be invertible.

We further note a corollary of the proof:

Corollary 1.1.4. *If X is an n-quasi-category and $\Sigma \hookrightarrow \Theta$ is a monomorphism of simplicial sets which is an isomorphism on $(n+1)$-skeleta, any map $\Sigma \to X$ can be extended uniquely to Θ.*

Proof. Immediate. □

1.1.3 Categories as Quasi-categories

Recall that given a small ordinary category $\mathcal{C}$ we may produce a simplicial set $N(\mathcal{C})$ (the nerve of $\mathcal{C}$) whose 0-simplices are the objects of $\mathcal{C}$, and whose n-simplices for $n > 0$ are composable sequences of morphisms

$$x_0 \to x_1 \to x_2 \to \cdots \to x_n.$$

The k^{th} boundary map is given by eliminating the object x_k from a sequence as above, either by composing the two incident morphisms or omitting the one incident morphism. The k^{th} degeneracy is given by inserting an identity morphism at the k^{th} place. This assignment of sets and functions is functorial because of the associativity of morphism composition.

It is not hard to see from the proof of the last proposition that the simplicial set $N(\mathcal{C})$ is actually the 1-quasi-category associated to the 2-skeleton where the 0-simplices are the objects of $\mathcal{C}$, the 1-simplices are the morphisms, and the 2-simplices are commutative triangles. This satisfies the condition from the proposition because e.g. a Λ^3_1 has edges f, g, h, gf, hg, $(hg)f$ (where f, g, and h are morphisms); the missing face in the shell has edges gf, h, and $(hg)f$, and so that it is fillable says that morphism composition in $\mathcal{C}$ is associative (similarly for Λ^3_2). The fact that $N(\mathcal{C})$ is just this 1-quasi-category follows upon noticing that in the nerve, every compatible configuration of the 2-skeleton of an m-simplices is the 2-skeleton of an m-simplex in the nerve.

Conversely, the same argument in reverse produces a category from a 1-quasi-category. The 0- and 1-simplices give the objects and morphisms with source, target, and identity maps. The 2-simplices define composition, in that a composable pair of morphisms is nothing more than a Λ^2_1, and so has a unique filler, defining a composite. That the identities respect composition follows from basic facts about degeneracy maps, namely that a degenerate 2-simplex consists of two identical 1-simplices (sharing a source or a target) and a degenerate 1-simplex between them.

Summarizing this discussion, we have the following corollary.

Corollary 1.1.5. *A simplicial set X is isomorphic to some $N(\mathcal{C})$ where $\mathcal{C}$ is a small category if and only if X is a 1-quasi-category.*

An analogous statement is true for groupoids and 1-quasi-groupoids.

Corollary 1.1.6. *A simplicial set X is isomorphic to one of the form $N(\mathcal{G})$ where $\mathcal{G}$ is a small groupoid if and only if X is a 1-quasi-groupoid.*

Proof. First suppose that $X = N(\mathcal{G})$, where $\mathcal{G}$ is a groupoid. By the preceding corollary, X is a 1-quasi-category, so by the proposition we need only verify that non-inner horns are fillable in dimensions 2, and that non-inner 3-horns can be completed to shells. Suppose we are given a Λ^2_2 whose zeroth face is f and whose first face is g; then this is filled by making the second face $f^{-1}g$. Similar remarks apply for a Λ^2_0. Now suppose we are given a Λ^3_3; we will for simplicity let f_{ij} $(i < j)$ denote the morphism in this horn from i to j. We wish to show that $f_{12}f_{01} = f_{02}$. But $f_{13}f_{01} = f_{03}$, $f_{23}f_{12} = f_{13}$, and $f_{23}f_{02} = f_{03}$, so

$$f_{12}f_{01} = \left(f_{23}^{-1}f_{13}\right)\left(f_{13}^{-1}f_{03}\right) = f_{23}^{-1}f_{03} = f_{02}.$$

Similar remarks apply to a Λ^3_0.

Now suppose that X is a 1-quasi-groupoid. X is a 1-quasi-category, and so is of the form $N(\mathcal{C})$ for some category $\mathcal{C}$. I claim that $\mathcal{C}$ is a groupoid. Indeed, if $f : x \to y$ is any morphism in $\mathcal{C}$, then we can produce a (non-inner) 2-horn which has f as its zeroth face and id_y as its first face; the filler will give a left inverse. Similar remarks produce a right inverse, so we are done. □

We also clearly have that $N(\mathcal{C}^{\mathrm{op}}) = N(\mathcal{C})^{\mathrm{op}}$.

1.2 Some Composable Shapes in a Quasi-Category

In an ordinary category, composition is a relatively simple business, as it only happens in one dimension (at the morphism level). In higher categories, the higher dimensional morphisms can be composed as well, and so it will be useful for us to have a cornucopia of shapes which can be composed. We first produce a small class of shapes which are useful for all quasi-categories, then a much broader class which is useful for 2-quasi-categories.

To accompany these shapes, we will occasionally utilize some notation. In the m-simplex Δ^m, whose vertices are labelled 0 through m, we will use $(a_0 a_1 \cdots a_r)$ for $0 \le a_0 < \cdots < a_r \le m$ to denote the r-simplex in Δ^m with vertices $a_0, \ldots, a_r$. For the purpose of the following definition, we set $Faces(\alpha)$ for a simplex $\alpha \in \Delta^m$ to be the set of indices i for which α is contained in the i^{th} face of Δ^m.

Definition 1.2.1. Let $m > 0$ be an integer, $S \subseteq [m]$ a subset. Define the *partial horn* Λ^m_S to be the simplicial set which is Δ^m minus every simplex α for which $Faces(\alpha) \subseteq S$. Equivalently,

- it is the subfunctor of Δ^m consisting of all simplices which do not include all of $[m] - S$, or
- it consists of the faces of Δ^m whose indices are in $[m] - S$, or
- it is Δ^m minus all simplices whose boundary contains (or which is) the $(m-|S|)$-simplex corresponding to $[m] - S$.

Lemma 1.2.2. *Let $m > 0$ be an integer and $S \subsetneq [m]$ a subset. Then:*

(i) If $[m] - S$ is not a string of consecutive integers, then Λ^m_S can be filled out to Δ^m by a sequence of inner horn fillings where the horns all have dimension greater than $m - |S|$.

(ii) If $[m] - S$ is not a string of consecutive integers containing 0, then Λ^m_S can be filled out to Δ^m by a sequence of left horn fillings where the horns all have dimension greater than $m - |S|$.

(iii) If $[m] - S$ is not a string of consecutive integers containing m, then Λ^m_S can be filled out to Δ^m by a sequence of right horn fillings where the horns all have dimension greater than $m - |S|$.

(iv) If S is arbitrary then Λ^m_S can be filled out to Δ^m by a sequence of arbitrary horn fillings.

Proof. Notice that all statements are vacuously true if $S = \emptyset$. We thus assume henceforth that $|S| > 0$.

For part (i), we induct on $|S|$. For $|S| = 1$, the statement follows from the definition of inner horn. Suppose $|S| > 1$ and we've proven the statement for $|S| - 1$; let $a \in S$ be such that $a = 0$ or $a = m$ if 0 or m is in S; otherwise a can be anything. The set $S' = S - \{a\}$ then satisfies the hypotheses both as a subset of the ordinal $[m] - \{a\}$ and as a subset of $[m]$.

Consider simplices in Δ^m which do not include all of $[m] - S'$ but which do include all of $[m] - S$ (i.e., simplices we must fill to reduce to the inductive hypothesis). These are precisely the simplices which do not contain a but which contain all of $[m] - S$, and in fact these are the simplices missing in $\Lambda^{m-1}_{S'}$, where here we consider S' to be a set of 0-simplices in the $(m-1)$-simplex $\partial_a \Delta^m$. Since this last is dimension $m - 1$, $|S'| = |S| - 1$ and $(m-1) - |S'| = m - |S|$, the inductive hypothesis gives us a filling to our liking.

We conclude that we can partially fill in the original Λ^m_S to a $\Lambda^m_{S'}$, where S' is a set of size $|S| - 1$ in $[m]$. Applying the inductive hypothesis a second time, we get a filling over all of Δ^m (by simplices of dimension greater than $m - |S| + 1$, so we're still fine).

For part (ii), notice that as inner horns are right horns it is enough to treat the case where $[m] - S$ is a string of consecutive integers which does not contain 0. Again, we induct on $|S|$. If $|S| = 1$, the horn $\Lambda^m_S = \Lambda^m_0$ and is itself a left m-horn. Assume the statement proven for all smaller sets than S. Let a be the greatest element of S, so that as $|S| > 1$, $a > 0$ and hence $S' = S - \{a\}$ satisfies the hypotheses both as a subset of $[m]$ and as a subset of $[m] - \{a\}$.

Proceeding as before, the difference between Λ^m_S and $\Lambda^m_{S'}$ consists of the simplices missing in $\Lambda^{m-1}_{S'}$, but this last can be filled by the induction hypothesis, so that we fill Λ^m_S to $\Lambda^m_{S'}$ and fill this last to Δ^m, again by the induction hypothesis.

Part (iii) is argued similarly to part (ii).

For part (iv), since S is nonempty, the complement $[m] - S$ will either be nonconsecutive or consist of a consecutive string of integers which cannot contain both 0 and m, so that either (ii) or (iii) apply. $\square$

Corollary 1.2.3. *Let X be a n-quasi-category, and let $S \subsetneq [m]$ be a subset such that $[m] - S$ is not a sequence of consecutive integers. Then any morphism $\Lambda^m_S \to X$ can be filled to all of Δ^m, and uniquely if $m - |S| \geq n$. Moreover, if X is an n-quasi-groupoid, then the statement is true for arbitrary proper subsets S of $[m]$.*

Proof. This follows immediately from Lemma 1.2.2. □

This corollary also of course has a strengthened version (following Corollary 1.1.4):

Corollary 1.2.4. *Let X be an n-quasi-category, and $(\Lambda^m_S)_{n+1}$ the $(n+1)$-skeleton of Λ^m_S (where here again $[m] - S$ is not a string of consecutive integers). Then a morphism $(\Lambda^m_S)_{n+1} \to X$ may be extended to Δ^m, uniquely if $m - |S| \geq n$.*

Proof. Any $(n+2)$-simplex in Λ^m_S will have all of its boundary $(n+1)$-simplices in $(\Lambda^m_S)_{n+1}$, and so we may extend the given morphism uniquely to $(\Lambda^m_S)_{n+2}$ by Corollary 1.1.4. Inducting, we fill out the morphism uniquely to Λ^m_S, and then finish the proof with Lemma 1.2.2. □

In the next two lemmas, we will be considering subcomplexes of a product of simplices such as $\Delta^r \times \Delta^m$, and will need to identify specific simplices. A j-simplex $\Delta^j \to \Delta^r \times \Delta^m$ consists of an order-preserving map $[j] \to [r]$ and an order-preserving map $[j] \to [m]$.

Lemma 1.2.5. *Let m and r be nonnegative integers. There is a sequence of inner horn fillings in $\Delta^m \times \Delta^r$ which will fill out the simplicial complex $\partial\Delta^m \times \Delta^r \cup \Delta^m \times \partial\Delta^r$ to all of $\Delta^r \times \Delta^m$ except for σ, where σ is the $(m+r)$-simplex of corresponding to the unique order-preserving maps $s : [m+r] \to [m]$ and $t : [m+r] \to [r]$ with $s(i) = i$ for $i \leq m$ and $t(i) = i - m$ for $i \geq m$.*

Proof. As remarked above, a j-simplex of $\Delta^m \times \Delta^r$ is just a pair (s,t) where $s : [j] \to [m-1]$ and $t : [j] \to [r]$. We only should consider nondegenerate j-simplices. In general, s and t might be degenerate as simplices in Δ^m and Δ^r, as long as they are not degenerate at the same index of $[j]$ (in which case they would be degenerate in $\Delta^m \times \Delta^r$). In a nondegenerate j-simplex, therefore, we see that for each i either $s(i) < s(i+1)$ or $t(i) < t(i+1)$, and so j is at most $m + r$ (as we would expect).

Notice moreover that a simplex (s,t) is in the boundary $\partial\Delta^m \times \Delta^r \cup \Delta^m \times \partial\Delta^r$ of $\Delta^m \times \Delta^r$ if and only if s or t is not surjective. We are assuming that the boundary (and only the boundary) has been filled in by the induction hypothesis, and so the simplices (s,t) we wish to fill are precisely those which are nondegenerate with s and t surjective; call these *interior simplices*, and the others *boundary simplices.*

Let (s,t) be a nondegenerate interior j-simplex, and let i be such that $0 < i < j$. We call i a *pivot* if $s(i-1) = s(i)$ and $t(i) = t(i+1)$ (so that by surjectivity and nondegeneracy $s(i+1) = s(i) + 1$ and $t(i) = t(i-1) + 1$); if the roles of s and t are reversed, call i an *antipivot.* It is immediate that $\partial_i(s,t)$ is an interior $(j-1)$-simplex if and only if i is a pivot or an antipivot. Moreover, if (s,t) has a pivot i (and we think of s and t as j-tuples), then it looks like:

$$\begin{array}{c|ccccc} s & \cdots & s(i) & s(i) & s(i)+1 & \cdots \\ t & \cdots & t(i)-1 & t(i) & t(i) & \cdots \end{array}$$

and so there is a unique j-simplex sharing the i^{th} face of (s,t), namely the following one:

$$\begin{array}{c|ccccc} s & \cdots & s(i) & s(i)+1 & s(i)+1 & \cdots \\ t & \cdots & t(i)-1 & t(i)-1 & t(i) & \cdots \end{array}.$$

We refer to this process as *moving along a pivot,* and analogously define it for antipivots.

Let (s,t) be a nondegenerate $(m+r)$-simplex (a nondegenerate simplex of maximal dimension in $\Delta^m \times \Delta^r$, necessarily interior). We know that for each i with $0 \leq i < m+r$, either $s(i) = s(i+1)$ and $t(i)+1 = t(i+1)$ or vice versa (by the maximality of dimension combined with nondegeneracy). Therefore, since both s and t must increase at some point, there is only one $(m+r)$-simplex without a pivot, namely the following one, described in the statement of the lemma as the unique simplex we will not fill, and which henceforth we call τ:

$$\begin{array}{c|cccccccccc} s & 0 & 1 & 2 & \cdots & m-1 & m & m & \cdots & m \\ t & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & r \end{array}.$$

For an $(m+r)$-simplex (s,t), define the *level* of (s,t) to be

$$\left(\sum_{i=0}^{m+r} s(i)\right) - \frac{m(m-1)}{2}.$$

We see immediately that levels range from 0 to rm, and moving along a pivot (antipivot) increases (decreases) the level by 1. Moreover, τ has level rm, and in fact is the only simplex with that level. Furthermore, there is only one simplex β of level 0, namely

$$\begin{array}{c|cccccccccc} s & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & m \\ t & 0 & 1 & 2 & \cdots & r-1 & r & r & \cdots & r \end{array};$$

we observe that β has one pivot and no antipivots.

By induction on $\ell < rm$, we fill in all nondegenerate $(m+r)$-simplices at levels $\leq \ell$. I claim this will fill in all of $\Delta^m \times \Delta^r$ except for τ. Indeed, the only interior face of τ is the one it shares with the unique $(m+r)$-simplex at level $rm-1$; thus all faces of τ will be filled. Since every nondegenerate j-simplex in $\Delta^m \times \Delta^r$, $j < m+r$, is contained in the boundary of some nondegenerate $(m+r)$-simplex, we conclude that τ will be the only simplex missing from the filling.

For the base case $\ell = 0$, we need only fill in β. But r is the only pivot of β, so its only interior face is $\partial_r\beta$; the rest is boundary, and so has been filled. Applying the inner horn-filling axiom on Y, we can fill in β (and its interior face).

Now suppose we have filled in all $(m+r)$-simplices at levels $< \ell$, and let $\alpha = (s,t)$ be at level ℓ. Consider the i^{th} face of α. If i is an antipivot, then $\partial_i\alpha$ has been filled in (moving along i gives an $(m+r)$-simplex of level $\ell - 1$ whose i^{th} face is $\partial_i\alpha$). If i is neither a pivot nor an antipivot, then $\partial_i\alpha$ is a boundary simplex, and so is filled in. Therefore, any unfilled faces must be at the pivots of α.

Let $i_1 < i_2 < \cdots < i_j$ be the pivots of α; we know that $j \geq 1$ because $\ell < rm$. Notice that for any $p < j$, we actually have $i_{p+1} \geq i_p + 2$; this is just because if i_p is a pivot, then $s(i_p) \neq s(i_p+1)$, and so $i_p + 1$ cannot be a pivot. But this means that any other $(m+r)$-simplex agreeing with α away from $i_1, \ldots, i_j$ must have a greater level: since $s(i_p - 1) = s(i_p)$ and $s(i_p) + 1 = s(i_p + 1)$, we can only put $s(i_p)$ or $s(i_p) + 1$ at the i_p^{th} place, thereby increasing the level altogether. We conclude that any simplex (of any dimension) in α which contains all non-pivots of α is unfilled, as such simplices are interior and not contained in any $(m+r)$-simplex of level $\leq \ell$ except for α.

Applying Lemma 1.2.2, we find a filling of α by inner horns (as the non-pivots include 0 and $m+r$). The induction is complete. □

Lemma 1.2.6. *Let m be a positive integer, k an integer with $0 \leq k \leq m$, and r a nonnegative integer. Then there is a filling of the subcomplex*

$$\Delta^m \times \partial\Delta^r \cup \Lambda^m_k \times \Delta^r$$

of $\Delta^m \times \Delta^r$ to all of $\Delta^m \times \Delta^r$ where all fillings but (possibly) the last are inner horn fillings, and where the last is a filling of a Λ^{m+r}_k, unless $k = m$ in which case the last horn filling is a filling of a Λ^{m+r}_{m+r}.

Proof. To begin with, assume $k < m$. Notice that the intersection of the shape we are given with $\partial_k\Delta^m \times \Delta^r$ is a $\partial\Delta^{m-1} \times \Delta^r \cup \Delta^{m-1} \cup \partial\Delta^r$, and so Lemma 1.2.5 applies. We are left with one unfilled simplex which we again name τ.

Note that (in the terminology of the proof of Lemma 1.2.5) the simplex τ is a boundary simplex in $\Delta^m \times \Delta^r$, in fact the only boundary simplex not now filled. The boundary simplex τ is not, however, contained in any interior $(m+r)$-simplex in $\Delta^m \times \Delta^r$ except the one of maximal level (again using the terminology from before). This follows upon noticing that there is only one place to insert k into the values of $s : [m + r - 1] \to [m]$ (this is because $k < m$; otherwise, the unique such $(m + r)$-simplex would not be maximal).

But then (since missing τ will be no obstruction) we may apply Lemma 1.2.5 again to conclude that we may fill all simplices of $\Delta^m \times \Delta^r$ except the maximal interior $(m + r)$-simplex. Finally, we observe that this last missing simplex together with τ make for a Λ^{m+r}_k, which of course is what we wanted.

If $k = m$, we may apply the op functor to produce a situation where $k = 0$; applying op to go back, we see that the last horn to fill is a Λ^{m+r}_{m+r}. $\square$

Thus we obtain a number of shapes which are fillable in any quasi-category.

Remark 1.2.7. The preceding proposition is probably more properly subsumed by a theory of *anodyne extensions* (cf. e.g. [GJ99] I.4) suitably adapted to this context.

The following shapes will only really assist us in working with 2-quasi-categories.

Lemma 1.2.8. *Let $\Pi \subseteq \Delta^n$ be a simplicial complex obtained from a single vertex by successive (not necessarily inner) horn fillings of horns of dimension 1 or higher. Then a sequence of horn fillings will enlarge Π to contain the 1-skeleton of Δ^n.*

Proof. Clearly it is enough to produce a horn filling which adds at least one missing simplex in the 1-skeleton of Δ^n. If one of the vertices of Δ^n is missing, we may fill any 1-horn whose missing vertex is this one and whose given vertex is any vertex in Π (such exists since Π is nonempty). If all the vertices have been filled, choose a missing 1-simplex (ab) of Δ^n such that the shortest sequence of 1-simplices travelling from a to b is of minimal length amongst all missing 1-simplices; choose in addition such a minimal path. I claim that this minimal path has exactly two 1-simplices, making it a 2-horn, and fillable to complete the proof. Indeed, if it were longer then any proper subpath of length at least 2 would also have to be minimal amongst all paths from its head to its target, and so it would be a minimal path shorter than the given one. Thus, we have what we want. $\square$

Proposition 1.2.9. *Let X be a 2-quasi-category, $m > 0$ an integer, $\Pi \subseteq \Delta^m$ a subcomplex, and $\Pi \to X$ a diagram with shape Π. Suppose moreover that Π has the property that it can be assembled out of successive fillings of inner horns, starting with the string of 1-simplices which moves consecutively from 0 to m in Δ^m. Then the morphism $\Pi \to X$ can be extended to Δ^m, and uniquely if Π contains all 1-simplices of Δ^m.*

Proof. We will show that Π can be filled by means of inner horn filling to the 3-skeleton of Π, which will obtain what we want by Corollary 1.1.4.

Let us prove this by induction on m. The case $m = 1$ is immediate. For an $m > 1$, we may then assume that the 0^{th} and m^{th} faces of Δ^m have intersection with Π which contains the 3-skeleta of these faces, as the intersections of these $(m-1)$-dimensional simplices with Π satisfy the same hypotheses and being in the filling of an inner horn in one of these faces is the same thing as being in the filling of an inner horn in all of Δ^m.

Notice that by assumption (and by the previous paragraph) all of the inner horn filling that we must do concerns horns of dimension 2 or higher which contain the vertices 0 and m, and moreover all outer faces of such horns have been filled. Define a subcomplex $\Pi' \subseteq \Delta^{m-2}$ to have as k-simplices all the $k + 2$-simplices of Π which contain 0 and m (this is a subcomplex of Δ^{m-2} as one can associate to a simplex σ of Π' the simplex of Δ^{m-2} whose vertices are those of σ minus 0 and m, and shifted down 1).

I claim that Π' has the property that a filling of an arbitrary horn in Π' is the same thing as filling an inner horn in Π. This is because horns in Π' give rise to inner horns in Π by adding the vertices 0 and m (and all incident simplices, which are present in Π by the previous paragraph); the reverse procedure is valid since we have filled all inner horns in Π except these. Notice that if the 1-simplex $(0m)$ is not present, we may fill say $(01m)$ to put it there, and then Π' will be obtained by a sequence of horn fillings off a single vertex.

Therefore a filling of Π' to the 1-skeleton by arbitrary horns gives rise to a filling of Π by inner horns. Since the former exists by Proposition 1.2.8, we have the filling to the 3-skeleton, hence the filling in our

2-quasi-category. The uniqueness statement is clear as all we can ever fill in this case are inner horns of dimension 3 or higher. ☐

1.3 Comparing (2,1)-Categories and 2-Quasi-Categories

We may also consider (weak) (2,1)-categories (defined below; a special case of weak 2-categories) to be quasi-categories in a natural way.

Definition 1.3.1. A *(2,1)-category* $\mathcal{C}$ consists of the data of a set of objects $|\mathcal{C}|$ and for every pair of objects $X, Y \in |\mathcal{C}|$ a groupoid $\mathrm{Hom}(X,Y)$ together with an "identity" functor for each X

$$1 \to \mathrm{Hom}(X,X),$$

the image being called 1_X, and for each $X, Y, Z \in |\mathcal{C}|$ composition functors

$$\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$$

which take (g,f) to something we will denote (gf). We also have the data of natural isomorphisms ("left and right unitors" and "associators") which relate the following functors respectively (where $f \in \mathrm{Hom}(X,Y)$, $g \in \mathrm{Hom}(Y,Z)$, and $h \in \mathrm{Hom}(Z,W)$):

$$\begin{aligned} [f \mapsto (1_Y f)] &\quad\Rightarrow\quad [(f \mapsto f)] \\ [f \mapsto (f 1_X)] &\quad\Rightarrow\quad [(f \mapsto f)] \\ [(h,g,f) \mapsto (h(gf))] &\quad\Rightarrow\quad [(h,g,f) \mapsto ((hg)f)] \end{aligned}$$

Abbreviating notation a bit, these natural isomorphisms are required to satisfy the "triangle identity:"

$$(g(1_Y f)) \Rightarrow (gf) = (g(1_Y f)) \Rightarrow ((g1_Y)f) \Rightarrow (gf)$$

as well as the "Stasheff pentagon identity:"

$$(i(h(gf))) \Rightarrow (i((hg)f)) \Rightarrow ((i(hg))f) \Rightarrow (((ih)g)f) = (i(h(gf))) \Rightarrow ((ih)(gf)) \Rightarrow (((ih)g)f).$$

This definition differs from the usual notion of weak 2-category (cf. e.g. [Lei02]) only insofar as the Hom-categories here are actually groupoids. It should also be noted that a strict 2-category with all 2-morphisms isomorphisms is a special case of a (2,1)-category; in a strict 2-category, the unitors and associators are just identities, and so the triangle identity and the Stasheff pentagon are satisfied automatically. Moreover, every weak 2-category is in fact weakly equivalent (in a sense to be discussed below) to a strict 2-category.

Notice that in a $(2,1)$-category there are two other "triangle identities" which are always satisfied:

Lemma 1.3.2. *Let $f : X \to Y$ and $g : Y \to Z$ be composable 1-morphisms in a (2,1)-category. The following two identities then always hold (where all 2-morphisms are obtained from associators and unitors in the obvious way):*

$$\begin{aligned} ((gf)1_X) \Rightarrow (gf) &\quad=\quad ((gf)1_X) \Rightarrow (g(f1_X)) \Rightarrow (gf) \\ (1_Z(gf)) \Rightarrow (gf) &\quad=\quad (1_Z(gf)) \Rightarrow ((1_Z g)f) \Rightarrow (gf). \end{aligned}$$

Proof. To see why this is so, we will compute the first identity (the second is dual). We abbreviate 1_X to simply "1." By naturality of right unitors, it is enough to check that

$$(((gf)1)1) \Rightarrow ((gf)1) = (((gf)1)1) \Rightarrow ((g(f1))1) \Rightarrow ((gf)1).$$

But the triangle identity gives us:

$$(((gf)1)1) \Rightarrow ((gf)1) = (((gf)1)1) \Rightarrow ((gf)(11)) \Rightarrow ((gf)1),$$

so that by naturality of the associator we can conclude:

$$(((gf)1)1) \Rightarrow ((gf)1) = (((gf)1)1) \Rightarrow ((gf)(11)) \Rightarrow (g(f(11))) \Rightarrow (g(f1)) \Rightarrow ((gf)1).$$

Now, applying the triangle identity again, the above expression

$$= (((gf)1)1) \Rightarrow ((gf)(11)) \Rightarrow (g(f(11))) \Rightarrow (g((f1)1)) \Rightarrow (g(f1)) \Rightarrow ((gf)1).$$

By the Stasheff pentagon, this is

$$= (((gf)1)1) \Rightarrow ((g(f1))1) \Rightarrow (g((f1)1) \Rightarrow (g(f1)) \Rightarrow ((gf)1).$$

Finally, by naturality of the associator again, this

$$= ((gf)1)1 \Rightarrow ((g(f1))1) \Rightarrow ((gf)1).$$

which of course is what we wanted. □

1.3.1 From (2,1)-Categories to 2-Quasi-Categories

Let $\mathcal{C}$ be a $(2,1)$-category. We construct from $\mathcal{C}$ a 2-quasicategory $N(\mathcal{C})$ by means of the method which follows.

The 0-simplices of $N(\mathcal{C})$ are the objects $|\mathcal{C}|$ of $\mathcal{C}$; the 1-simplices are the objects of the groupoids $\mathrm{Hom}(X,Y)$ (letting X and Y vary) with the obvious boundary maps. The 2-simplices are (iso)morphisms $(gf) \Rightarrow h$, and the 3-simplices are tetrahedra with edges f_{ij}, $0 \le i < j \le 3$ and faces from the 2-simplices so that

$$(f_{23}(f_{12}f_{01})) \Rightarrow ((f_{23}f_{12})f_{01}) \Rightarrow (f_{13}f_{01}) \Rightarrow f_{03} = (f_{23}(f_{12}f_{01})) \Rightarrow (f_{23}f_{02}) \Rightarrow f_{03}.$$

Again, boundary maps are clear. Degeneracies of 0-simplices are given by the identity functors. Degeneracies of 1-simplices are given by the left and right unitors. Degeneracies of 2-simplices are given by the (axiomatic and proven) triangle identities in the sense that for a 2-simplex $(gf) \Rightarrow h$ whose vertices are X, Y, Z, the following equations are satisfied:

$$\begin{aligned}
(g(f1_X)) \Rightarrow (gf) \Rightarrow h &= (g(f1_X)) \Rightarrow ((gf)1_X) \Rightarrow (gf) \Rightarrow h \\
(g(1_Y f)) \Rightarrow (gf) \Rightarrow h &= (g(1_Y f)) \Rightarrow ((g1_Y)f) \Rightarrow (gf) \Rightarrow h \\
(1_Z(gf)) \Rightarrow (gf) \Rightarrow h &= (1_Z(gf)) \Rightarrow ((1_Z g)f) \Rightarrow (gf) \Rightarrow h;
\end{aligned}$$

each of these then guarantees a (unique) 3-simplex with the required boundary maps for a corresponding degeneracy of the original 2-simplex.

Given this construction, we can state a proposition.

Proposition 1.3.3. *The data for $N(\mathcal{C})$ described above characterizes a 2-quasicategory (which we also denote $N(\mathcal{C})$).*

Proof. We need to check that this data characterizes a 2-quasicategory. To see this, first note that the inner horn-filling conditions hold for 2-horns, as every two composable 1-morphisms can be, well, composed. The inner horn-filling conditions for 3-horns hold because given (say) the first 3-horn of a tetrahedron with edges labelled as above, we may fill it with the isomorphism

$$(f_{23}f_{02}) \Rightarrow (f_{23}(f_{12}f_{01})) \Rightarrow ((f_{23}f_{12})f_{01}) \Rightarrow (f_{13}f_{01}) \Rightarrow f_{03},$$

and similarly for the second 3-horns.

Finally, consider the condition that inner 4-horns should be completable to 4-shells. Fix the 2-skeleton of a 4-simplex (all 4-horns will contain the 2-skeleton of the desired 4-shell); we enumerate its 0-simplices

x_i, $0 \leq i \leq 4$, its 1-simplices f_{ij}, $0 \leq i < j \leq 4$, and note that as this is a 4-horn, we also have for each $i < j < k$ a given morphism $(f_{jk}f_{ij}) \Rightarrow f_{ik}$.

The fact of a face of this 4-simplex being (uniquely) fillable is expressed as an equation (we have defined it so). We thus have five equations, corresponding to the faces (in order from source to target):

$$\begin{aligned}
\Big[(f_{23}(f_{12}f_{01})) \Rightarrow ((f_{23}f_{12})f_{01}) \Rightarrow (f_{13}f_{01}) \Rightarrow f_{03}\Big] &= \Big[(f_{23}(f_{12}f_{01})) \Rightarrow (f_{23}f_{02}) \Rightarrow f_{03}\Big] \\
\Big[(f_{24}(f_{12}f_{01})) \Rightarrow ((f_{24}f_{12})f_{01}) \Rightarrow (f_{14}f_{01}) \Rightarrow f_{04}\Big] &= \Big[(f_{24}(f_{12}f_{01})) \Rightarrow (f_{24}f_{02}) \Rightarrow f_{04}\Big] \\
\Big[(f_{34}(f_{13}f_{01})) \Rightarrow ((f_{34}f_{13})f_{01}) \Rightarrow (f_{14}f_{01}) \Rightarrow f_{04}\Big] &= \Big[(f_{34}(f_{13}f_{01})) \Rightarrow (f_{34}f_{03}) \Rightarrow f_{04}\Big] \\
\Big[(f_{34}(f_{23}f_{02})) \Rightarrow ((f_{34}f_{23})f_{02}) \Rightarrow (f_{24}f_{02}) \Rightarrow f_{04}\Big] &= \Big[(f_{34}(f_{23}f_{02})) \Rightarrow (f_{34}f_{03}) \Rightarrow f_{04}\Big] \\
\Big[(f_{34}(f_{23}f_{12})) \Rightarrow ((f_{34}f_{23})f_{12}) \Rightarrow (f_{24}f_{12}) \Rightarrow f_{14}\Big] &= \Big[(f_{34}(f_{23}f_{12})) \Rightarrow (f_{34}f_{13}) \Rightarrow f_{14}\Big]
\end{aligned}$$

We may as well assume that the source and target expressions hold, as this will happen no matter which inner 4-horn we look at.

Let us simplify notation a bit by writing $\alpha_{ijk\ell}$ for the associator isomorphism

$$(f_{k\ell}(f_{jk}f_{ij})) \Rightarrow ((f_{k\ell}f_{jk})f_{ij}),$$

writing β_{ijk} for the given morphism $(f_{jk}f_{ij}) \Rightarrow f_{ik}$. Moreover, given $g, h : x_i \to x_j$ and a morphism $\gamma : g \Rightarrow h$, γ^k shall denote the morphism obtained by acting on the left by f_{jk} or on the right by f_{ki}, whichever makes sense. The five equations above then become:

$$\begin{aligned}
\beta_{013}\beta^0_{123}\alpha_{0123} &= \beta_{023}\beta^3_{012} \\
\beta_{014}\beta^0_{124}\alpha_{0124} &= \beta_{024}\beta^4_{012} \\
\beta_{014}\beta^0_{134}\alpha_{0134} &= \beta_{034}\beta^4_{013} \\
\beta_{024}\beta^0_{234}\alpha_{0234} &= \beta_{034}\beta^4_{023} \\
\beta_{124}\beta^1_{234}\alpha_{1234} &= \beta_{134}\beta^4_{123}
\end{aligned}$$

The Stasheff pentagon will provide the relation amongst these that we desire. The Stasheff pentagon, however, contains the following three morphisms:

$$\begin{aligned}
(f_{34}(f_{23}(f_{12}f_{01}))) &\Rightarrow ((f_{34}f_{23})(f_{12}f_{01})) \\
((f_{34}f_{23})(f_{12}f_{01})) &\Rightarrow (((f_{34}f_{23})f_{12})f_{01}) \\
(f_{34}((f_{23}f_{12})f_{01})) &\Rightarrow ((f_{34}(f_{23}f_{12}))f_{01})
\end{aligned}$$

which do not occur in the α's and β's alone. To represent these, we need to use the naturality of the associator transformations. Using the usual relation for naturality, the first morphism above can be represented as

$$f_{34}(f_{23}(f_{12}f_{01})) \Rightarrow f_{34}(f_{23}f_{02}) \Rightarrow (f_{34}f_{23})f_{02} \Rightarrow (f_{34}f_{23})(f_{12}f_{01}).$$

The first two arrows here are just $\alpha_{0234}(\beta^3_{012})^4$; leave the third aside for a moment. The second arrow above can be represented (by naturality) as

$$(f_{34}f_{23})(f_{12}f_{01}) \Rightarrow f_{24}(f_{12}f_{01}) \Rightarrow (f_{24}f_{12})f_{01} \Rightarrow ((f_{34}f_{23})f_{12})f_{01};$$

here the second and third arrows are $[(\beta^1_{234})^0]^{-1}\alpha_{0124}$. Finally, notice that again by naturality

$$\Big[(f_{34}f_{23})f_{02} \Rightarrow (f_{34}f_{23})(f_{12}f_{01}) \Rightarrow f_{24}(f_{12}f_{01})\Big] = \Big[(f_{34}f_{23})f_{02} \Rightarrow f_{24}f_{02} \Rightarrow f_{24}(f_{12}f_{01})\Big],$$

so that in fact the composition of the first two Stasheff arrows above is

$$[(\beta^1_{234})^0]^{-1}\alpha_{0124}[\beta^4_{012}]^{-1}\beta^0_{234}\alpha_{0234}(\beta^3_{012})^4.$$

The third Stasheff arrow above can be handled similarly, as it is just

$$\Big[f_{34}((f_{23}f_{12})f_{01}) \Rightarrow f_{34}(f_{13}f_{01}) \Rightarrow (f_{34}f_{13})f_{01} \Rightarrow (f_{34}(f_{23}f_{12}))f_{01}\Big] = [(\beta^4_{123})^0]^{-1}\alpha_{0134}(\beta^0_{123})^4.$$

Putting this all together, we see that the Stasheff pentagon relation is "just"

$$\alpha^0_{1234}[(\beta^4_{123})^0]^{-1}\alpha_{0134}(\beta^0_{123})^4\alpha^4_{0123} = [(\beta^1_{234})^0]^{-1}\alpha_{0124}[\beta^4_{012}]^{-1}\beta^0_{234}\alpha_{0234}(\beta^3_{012})^4.$$

Rearranging this expression a bit, we can write it as

$$[(\beta^1_{234})\alpha_{1234}(\beta^4_{123})^{-1}]^0\alpha_{0134}[\beta^0_{123}\alpha_{0123}(\beta^3_{012})^{-1}]^4 = \alpha_{0124}[\beta^4_{012}]^{-1}\beta^0_{234}\alpha_{0234}.$$

Now, on the left-hand side, the two bracketed expressions can be replaced by equivalent expressions by means of the source and target 3-simplex expressions above (both of which hold no matter which inner 4-horn we are looking at), so we obtain

$$[(\beta_{124})^{-1}\beta_{134}]^0\alpha_{0134}[(\beta_{013})^{-1}\beta_{023}]^4 = \alpha_{0124}[\beta^4_{012}]^{-1}\beta^0_{234}\alpha_{0234}.$$

But this last expression can be written

$$[\beta_{024}\beta^4_{012}\alpha^{-1}_{0124}(\beta^0_{124})^{-1}\beta^{-1}_{014}][\beta_{014}\beta^0_{134}\alpha_{0134}\beta^{-1}_{034}(\beta^4_{013})^{-1}\beta^{-1}_{034}] = [\beta_{024}\beta^0_{234}\alpha_{0234}(\beta^4_{023})^{-1}\beta^{-1}_{034}].$$

Each bracketed expression here corresponds to an inner 3-simplex, in that it is null if and only the corresponding 3-shell can be filled. This last relation thus expresses that every inner 4-horn can be completed to a 4-shell, and we conclude that the data of $N(\mathcal{C})$ characterize a 2-quasicategory, as we wished to show. Alternatively, the above discussion can be understood in terms of the following diagram:

$$\begin{tikzcd}
& & & f_{04} & & & \\
\partial_2 & & \partial_3 & f_{24}f_{02} \arrow[u, Rightarrow] & \partial_1 & & \\
& & f_{24}(f_{12}f_{01}) \arrow[ur, Rightarrow] \arrow[dl, Rightarrow] & & (f_{34}f_{23})f_{02} \arrow[ul, Rightarrow] & & \\
f_{14}f_{01} \arrow[uuurrr, Rightarrow, bend left] & (f_{24}f_{12})f_{01} \arrow[l, Rightarrow] & & (f_{34}f_{23})(f_{12}f_{01}) \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] \arrow[dl, Rightarrow] & & f_{34}(f_{23}f_{02}) \arrow[ul, Rightarrow] \arrow[r, Rightarrow] & f_{34}f_{03} \arrow[uuulll, Rightarrow, bend right] \\
& (\partial_0)f_{01} & ((f_{34}f_{23})f_{12})f_{01} \arrow[ul, Rightarrow] & \circledS & f_{34}(f_{23}(f_{12}f_{01})) \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] \arrow[d, Rightarrow] & f_{34}(\partial_4) & \\
& & (f_{34}(f_{23}f_{12}))f_{01} \arrow[u, Rightarrow] \arrow[d, Rightarrow] & & f_{34}((f_{23}f_{12})f_{01}) \arrow[ll, Rightarrow] \arrow[d, Rightarrow] & & \\
& & (f_{34}f_{13})f_{01} \arrow[uuull, Rightarrow, bend left] & & f_{34}(f_{13}f_{01}) \arrow[ll, Rightarrow] \arrow[uuurr, Rightarrow, bend right] & &
\end{tikzcd}$$

Here ∂_i refers to the relation corresponding to the i^{th} boundary of a 4-simplex, and $\circledS$ refers to the Stasheff pentagon identity; the other squares always commute by naturality and functoriality, as above. $\square$

1.3.2 From 2-Quasi-Categories to (2,1)-Categories

For the other direction, suppose we are given a 2-quasicategory C. We will make (nonuniquely) a $(2,1)$-category $\widetilde{C}$ from this.

The objects of $\widetilde{C}$ are the 0-simplices C_0. Given two objects X and Y, we may define a category $\mathrm{Hom}(X,Y)$ whose objects are the 1-simplices with source and target X and Y respectively, and whose morphisms are the 2-simplices whose three vertices are X, Y, Y (in that order) and whose zeroth face is 1_Y.

We have a composition law on this collection of objects and morphisms because given two composable morphisms φ and ψ in $\mathrm{Hom}(X,Y)$, we may form a 3-horn whose filled faces are φ, ψ, and the degeneracy of Y; the unique filler to this horn gives the composition (call it "$(\psi\varphi)$") on the remaining side. Moreover, this composition law is associative; given composable φ, ψ, and χ, we may form a 4-horn whose four filled 3-simplices are the 3-simplex witnessing the composition of φ and ψ, the 3-simplex witnessing the composition of ψ and χ, the 3-simplex witnessing the composition of $(\psi\varphi)$ and χ, and the degenerate 3-simplex over Y. When this 4-horn is filled, the new 3-simplex has as faces $(\chi\psi)$, φ, $(\chi(\psi\varphi))$, and the degeneracy over Y; by uniqueness, then, $\chi(\psi\varphi)$ must be the composition of φ and $(\chi\psi)$. Identity morphisms are just the first degeneracies of 1-cells.

Notice that this construction of $\mathrm{Hom}(X,Y)$ would go through just as well if we made the morphisms in this category be 2-simplices with vertices X, X, Y and second face 1_X. Call this latter category $\mathrm{Hom}'(X,Y)$.

Lemma 1.3.4. *Let $\varphi : f \to g$ and $\psi : f \to g$ be morphisms (2-simplices) in $\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}'(X,Y)$ respectively. The following are equivalent:*

(i) There is a 3-simplex with faces φ, ψ, and the zeroth and first degeneracy of f.

(ii) There is a 3-simplex with faces φ, ψ, and the zeroth and first degeneracy of g.

Proof. Assume that the first condition holds. Consider the 4-horn with vertices X, X, Y, Y, Y and whose zeroth 3-simplex is the first degeneracy of φ, whose first 3-simplex is the second degeneracy of φ, whose third 3-simplex is the zeroth degeneracy of φ, and whose fourth 3-simplex is the 3-simplex we are assuming exists. Filling the horn, we obtain a 3-simplex as in the second condition.

By using a 4-horn with vertices X, X, X, Y, Y we may more or less reverse the roles of φ and ψ in the above argument, obtaining the other direction. □

Notice that for a given φ as in the lemma, there is exactly one ψ such that the conditions of the lemma hold (by uniqueness of 3-horn fillers). In the sequel, we shall refer to φ and ψ as *twin* to one another. Witnesses to twinness as in the first condition will be called simply "witnesses," if they are as in the second condition we will call them witnesses of the second type.

Proposition 1.3.5. *There is a canonical* <u>*isomorphism*</u> $\mathrm{Hom}(X,Y) \to \mathrm{Hom}'(X,Y)$ *which acts identically on objects and sends a morphism to its twin.*

Proof. We need only check that the operation described in the statement of the proposition is in fact a functor.

Identities are clearly sent to identities. Let $\varphi : f \to g$ and $\psi : g \to h$ be composable morphisms in $\mathrm{Hom}(X,Y)$. We wish to show that $(\psi\varphi)' = \psi'\varphi'$. We will form a $\Lambda^5_{1,4}$ whose vertices are X, X, X, Y, Y, Y. The zeroth face is the third degeneracy of the 3-simplex witnessing that φ and φ' are twins, and the fifth face is the zeroth degeneracy of this same 3-simplex. To produce the second and third faces we perform some auxiliary horn-fillings. To wit, consider the 4-horn whose vertices are X, X, Y, Y, Y and whose zeroth, first, second, and fourth faces are the second degeneracy of φ, the witness to the composition of φ and ψ, the witness to the twinness of ψ and ψ', and the zeroth degeneracy of φ; the unique filler of this horn will be the second face of our $\Lambda^5_{1,4}$. The third face is obtained analogously.

Now, this $\Lambda^5_{1,4}$ has a unique filler (because $5 - 2 > 2$). Let us look at the (only) filled 3-simplex: a quick check reveals that its faces (in order) are the first degeneracy of f, $\psi\varphi$, $\psi'\varphi'$, and the zeroth degeneracy of f. Thus this 3-simplex witnesses that $\psi\varphi$ and $\psi'\varphi'$ are twins, which is what we wanted. □

From now on we identify $\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}'(X,Y)$; when passing from morphisms in $\mathrm{Hom}(X,Y)$ to 2-simplices, we will say explicitly whether we are thinking of a morphism as living in $\mathrm{Hom}(X,Y)$ or $\mathrm{Hom}'(X,Y)$.

Corollary 1.3.6. *The category* $\mathrm{Hom}(X,Y)$ *is a groupoid.*

Proof. Given a morphism $\varphi : f \to g$, we first think of it as in $\mathrm{Hom}(X,Y)$, and we may form a 3-horn whose zeroth face is the degeneracy of Y, whose second face is the first degeneracy of f, and whose third is φ itself. Filling the horn, we find a morphism (in the sense of $\mathrm{Hom}(X,Y)$) which is a left inverse of φ. Repeating the discussion, this time thinking of φ as a morphism in $\mathrm{Hom}'(X,Y)$, we find that φ has a right inverse as well. $\square$

Now, once and for all, choose a subset $\widetilde{C_2}$ of C_2 with the property that every inner 2-horn (i.e. pair of composable 1-morphisms) is the inner 2-horn of exactly one 2-simplex in $\widetilde{C_2}$. This data will give us enough information to force a (2,1)-category.

For two composable 1-morphisms f and g, we will denote by (gf) ("the composition of f and g") the third side of the unique 2-simplex in $\widetilde{C_2}$ with f and g comprising its inner horn. If X is an object of $\widetilde{C}$, 1_X will just denote the (zeroth) degeneracy of X.

To define the composition law, our "composition" notation above gives us a map at the object level

$$\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z).$$

This in fact extends naturally to a map on morphisms. Given a morphism $\varphi : f \to f' \in \mathrm{Hom}(X,Y)$ and an object $g \in \mathrm{Hom}(Y,Z)$, we can produce naturally a morphism in $\mathrm{Hom}(X,Z)$, by considering the 3-horn whose faces are φ (considered in Hom') and the witnesses (in $\widetilde{C_2}$) to the compositions of (gf) and (gf'). The unique filler will give a morphism $(gf) \to (gf') \in \mathrm{Hom}(X,Z)$. Similarly, we may switch the roles of X and Z to obtain maps on morphisms in $\mathrm{Hom}(Y,Z)$.

Proposition 1.3.7. *There is a one-to-one correspondence between 2-simplices with a fixed boundary* (g,h,f) *(where* $g \in \mathrm{Hom}(Y,Z)$*,* $h \in \mathrm{Hom}(X,Z)$*, and* $f \in \mathrm{Hom}(X,Y)$*) and morphisms* $(gf) \to h$ *in* $\mathrm{Hom}(X,Z)$*.*

Proof. Suppose we are given a 2-simplex with boundary as in the proposition. Form a 3-horn whose vertices are X,Y,Z,Z and whose zeroth, second, and third faces are the first degeneracy of g, the given 2-simplex, and the witness to (gf) being the composition of f and g, respectively. Then, filling this (uniquely), we obtain a 2-simplex which is the same thing as a morphism $(gf) \to h$, as we've defined things.

Conversely, the same shape works; fill in the zeroth, first, and third faces, and the filled-in second gives the desired 2-simplex. $\square$

Notice that in particular this gives us left and right unitors from the first and zeroth degeneracies of a 1-morphism f, respectively.

Lemma 1.3.8. *Consider a 3-shell with vertices* X_i*,* i *ranging from* 0 *to* 3*, and 1-simplices* $f_{ij} : X_i \to X_j$*. Suppose, moreover, that* $X_0 = X_1$ *and* f_{01} *is the degenerate 1-simplex. Then this 3-shell can be filled to a 3-simplex if and only if:*

$$(f_{23}f_{12}) \to f_{13} \to f_{03} = f_{23} \circ [f_{12} \to f_{02}] \to f_{03},$$

where $f_{23}\circ$ *refers to the action of* f_{23} *on* $\mathrm{Hom}(X_0, X_2)$*, and all morphisms are as in Lemma 1.3.7. Analogous statements hold if* $X_2 = X_3$ *and* f_{23} *is degenerate.*

In particular, if both f_{01} *and* f_{23} *are degenerate, then the shell fills if and only if the associated square of morphisms in* $\mathrm{Hom}(X_1, X_2)$ *commutes.*

Proof. For the first statement, first assume that the shell fills. We will form a $\Lambda^5_{0,2,4}$. The vertices will be $X_0, X_0, X_1, X_1, X_2, X_3$. The first face is formed from a $\Lambda^4_{1,3}$, whose zeroth 3-simplex is the witness to the composition $(f_{23}f_{12}) \to f_{13}$ as in Lemma 1.3.7, whose second 3-simplex is the given one, and whose fourth 3-simplex is the first degeneracy of the given witness to $f_{12} \to f_{02}$. The third face is formed from the $\Lambda^4_{0,3}$

whose first face is the given 3-simplex, whose second face is the witness to $(f_{23}f_{02} \to f_{03}$ from the lemma, and whose fourth face is the zeroth degeneracy of the given $f_{12} \to f_{02}$. The fifth face is given by the zeroth degeneracy of the first degeneracy of the witness to $f_{12} \to f_{02}$.

Filling this (uniquely, as $5-3 \geq 2$) to a 5-simplex, we find inside two 3-simplices, one witnessing the morphism $(f_{23}f_{12}) \to f_{13} \to f_{03}$, the other witnessing the morphism $(f_{23}f_{12}) \to (f_{23}f_{02}) \to f_{03}$, and which share the 2-simplex which witnesses both compositions. This proves that both compositions are equal, as desired.

For the other direction, we will form a $\Lambda^5_{1,3}$ which will fill to the same 5-simplex (so that, e.g., its vertices will also be $X_0, X_0, X_1, X_1, X_2, X_3$). The zeroth face will be formed from a $\Lambda^4_{2,3}$ whose zeroth 3-simplex is the witness to $(f_{23}f_{12}) \to f_{13}$ as in the lemma, whose first 3-simplex witnesses the action of f_{23} on $f_{12} \to f_{02}$, and whose fourth 3-simplex is the first degeneracy of $f_{12} \to f_{02}$. The second face will be formed from a $\Lambda^4_{1,3}$, one whose zeroth face witnesses the action of f_{23} on $f_{12} \to f_{02}$, whose second face witnesses the morphism $(f_{23}f_{02}) \to f_{03}$ (as in Lemma 1.3.7), and whose fourth face is just the zeroth degeneracy of the 2-simplex $f_{12} \to f_{02}$.

The fourth face will be formed from a $\Lambda^4_{0,3}$ whose first face is the witness to the composition $(f_{23}f_{12}) \to f_{13} \to f_{03}$, whose second face is the witness to the composition $(f_{23}f_{12}) \to (f_{23}f_{02}) \to f_{03}$ (these agree on their shared face by assumption), and whose fourth face is the degenerate 3-simplex on $X_0 = X_1$. Finally, the fifth face as before is the zeroth degeneracy of the first degeneracy of the witness to $f_{12} \to f_{02}$. Filling this to a full 5-simplex, we recover (as the unique filled 3-simplex) the 3-simplex we desire.

The second case in the statement of the lemma follows analogously. $\square$

Proposition 1.3.9. *The data $\widetilde{C_2}$ and structures on the* Hom*-categories described above may be extended (necessarily uniquely) to a functor*

$$\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z).$$

Proof. Let us first prove functoriality on separate factors. Indeed, that identities map to identities is a consequence of uniqness of fillers of 3-horns. Without loss of generality, if $\varphi : f \to f'$ and $\varphi' : f' \to f''$ are morphisms in $\mathrm{Hom}(X, Y)$, g an object of $\mathrm{Hom}(Y, Z)$, then we may assemble a $\Lambda^4_{1,3}$ whose objects are X, X, X, Y, Z whose 3-face with 0-simplices X, X, X, Y (the fourth) is the witness to composition of φ and φ' and whose zeroth and second 3-simplices witness the application of g to φ' and φ, respectively. Filling this $\Lambda^4_{1,3}$ in the unique way, we obtain as its first face the witness to application of g to $\varphi'\varphi$ and as its third face the witness to composition of the applications of g to φ and to φ'. These two (uniquely characterized) 3-simplices share the 2-simplex which witnesses that both processes yield the same result, so that we have functoriality on separate factors.

Note that if the above data are to be a part of a functor, a morphism $(\psi, \varphi) : (g, f) \to (g', f')$ in $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y)$ must map to

$$(gf) \to (gf') \to (g'f') = (gf) \to (g'f) \to (g'f'),$$

where the morphisms here are as defined above, so that the above identity must be verified in addition to functoriality in both factors. This is in fact enough, for the map will then be defined on all morphisms and functoriality for all morphisms follows easily from functoriality on both factors and the above commutativity.

For the commutativity statement, we use Lemma 1.3.8. Indeed, let $\varphi : f \to f'$ and $\psi : g \to g'$ be morphisms in $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(Y, Z)$, respectively. Form a 4-horn whose 0-simplices are X, X, Y, Z, Z, whose zeroth and first face witness the applications of f and f' respectively to ψ, and whose third and fourth faces witness the applications of g and g' respectively to φ. Filling the horn, we obtain a 3-simplex which (according to Lemma 1.3.8) witnesses the commutativity desired. $\square$

We should check that left and right unitors are natural transformations.

Proposition 1.3.10. *Unitors (right and left) are natural transformations of functors.*

Proof. Let us first show that right unitors as described above are natural in f; left unitors are treated analogously. To wit, let $\varphi : f \to f'$ be a morphism in $\mathrm{Hom}(X,Y)$, considered as a 2-simplex in Hom (as opposed to Hom$'$). The zeroth degeneracy of φ is a 3-simplex which (after Lemma 1.3.8) witnesses that

$$(f1_X) \to f \to f' = (f1_X) \to (f'1_X) \to f',$$

showing that right unitors are natural transformations, as desired. $\square$

Now let f, g, and h be objects in $\mathrm{Hom}(X,Y)$, $\mathrm{Hom}(Y,Z)$, and $\mathrm{Hom}(Z,W)$. Form a 3-horn whose vertices are X, Y, Z, W and whose zeroth, second and third faces witness the compositions (hg), $((hg)f)$, and (gf); after filling, the filled face will be a morphism $(h(gf)) \to ((hg)f)$, which will be our associator.

Notice that we might have performed the operations in the above paragraph in the other order, producing a morphism $((hg)f) \to (h(gf))$. Our first task will be to show that these two are in fact inverse to one another.

Lemma 1.3.11. *The two methods for producing an associator, descibed above, are inverse to one another.*

Proof. Let f, g, h be as in the definition of these associators. Form a $\Lambda^5_{3,4}$ with vertices

$$X, X, X, Y, Z, W.$$

The zeroth face witnesses the associator $(h(gf)) \to ((hg)f)$ as a 2-simplex in $\mathrm{Hom}'(X,W)$, the first face is the zeroth degeneracy of the 3-simplex from which the alternate associator $((hg)f) \to (h(gf))$ is derived, the second face witnesses the alternate associator $((hg)f) \to (h(gf))$ as a 2-simplex in Hom$'$, and the fifth face is the zeroth degeneracy of the zeroth degeneracy of the 2-simplex which composes f and g. Filling this to a 5-simplex, we see that the filled 3-simplex witnesses that the composition of the two associator candidates is the identity, as desired. $\square$

Proposition 1.3.12. *The associators are natural in all three variables.*

Proof. First let us prove naturality in h. Let $h \to h'$ be a morphism in $\mathrm{Hom}(Z,W)$, seen as a 2-simplex in $\mathrm{Hom}(Z,W)$. We form a 4-horn with objects X, Y, Z, W, W whose zeroth faces witnesses application of g to $h \to h'$, whose second face witnesses the application of f to $(hg) \to (h'g)$, and whose third and fourth faces witness the associators of (f, g, h') and (f, g, h), respectively. Filling this, the first face witnesses (after Lemma 1.3.8) that

$$h(gf) \to h'(gf) \to (h'g)f = h(gf) \to (hg)f \to (h'g)f,$$

as desired. The same argument applies to naturality in f, as the argument would give naturality in the "alternate" associator, which as we proved in Lemma 1.3.11, is just inverse to the usual associator.

We are left with proving naturality in g. Fix a morphism $g \to g'$, considered as in Hom$'$. Let us form a composition diagram with vertices X, X, Y, Y, Z, W, labelled 0 through 5. We fill the 3-simplex (0123) with the fitting degeneracy of f, and then fill (234) with the morphism $g \to g'$, (345) with the composition of g and h, (245) with the composition of g' and h, (134) with the composition of g and f, (024) with the composition of g' and f. Finally, put the composition of (hg) and f in (135) and the composition of (hg') and f in (025). Composing this diagram, we find that (0145) witnesses (after Lemma 1.3.8) that

$$(h(gf)) \to (h(g'f)) \to ((hg')f) = (h(gf)) \to ((hg)f) \to ((hg')f),$$

as we wanted. $\square$

Proposition 1.3.13. *There is a one-to-one correspondence between 3-simplices in our 2-quasi-category C and families of 1-morphisms*

$$\left\{ X_i \xrightarrow{f_{ij}} X_j \right\}_{0 \le i < j \le 3}$$

together with morphisms

$$\left\{ (f_{jk} f_{ij}) \xrightarrow{\varphi_{ijk}} f_{ik} \right\}_{0 \le i < j < k \le 3}$$

such that

$$\begin{tikzcd}
(f_{23}(f_{12}f_{01})) \arrow[r, "f_{23}\circ\varphi_{012}"] \arrow[d] & (f_{23}f_{02}) \arrow[dr, "\varphi_{023}"] & \\
((f_{23}f_{12})f_{01}) \arrow[r, "\varphi_{123}\circ f_{01}"] & (f_{13}f_{01}) \arrow[r, "\varphi_{013}"] & f_{03}
\end{tikzcd}$$

commutes.

Proof. In either case (i.e. given all four faces of the 3-simplex), we concoct a monumental composition diagram, with nine vertices

$$X_0, X_1, X_2, X_2, X_3, X_3, X_3, X_3, X_3,$$

labelled as usual 0 through 8.

The 2-simplex (012) is the composition of f_{01} and f_{12}; the 6-simplex (2345678) is the fitting degeneracy of f_{23}. The remaining 1-simplices are the those whose first vertex is 0 or 1 (and which have not yet been filled). Moving along, the 2-simplex (123) is the degeneracy of f_{12}, (013) is the morphism $(f_{12}f_{01}) \to f_{02}$, and (124) will be the composition of f_{12} and f_{23}. The 2-simplex (024) is the composition of $(f_{12}f_{01})$ and f_{23}. The simplex (125) is the composition of f_{12} and f_{23}; the simplex (015) is the composition of f_{01} and $(f_{23}f_{12})$. The simplex (126) is the morphism $(f_{23}f_{12}) \to f_{13}$; the simplex (016) is the composition of f_{01} and f_{13}. The simplex (137) is the composition of f_{12} and f_{23}; the simplex (037) is the composition of f_{02} and f_{23}. Finally, the simplex (138) is the morphism $(f_{23}f_{12}) \to f_{13}$, and the simplex (068) is the morphism $(f_{13}f_{01}) \to f_{03}$.

Fill this (uniquely).

To prove $\Leftarrow$, we see that we obtain a 5-simplex (045678) which contains (in (078), as an element of Hom) a morphism $(f_{23}f_{02}) \to f_{03}$ that satisfies the same identity as in the statement of the proposition, so that it must be the same morphism as given by the data, and thus also be witnessed by (038). But then (0138) is evidently the 3-simplex we were looking for.

Conversely, by uniqueness of fillers of 3-simplices, we know that (0138) must agree with our given 3-simplex, so that (038) and thus (078) correspond to the given $(f_{23}f_{02}) \to f_{03}$. But then the identity holds in (045678). ☐

We now have the main result.

Proposition 1.3.14. *The data for $\widetilde{C}$ define a (2,1)-category.*

Proof. All that remains is to show that the triangle and pentagon identities hold.

For the triangle identity, let $f : X \to Y$ and $g : Y \to Z$ be 1-morphisms. Notice that the 3-simplex which is the first degeneracy of the composition of f and g actually witnesses (according to Proposition 1.3.13) that

$$g(1f) \to gf \to gf = g(1f) \to (g1)f \to gf \to gf,$$

where $gf \to gf$ is the identity, so that we have the triangle identity.

For the pentagon, we suppose that we are given 1-morphisms $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$, and $i : W \to V$, and we concoct a 9-simplex, this one with vertices

$$X, Y, Z, W, W, V, V, V, V, V.$$

The 2-simplex (012) will be the composition of f and g, while the 7-simplex (23456789) will be the degeneracy of the composition of h and i. The simplices (123) and (124) will both be the composition of g and h (so that (1234) is the a degeneracy of this composition). Let (023) and (024) be the compositions of (gf) and h and of f and (hg), respectively. Let (125) be the composition of g and (ih), and (035) be the composition of $(h(gf))$ and i. Let (126) be the composition of g and (ih), and (046) the composition of $((hg)f)$ and i. Let (137) be the composition of (hg) and i, and (017) the composition of f and $(i(hg))$. Let (128) be the composition of g and (ih), with (028) the composition of (gf) and (ih). Finally, let (129) be the composition of g and (ih), with (019) the composition of f and $((ih)g)$.

Filling the diagram, we obtain a 6-simplex (056789) which witnesses the pentagon identity. ☐

1.4 Morphisms of Quasi-Categories

1.4.1 Quasi-Functors

The elegance of the following definition is another reason why the simplicial sets approach to higher category theory is pedagogically useful.

Definition 1.4.1. Let X and Y be quasi-categories. A *quasi-functor* (sometimes a *(1)-morphism*) $f : X \to Y$ is a morphism of simplicial sets.

We define the categories nQCat, and nQGpd to be the categories of n-quasi-categories, and n-quasi-groupoids respectively. Define QCat $= \infty$QCat and QGpd $= \infty$QGpd.

It will be useful to have a finite characterization of what the data of a morphism entails for n-quasi-categories.

Proposition 1.4.2. *Let X be a quasi-category, Y an n-quasi-category. A quasi-functor $f : X \to Y$ is determined by what it does on the n-skeleton of X. Moreover, a morphism of simplicial sets $g : X_n \to Y$ extends to a quasi-functor $X \to Y$ if and only if for each $(n+1)$-simplex x in X, g takes the boundary of x to a fillable shell in Y.*

Proof. Given a morphism $f : X_n \to Y$, we first show that this has at most one extension to X_{n+1}; this shows the first part of the proposition by induction (because Y is an m-category for each $m \geq n$). So let x be an $(n+1)$-simplex in X. The boundary of x maps to an $(n+1)$-shell in Y; such an $(n+1)$-shell has at most one filler since Y is an n-quasi-category. Therefore, there is at most one place x can map, and we are done with the first part.

For the second part, we show that we may extend any g as in the proposition statement to X_{n+1} in such a way that each boundary of an $(n+2)$-simplex in X maps to a fillable $(n+2)$-shell in Y; this will prove the result by induction. To wit, the condition in the proposition immediately gives us an extension to X_{n+1}. Suppose that x is an $(n+2)$-simplex in X; consider the (inner) horn of type Λ_1^{n+2} in the boundary of x. This maps to an inner horn in Y, which has a filler y. But $\partial_1 y$, being an $(n+1)$-simplex in the n-quasi-category Y sharing a boundary with the image of $\partial_1 x$, must actually *be* the image of $\partial_1 x$, so that the image of the boundary of x is fillable (by y). $\square$

Notice what the preceding proposition says about quasi-functors between nerves of categories. A quasi-functor consists of the data of a map on objects and a map on morphisms which respects the source, target, and identity maps and which takes commutative triangles to commutative triangles. This is of course the definition of a functor between categories, and so we conclude (combining with the corollary above):

Corollary 1.4.3. *There is a functor (of categories) N : Cat $\to$ SSets which takes a category to its nerve and a functor to its corresponding quasi-functor. This is a full embedding of categories, and it is essentially surjective onto the (full) subcategory of* SSets *consisting of the 1-quasi-categories.*

Recall that if X and Y are simplicial sets, then the (categorial) product $X \times Y$ is just the simplicial set whose m-simplices are defined

$$(X \times Y)_m = X_m \times Y_m.$$

It is immediate from the definition that if X and Y are n-quasi-categories or n-quasi-groupoids respectively, that $X \times Y$ is also an n-quasi-category or an n-quasi-groupoid. Thus because nQCat and nQGpd are full subcategories of SSets, these products in SSets are also products in nQCat and nQGpd, respectively.

Finally, note that there are forgetful functors nQCat $\to$ mQCat and nQGpd $\to$ mQGpd whenever $n \leq m$.

1.4.2 Quasi-Functor Quasi-Categories

We first make a rather general definition:

Definition 1.4.4. Suppose X and Y are simplicial sets. The simplicial set $[X, Y]$ is defined to have as its set m-simplices $[X, Y]_m$ the set of morphisms $X \times \Delta^m \to Y$, with boundary and degeneracy maps induced from those on the Δ^k.

This definition is clearly functorial in X and Y (contravariantly and covariantly, respectively). The following property characterizes the simplicial set $[X, Y]$.

Proposition 1.4.5. *For any simplicial set X, the functor $[X, -]$ is right adjoint to the functor $(-) \times X$ as endofunctors of* SSets.

Proof. Let Y and Z be simplicial sets; we wish to produce a natural bijection between $\mathrm{Hom}(Y \times X, Z)$ and $\mathrm{Hom}(Y, [X, Z])$. Indeed, given a morphism $f : Y \times X \to Z$ and an m-simplex $y : \Delta^m \to Y$ of Y, the composition $f \circ (y \times \mathrm{id}_X)$ defines an m-simplex in $[X, Z]$, and the correspondence $y \mapsto f \circ (y \times \mathrm{id}_X)$ is a morphism of simplicial sets. Conversely, given a morphism $g : Y \to [X, Z]$ we may produce a morphism $Y \times X \to Z$ by taking an m-simplex (y, x) to $g(y)(x, s_m)$ where s_m is the unique m-simplex in Δ^m. These operations are clearly inverse to one another, and functorial, so we have what we want. $\square$

The operation $[X, Y]$ is thus an "internal Hom" in SSets. Moreover, we see that the 0-simplices of $[X, Y]$ are just the morphisms $X \to Y$, and the 1-simplices are just homotopies of morphisms, in the obvious sense.

We now relate this notion to quasi-categories.

Lemma 1.4.6. *Let X be a simplicial set and Y an object of nQCat (resp. nQGpd). Then the simplicial set $[X, Y]$ is an object of nQCat (resp. nQGpd).*

Proof. Suppose we are given a horn of $[X, Y]$, i.e. a morphism $\Lambda^m_k \to [X, Y]$, or equivalently a morphism $h : \Lambda^m_k \times X \to Y$. We would like to extend this to a morphism $g : \Delta^m \times X \to Y$ (uniquely, if $m > n$). We will build a morphism $g_r : \Delta^m \times X_r \to Y$ for each r so that the g_r's form a chain of extensions and such that each agrees with h as far as it goes; their union will be the g we want.

The base case g_0 is simple. X_0 consists of a set of points, and so for each $p : * \to X$ we need only map p to a filler of the horn $h \circ (\mathrm{id} \times p) : \Lambda^m_k \to Y$; such a filler is unique if $m > n$. Now suppose that we have defined g_{r-1}. Given an r-simplex $s : \Delta^r \to X$ in X_r, we have already defined the map

$$g' = g_{r-1} \circ (\mathrm{id} \times s|_{\partial\Delta^r}) : \Delta^m \times \partial\Delta^r \to Y,$$

as well as wanting the map

$$h' = h \circ (\mathrm{id} \times s) : \Lambda^m_k \times \Delta^r \to Y$$

to agree with the g_r we will construct; both g' and h' already agree on the $(r-1)$-skeleton of X by the induction hypothesis. We endeavor to extend the union of these maps to $\Delta^m \times \Delta^r$; this will finish the proof by induction on r.

But now we have reduced to the case of filling a morphism

$$\Lambda^m_k \times \Delta^r \cup \Delta^m \times \partial\Delta^r \to Y$$

to $\Delta^m \times \Delta^r$. According to Lemma 1.2.6, we may fill this by a succession of inner horn fillings followed by a filling of a Λ^{m+r}_k (or a Λ^{m+r}_{m+r} if $k = m$). Thus, the diagram fills (in the case of quasi-groupoids and quasi-categories both). Moreover, the diagram as given already has all simplices of dimension $\max(r-1, m-2)$ (as all such simplices either have a Δ^m-component in the $(m-2)$-skeleton or a Δ^r-component in the boundary). Therefore, if $m > n$, all $n-1$-simplices were given already, and so all the horns we fill must fill uniquely. We thus finish the induction, and the proof. $\square$

Using again the fact that QCat, nQCat, QGpd, and nQGpd are full subcategories of SSets, we see that the adjunction between $(-) \times X$ and $[X, -]$ holds in these categories as well; it therefore makes sense to refer to $[X, Y]$ as an internal Hom in these categories.

1.5 Equivalence

There are two major notions of equivalence which concern us: that of equivalence of objects in a quasi-category, and equivalence of quasi-categories themselves. We deal with both in turn, but first we look at the key new notion for this discussion, what we will call very surjective morphisms (also known as trivial Kan fibrations).

1.5.1 Very Surjective Morphisms

A very surjective morphism will serve for us to be a prototypical equivalence of quasi-categories. That trivial Kan fibrations serve this purpose is an adaptation of the "main new concept" in [Mak97]. We use the terminology from this paper, to point to the "logical" (as opposed to "topological") source of inspiration.

Definition 1.5.1. Let $f : X \to Y$ be a map of simplicial sets and $0 < n \leq \infty$. We say that f is *n-very surjective* if for every m-simplex $\Delta^m \to Y$ and every lift of its boundary to a shell $\sigma : \partial\Delta^m \to X$ in X, there is an extension of σ to an m-simplex $\Delta^m \to X$, and moreover that these extensions are unique whenever $m \geq n$.

In the sequel, we may abbreviate "∞-very surjective" to "very surjective." Notice that a very surjective map of categories is the same thing as a functor which is fully faithful and surjective on objects. Also, a very surjective map of simplicial sets is known in the literature as a "trivial Kan fibration." We use the terminology from [Mak97] mainly because it is in Makkai's work that the connection of these maps to equivalences of categories is fleshed out.

We have the following "nicer" characterization of n-very surjective maps which subsumes the topological notion that a trivial Kan fibration is a morphism that possesses the right lifting property with respect to cofibrations.

Proposition 1.5.2. *Let $f : X \to Y$ be a morphism of simplicial sets. Then f is very surjective if and only if for every monomorphism of simplicial sets $\Sigma \hookrightarrow \Theta$ together with compatible morphisms $\Sigma \to X$ and $\Theta \to Y$, there is a compatible lift $\Theta \to X$. The map f is n-very surjective if and only if in addition whenever $\Sigma \hookrightarrow \Theta$ is an isomorphism on $(n-1)$-skeleta, then there is only one lift (this is also of course vacuously true when $n = \infty$).*

Proof. As $\partial\Delta^m \hookrightarrow \Delta^m$ is a monomorphism which is an isomorphism on $(n-1)$-skeleta for all $n \leq m$, one direction is clear in both parts of the proposition statement.

For the other direction, assume that $f : X \to Y$ is very surjective. We prove inductively that we may extend the map $\Sigma \to X$ to $\Sigma \cup \Theta_m$, where Θ_m is the m-skeleton of Θ and the union is taken in Θ. Indeed, since the very surjective condition allows us to lift vertices (as $\partial\Delta^0 = \emptyset$ and Δ^0 is a point), we may extend to $\Sigma \cup \Theta_0$. Now, suppose we are given a map on $\Sigma \cup \Theta_{m-1}$. Any missing m-simplex has its boundary already there, so we are given a map $\partial\Delta^m \to X$ with a filler $\Delta^m \to Y$ downstairs; the very surjective condition allows us to lift this (freely, as we have already lifted the $(m-1)$-skeleton of Θ). This completes the induction, and so the union of these extensions to $\Sigma \cup \Theta_m$ gives us the extension to Θ that we wanted.

For the n-very surjective condition, notice that every lift of $\Theta \to Y$ to X arises in the way described in the previous paragraph, and so as the lifts of m-simplices will be unique for $m \geq n$ (and all simplices of lesser dimension have already been filled in the condition), we have a unique lift, as desired. □

Thus, setting $B = Y$, we see that very surjective maps have the property that partial sections always extend to full sections; in particular, a section always exists. This motivates the terminology: the axiom of choice fails in general in the presheaf category SSets, and a very surjective map is a map for which choice holds in a very strong way.

Proposition 1.5.3. *The class of n-very surjective maps is stable under composition and base change. If $X \to Y$ is n-very surjective and S is a simplicial set, then $[S, X] \to [S, Y]$ is n-very surjective. For a morphism $X \to Y$ of simplicial sets, the following are equivalent:*

(i) The simplicial set X is an n-quasi-category (n-quasi-groupoid) and the morphism $X \to Y$ is very surjective.

(ii) The simplicial set Y is an n-quasi-category (n-quasi-groupoid) and the morphism $X \to Y$ is n-very surjective.

Proof. Stability under composition is clear. Suppose that $X \to Y$ is very surjective and $Z \to Y$ is any morphism. Let $\Delta^m \to Z$ be an m-simplex of Z and $\partial\Delta^m \to X \times_Y Z$ a lift of its boundary. Composing with the morphisms of base change, we get an m-simplex of Y and a lift of its boundary to X, whence a full lift of the simplex to X. Thus we get compatible maps $\Delta^m \to Z$ and $\Delta^m \to X$, whence a map $\Delta^m \to X \times_Y Z$ which is the desired extension. It is easy to check that if $m > n$, this extension is unique.

If $X \to Y$ is n-very surjective, let S be a simplicial set, $\partial\Delta^m \to [S, X]$ an m-simplex in $[S, X]$ with a filling $\Delta^m \to [S, Y]$ of its projection to Y. This the same as a morphism $\partial\Delta^m \times S \to X$ with an extension to $\Delta^m \times S \to Y$ in Y. But then the given $\partial\Delta^m \times S \to X$ extends to $\Delta^m \times S \to X$ over the given filler, and uniquely so if $m > n$ because in that case the n-skeleton of $\Delta^m \times S$ is contained in $\partial\Delta^m \times S$.

Suppose now that $X \to Y$ is very surjective, and assume that X is an n-quasi-groupoid (n-quasi-category). If $\Lambda^m_k \to Y$ is an (inner) horn, then by lifting simplices one by one according to the definition of n-very surjective, we may lift this horn to a horn $\Lambda^m_k \to X$. Filling it upstairs, we may map it downstairs and obtain the result. If $m > n$, any other filler downstairs lifts to a filler upstairs, but by uniqueness of fillers in X, these must be equal, hence equal downstairs, so we have uniqueness.

Lastly, we need to show that $X \to Y$ is n-very surjective. Suppose $\tau, \tau' : \Delta^m \to X$ are two m-simplices in X with the same boundary and which map to the same m-simplex $\bar{\tau}$ in Y, with $m \geq n$. Form an $(n+1)$-shell whose k^{th} face is $\sigma_{m-1}\partial_k\tau = \sigma_{m-1}\partial_k\tau'$, $0 \leq k \leq m-1$, and whose m^{th} and $(m+1)^{\text{th}}$ faces are τ and τ' respectively. This shell maps to the boundary of the $\sigma_m\bar{\tau}$, and so by very surjectivity the filler in Y lifts to X, showing that this shell fills. But the m^{th} horn of this filler can also be filled by the m^{th} degeneracy of τ'. As $m + 1 > n$, uniqueness of fillers above dimension n in the n-quasi-category X shows that $\tau = \tau'$, as desired.

For the other direction, assume that Y is a n-quasi-groupoid (n-quasi-category), $X \to Y$ is n-very surjective, and $\Lambda^m_k \to X$ is an (inner) horn. Composing with $X \to Y$, we obtain a horn $\Lambda^m_k \to Y$ which we may fill to an m-simplex. The k^{th} face of this m-simplex has been filled downstairs and its boundary has been lifted upstairs, so we may lift the interior by the definition of very surjective. Now, the m-simplex itself has been filled in downstairs and its boundary has been lifted upstairs, so we may fill that as well. Every filler of this horn clearly arises in this fashion, so if $m > n$ this filler is unique as both steps characterize the filler uniquely. □

1.5.2 Equivalence and Loose n-Quasi-Categories

Definition 1.5.4. Let X and Y be simplicial sets. We say that X and Y are *n-(quasi-)equivalent* if there is a simplicial set P and n-very surjective morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$ (so that if X or Y is e.g. an n-quasi-category then P is also an n-quasi-category).

As before, we will habitually omit "∞-" in the terms ∞-(quasi-)equivalent and ∞-equivalence.

Proposition 1.5.5. *The relation of n-quasi-equivalence is an equivalence relation on simplicial sets.*

Proof. The relation in quastion is clearly reflexive and symmetric. For transitivity, suppose that $P \to X$, $P \to Y$, $Q \to Y$, and $Q \to Z$ are n-very surjective. Then $P \times_Y Q \to P \to X$ and $P \times_Y Q \to Q \to Z$ are n-very surjective, so that X is equivalent to Z. □

Recalling the definition of this equivalence relation, it is clear that n-quasi-equivalence is the smallest equivalnce relation so that $P \sim X$ whenever there is an n-very surjective morphism $P \to X$.

The work in [Mak97] suggests that this notion of equivalence (which is "semantic" in nature) should be sharply faithful to an appropriate notion of "syntactic equivalence." We will not attempt to make this formal here, although at times it will be a guiding principle.

Definition 1.5.6. Let X be a quasi-category. We say that X is a *loose n-quasi-category* if X is equivalent to an n-quasi-category.

Before we prove the main result about loose n-quasi-categories, we will need a few combinatorial lemmas about general quasi-categories.

Lemma 1.5.7. *Let X be a quasi-category and $m \geq 1$ an integer. Define a relation $\sim$ on m-simplices by saying that $\mu \sim \mu'$ if there is an $(m+1)$-simplex τ with $\partial_m \tau = \mu$, $\partial_{m+1}\tau = \mu'$, and for k with $0 \leq k < m$ we have $\partial_k \tau = \sigma_{m-1}\partial_k \mu = \sigma_{m-1}\partial_k \mu'$. Then $\sim$ is an equivalence relation. Moreover, we obtain the same relation if we use the "j^{th}" condition ($0 \leq j \leq m$), namely that $\mu \sim \mu'$ if there is an $(m+1)$-simplex τ such that $\partial_j \tau = \mu$, $\partial_{j+1}\tau = \mu'$, and for k with $0 \leq k \leq m+1$ and $k \neq j, j+1$ we have $\partial_k \tau = \partial_k \sigma_j \mu$.*

Proof. Notice that equivalent m-simplices have equal boundaries.

Reflexivity is given by taking $\tau = \sigma_m \mu$. For symmetry, suppose that τ witnesses that $\mu \sim \mu'$. Define a Λ^{m+2}_m by letting its $(m+2)^{\text{th}}$ boundary be τ, its $(m+1)^{\text{th}}$ boundary be $\sigma_m \mu$, and by letting its k^{th} boundary with $0 \leq k < m$ be $\sigma_m \sigma_{m-1} \partial_k \mu$. Filling this inner horn, an extracting its m^{th} boundary, we obtain a witness to $\mu' \sim \mu$. Finally, for transitivity, let τ witness $\mu \sim \mu'$ and τ' witness $\mu' \sim \mu''$. Then we can form a Λ^{m+2}_{m+1} whose $(m+2)^{\text{th}}$ boundary is τ, whose m^{th} boundary is τ', and whose k^{th} boundary for $0 \leq k < m$ is $\sigma_m \sigma_{m-1} \partial_k \mu$. Filling this inner horn, the $(m+1)^{\text{th}}$ face of the filler gives us a witness to $\mu \sim \mu''$.

For the last part, we first check one direction. So, given τ witnessing $\mu \sim \mu'$ as in the initial definition of $\sim$, form a Λ^{m+2}_{m+1} whose $(m+2)^{\text{th}}$ face is $\sigma_j \mu$, whose j^{th} face is $\sigma_m \mu$, whose $(j+1)^{\text{th}}$ face is τ, and whose k^{th} face for $k \neq j, j+1, m+1, m+2$ is $\sigma_j \sigma_{m-1} \partial_{k-1} \mu$. Filling this inner horn, we can extract its $(m+1)^{\text{th}}$ face which will witness that $\mu \sim \mu'$ in the "j^{th}" sense.

Conversely, we can use essentially the same $(m+2)$-horn, only this time it will be a Λ^{m+2}_{j+1} whose $(m+1)^{\text{th}}$ face witnesses equivalence in the j^{th} sense. We are done. $\square$

In the sequel, we will also use the term "homotopic" to refer to this relation.

Given a quasi-category X and an integer $m \geq 1$, we can thus form an equivalence relation on r-simplices by saying that two r-simplices are equivalent if their m-skeleta are equivalent (in the sense of the lemma). We may then form a simplicial set $\pi_m(X)$ whose set of r-simplices is the set of r-simplices of X modulo this equivalence relation. This simplicial set has the property that if two r-simplices agree on their m-skeleton, then they are equal. Moreover, $\pi_m(X)^{\text{op}} = \pi_m(X^{\text{op}})$.

Lemma 1.5.8. *Let X be a quasi-category, $m \geq 1$ an integer, and $\sim$ the equivalence relation defined in the previous lemma. Let τ be an $(m+1)$-simplex of X. Moreover, assume that for each integer k with $0 \leq k \leq m+1$, we are given an m-simplex μ_k with $\mu_k \sim \partial_k \tau$. Then there is an $(m+1)$-simplex τ' with $\partial_k \tau' = \mu_k$ and for all k.*

Proof. We will prove this assuming that all the μ_k but one are equal to $\partial_k \tau$; the general case follows by iterating the special case, changing one face at a time. For the rest of the proof, let k denote that unique index such that $\mu_k \neq \partial_k \tau$.

After possibly passing to the opposite category, we may assume that $k > 0$. Let ξ witness that $\partial_k \tau \sim \mu$. Form a Λ^{m+2}_{m+1} where for r with $0 \leq r \leq m$, $r \neq k$, the r^{th} face is $\sigma_m \partial_r \tau$, the k^{th} face is ξ, and the $(m+2)^{\text{th}}$ face is τ. Filling this, the $(m+1)^{\text{th}}$ face will have the properties we desire. $\square$

Proposition 1.5.9. *Let X be a quasi-category. The morphism of simplicial sets $X \to \pi_n(X)$ has the property that any morphism $X \to Y$ with Y an n-quasi-category factors uniquely as $X \to \pi_n(X) \to Y$.*

Proof. The morphism $X \to \pi_n(X)$ is an epimorphism, so uniqueness is immediate. Suppose that $X \to Y$ is a morphism, with Y an n-quasi-category. We need only show that if two r-simplices ξ and ξ' in X are equivalent, then they map to the same simplex in Y. In the case $r = n$, we have an $(n+1)$-simplex τ which agrees with $\sigma_n \xi$ on Λ^{n+1}_n, but whose n^{th} face is ξ'. Mapping this Λ^{n+1}_n forward to Y, it has only one filling, which must be both the image of τ and the image of $\sigma_n \xi$. We conclude that ξ and ξ' map to the same n-simplex in Y. In the case $r > n$, two equivalent r-simplices ξ and ξ' map to simplices in Y with equal n-skeleta, whence equal simplices. $\square$

Theorem 1.5.10. *Let X be a quasi-category. The following are equivalent:*

(i) The quasi-category X is a loose n-quasi-category.

(ii) There is an n-quasi-category Y and a very surjective morphism $X \to Y$.

(iii) For every monomorphism of simplicial sets $\Sigma \hookrightarrow \Theta$ which is an isomorphism on $(n+1)$-skeleta, every map $\Sigma \to X$ extends to a map $\Theta \to X$.

(iv) For every $k \geq n+2$, every morphism $\partial\Delta^k \to X$ can be extended to a morphism $\Delta^k \to X$.

Moreover, in this case $\pi_n(X)$ is an n-quasi-category and the morphism $X \to \pi_n(X)$ is very surjective.

Proof. The implication (ii)⇒(i) is immediate from the definition of equivalence.

For (i)⇒(iii), let P admit very surjective maps to X and to an n-quasi-category Y, say π_X and π_Y respectively. Let $\Sigma \to \Theta$ be a monomorphism of simplicial sets which is an isomorphism on $(n+1)$-skeleta. If $\Sigma \to X$ is a morphism, we may lift this to a morphism $\widetilde{\sigma} : \Sigma \to P$, which then projects down to a morphism $\pi_Y \circ \widetilde{\sigma} : \Sigma \to Y$. By Corollary 1.1.4, we may extend this map to Θ, and so by Proposition 1.5.2 we may extend $\widetilde{\sigma}$ to a morphism $\tau : \Theta \to P$, so that $\pi_X \circ \tau$ extends σ to Θ, as desired.

The implication (iii)⇒(iv) is clear.

Finally, we prove (iv)⇒(ii). Let X satisfy (iv). I claim that $X \to \pi_n(X)$ is very surjective. Indeed, let $\partial\Delta^r \to X$ be an r-shell and $\Delta^r \to \pi_n(X)$ a filling of its projection in $\pi_n(X)$. If $r \geq n+2$, then by assumption we may fill the shell in X, and the projection of this filler agrees with the given filler along n-skeleta, so this filler is in fact a lift. If $r \leq n$, then as $X \to \pi_n(X)$ is an isomorphism along $(n-1)$-skeleta, any representative of the given r-simplex in $\pi_n(X)$ (which is an equivalence class of r-simplices in X) will fill the given shell in X. So the only case left to consider is $r = n+1$. In this case, fix one representative $\widetilde{\sigma}$ of the given $(n+1)$-simplex in $\pi_n(X)$. The boundary of $\widetilde{\sigma}$ is equivalent to the given shell in X, so by Lemma 1.5.8 the given shell can as well be filled in X. This completes the proof of the claim.

Now, as $X \to \pi_n(X)$ is very surjective, we conclude that $\pi_n(X)$ is a quasi-category. I claim that $\pi_n(X)$ is an n-quasi-category, which will complete the proof that (iv)⇒(ii). Indeed, suppose we are given two fillings $\tau, \tau' : \Delta^r \to \pi_n(X)$ of the same inner horn $\Lambda^r_k \to \pi_n(X)$, with $r > n$. If $r \geq n+2$, τ and τ' agree on their n-skeleta, and so must be equal. If $r = n+1$, we may (by lemma 1.5.8) lift τ and τ' to $(n+1)$-simplices $\widetilde{\tau}$ and $\widetilde{\tau}'$ in X whose k^{th} horns agree. We may then form a Λ^{n+2}_k in X whose m^{th} face is $\sigma_{m+1}\partial_m\widetilde{\tau}$ for $0 \leq m \leq n$, $m \neq k$, and whose $(n+1)^{\text{th}}$ and $(n+2)^{\text{th}}$ faces are $\widetilde{\tau}$ and $\widetilde{\tau}'$ respectively. Filling the horn and extracting its k^{th} face, we see that $\partial_k\widetilde{\tau'} \sim \partial_k\widetilde{\tau}$, so that $\partial_k\tau' = \partial_k\tau$. Thus, τ and τ' agree on their n-skeleta, and so must be equal in this case as well. We are done. □

1.5.3 Truncation

We now use Theorem 1.5.10 to define a useful truncation operation on quasi-categories.

Corollary 1.5.11. *Let X be a quasi-category, $n \geq 1$ an integer. Then there is a loose n-quasi-category X^n and a monomorphism $X \hookrightarrow X^n$ such that every morphism $X \to Y$ with Y an n-quasi-category factors uniquely through X^n. In particular, the morphism $X \to \pi_n(X^n)$ is initial in the category of n-quasi-categories under X.*

Proof. Define $X^n_{n+1} = X$, and for $k > n+1$, inductively define X^n_k to be the simplicial set obtained from X^n_{k-1} by adding one additional k-simplex for every unfilled k-shell, and giving the new k-simplices their associated k-shells as boundaries. Let $X^n = \bigcup_{k=n+1}^{\infty} X^n_k$; by the theorem, X^n is a loose n-quasi-category. If $k \geq n+2$ and Y is an n-quasi-category, then every k-shell in Y has a unique filler, and so these new simplices will have (inductively) exactly one possible destination, showing that every morphism $X \to Y$ extends uniquely to $X^n \to Y$. This certainly shows that such morphisms also factor through $X \to \pi_n(X^n)$; for uniqueness, a different factorization through $\pi_n(X^n)$ would give a different factorization through X^n as $X^n \to \pi_n(X^n)$ is epimorphic, which would be a contradiction. □

In the sequel, for a quasi-category X, we will let $\Pi_n(X)$ denote the n-quasi-category under X which is initial in the category of all such (the corollary proved that $\Pi_n(X)$ exists and is canonically isomorphic to $\pi_n(X^n)$).

Corollary 1.5.12. *Let X be a quasi-category. The quasi-functor $X \to \Pi_n(X)$ is an isomorphism of simplicial sets on $(n-1)$-skeleta, and for $k = n, n+1$, any filler $\Delta^k \to \Pi_n(X)$ of the image of a shell $\partial\Delta^k \to X$ in X lifts to a filler of the shell in X.*

Proof. Given $\partial\Delta^k \to X$ with a filler of its image $\Delta^k \to \Pi_n(X) = \pi_n(X^n)$, we can lift the filler along the very surjective morphism $X^n \to \pi_n(X^n)$, and then the filler must live in X, as the loose n-quasi-category X^n only possesses additional simplices in dimensions $n+2$ and higher. Moreover, the passage from X^n to $\pi_n(X^n)$ only involves taking equivalence classes of simplices in dimensions n and higher, so $X \to \Pi_n(X)$ is an isomorphism along the $(n-1)$-skeleton. □

Corollary 1.5.13. *There is a natural extension of Π_n to a functor* QCat $\to n$QCat.

Proof. We need only say what Π_n does on morphisms. But given a morphism of quasi-categories $X \to Y$, the composition $X \to \Pi_n(Y)$ factors uniquely through $X \to \Pi_n(X)$, giving a map $\Pi_n(X) \to \Pi_n(Y)$. It is easy to check that this assignment is functorial. □

Corollary 1.5.14. *If $m \geq n$, the forgetful functor nQCat $\to m$QCat is right adjoint to Π_n.*

Proof. Immediate from the fact that $\Pi_n(X)$ is the initial n-quasi-category under X. □

Corollary 1.5.15. *If $m \geq n$, then $\Pi_n \circ \Pi_m$ is naturally isomorphic to Π_n.*

Proof. This follows immediately from the previous corollary and the fact that the forgetful functors nQCat $\to$ mQCat $\to$ QCat and nQCat $\to$ QCat are naturally isomorphic. □

1.5.4 Loose n-Quasi-Categories for $n < 1$

Another application of Theorem 1.5.10 is to sensibly extend the notion of loose n-quasi-category to nonpositive n-values, although only the case $n = 0$ is even remotely interesting, and so the following definition should probably only be seen as a integrative digression.

Definition 1.5.16. Let X be a quasi-category, n an integer (which could be nonpositive). We say that X is a loose n-quasi-category if for each $k \geq n+2$ and each k-shell $\partial\Delta^k \to X$ of X, there is a k-simplex $\Delta^k \to X$ which fills it.

Of course, Theorem 1.5.10 shows that this definition is compatible with what we have written earlier.

Recall that we may consider a poset to be a simplicial set by thinking of it as a category and then taking the nerve of this category. Characteristically, a poset is a category which is skeletal and in which all Hom-sets are empty or singletons.

Proposition 1.5.17. *Let X be a quasi-category. Then X is a loose 0-quasi-category if and only if X is equivalent to a poset, and in this case X admits a very surjective map to a poset (in fact, any poset it is equivalent to) which is unique up to isomorphism. Moreover, X is a loose (-1)-quasi-category if and only if X is equivalent to Δ^0 or the empty simplicial set, and X is a loose n-quasi-category for $n \leq -2$ if and only if X is equivalent to Δ^0.*

Proof. Suppose that X is equivalent to a poset Y. Let P be a simplicial set and $\pi_X : P \to X$, $\pi_Y : P \to Y$ very surjective maps. I claim that for any section $s : X \to P$, the composition $\pi_Y \circ s$ is very surjective. Indeed, let $\partial\Delta^k \to X$ be a shell in X and $\Delta^k \to Y$ a filler in Y. If $k \geq 2$, fillers exist in X and are unique in Y, so this case is easy. For $k = 0$, we need to show that objects in Y lift to X. Let y be a 0-simplex in Y, and lift it to an object p in P, projecting down to an object x of X. Since both p and $s(x)$ map to x, they have 1-morphisms between them in both directions (over 1_x in X). Projecting these down to Y, we see that $\pi_Y(s(x))$ is isomorphic to y, hence equal to y. Similarly, given a lift to X of a the boundary of a Δ^1 in Y, we can lift this boundary to P via s which will lie over the given boundary of the given Δ^1; filling it by very surjectivity of π_Y, we map the filler back to X and are done. But then X is a loose 0-quasi-category as fillers of k-shells exist in Y for all $k \geq 2$, and these all lift by very surjectivity. It is easy to see that any very surjective map between posets is an isomorphism, which gives uniqueness.

Conversely, if X is a loose 0-quasi-category, it is in particular a loose 1-quasi-category, and then $\pi_1(X)$ is a category with the property that for any three morphisms $f : x \to y$, $g : y \to z$, and $h : x \to z$, $h = gf$. This clearly implies that $\pi_1(X)$ has Hom-sets which are empty or singletons, so that a skeletal subcategory Y of $\pi_1(X)$ will be a poset. Moreover, the functor $\pi_1(X) \to Y$ is very surjective as it is surjective on objects and fully faithful. Therefore X is equivalent to a poset.

If X is a loose (-1)-quasi-category, the poset it is equivalent to will have the property that any two objects x and y have that $x \leq y$. This clearly implies that the poset is either empty or a singleton. Conversely, if X is equivalent to the empty poset or the singleton, then the existence of the very surjective map to either of these two cases will clearly imply that X is a loose (-1)-quasi-category. Being a loose (-2)-quasi-category in addition is the same thing as imposing the existence of a point, hence the result. □

After this proposition, it makes sense to define $\Pi_0(X)$ to be the initial poset under X, and it exists for the same reasons the other $\Pi_n(X)$ exist.

Definition 1.5.18. We say that a simplicial set X is *contractible* if X is equivalent to Δ^0, or (equivalently) a (-2)-quasi-category.

1.5.5 The Slice Construction

We now proceed with more constructions reminiscent of those from ordinary category theory. The first notion we generalize is that of slice (and coslice) categories.

Definition 1.5.19. Let X be a simplicial set, x an m-simplex of X. The *slice quasi-category* X/x is defined to be the simplicial set whose r-simplices are the $(r+m+1)$-simplices α of X whose target m-simplex (i.e. the boundary simplex whose vertices are $r+2, \ldots, r+m+2$) is x. These simplices carry an induced simplicial set structure from X. Define $x\backslash X$ dually, or what is the same thing, as $(X^{\mathrm{op}}/x^{\mathrm{op}})^{\mathrm{op}}$.

We will not return to "coslice" quasi-categories until the next section, although most if not all of what is done in this section would hold as well if we replaced uses of slice quasi-categories with coslice quasi-categories.

It is immediate from the definition that for each m-simplex x we have a "forgetful quasi-functor" $X/x \to X$ which just takes an r-simplex in X/x (which is an $(r+m+1)$-simplex in X) to its boundary face with vertices $0, \ldots, r$. Moreover, if x is a boundary simplex of f, then a similar procedure produces a morphism $X/f \to X/x$, and these morphisms are compatible with the correpsonding boundary morphisms in Δ. There are also degeneracy maps induced by corresponding degeneracies. The morphism $X/x \to X$ could be thought of as a morphism of this second type where the boundary simplex is "the unique (-1)-simplex in the boundary;" the null simplex.

In the case where X is an n-quasi-category or n-quasi-groupoid, we should make sure that taking slices respects these properties.

Proposition 1.5.20. *Let X be an n-quasi-category or an n-quasi-groupoid, x an m-simplex of X. Then X/x is an n-quasi-category or an n-quasi-groupoid, respectively.*

Proof. Let $\alpha : \Lambda^r_k \to X/x$ be a horn in X/x, inner in the n-quasi-category case. Then we can think of α as a morphism $\Lambda^{r+m+1}_{k,r+1,\ldots,r+m+1} \to X$, which has a filler in X (unique, if $r > n$) by Lemma 1.2.2; this of course gives a filler in X/x. □

1.5.6 Quasi-isomorphisms

Proposition 1.5.21. *Let X be a quasi-category and $f : x \to y$ a morphism in X. Then $X/f \to X/x$ (induced from x being the domain of f) is very surjective.*

Proof. An m-simplex of X/x together with a lift of its boundary to X/f amounts to the data of the $(m+2)^{\text{th}}$ face of a $(m+2)$-simplex in X, together with its zeroth through m^{th} faces. We thus have a horn, a Λ^{m+2}_{m+1} in X, which is inner and thus can be filled, giving the desired m-simplex in X/f. □

This proposition begs the question of what happens if we use the quasi-functor induced from the codomain of a morphism.

Definition 1.5.22. Let X be a quasi-category and $f : x \to y$ a morphism in X. We say that f is an *quasi-isomorphism* if the quasi-functor $X/f \to X/y$ (induced from y being the codomain of f) is very surjective.

The next lemma is key.

Lemma 1.5.23. *Let X be a quasi-category and x an object of X. Then the morphism 1_x is a quasi-isomorphism.*

Proof. The data of an m-simplex of X/x and a lift of its boundary to $X/1_x$ consists of an $(m+1)$-simplex σ of X with target x along with a $\Lambda^{m+2}_{m+1,m+2}$ whose target morphism is 1_x and whose intersection with the $(m+1)^{\text{th}}$ face of Δ^{m+2} agrees with σ. We thus are given an (arbitrary) Λ^{m+2}_{m+2} with target morphism 1_x, and desire to fill it. Call the given horn $\lambda : \Lambda^{m+2}_{m+2} \to X$.

Let $k \leq m$ be the largest integer such that for all subsets $S \subseteq [m]$ of size k, the simplex of λ with vertices $S \cup \{m+1, m+2\}$ is the k^{th} degeneracy of some k-simplex. We will prove the result by induction on $m - k$.

For the base of the induction, suppose $m = k$. Then $\sigma_{m+1}\partial_{m+1}\lambda$ (i.e. the $(m+1)^{\text{th}}$ degeneracy of the $(m+1)^{\text{th}}$ face of λ will fill the given horn.

Now, assume the induction hypothesis. We will form a morphism $\widetilde{\lambda} : \Lambda^{m+3}_{m+1,m+3} \to X$ whose $(m+1)^{\text{th}}$ (partial) boundary is λ; filling this horn (using Lemma 1.2.2) will thus fill λ. To this end, start with a partial filling of Δ^{m+3} wherein only λ is filled in on the $(m+1)^{\text{th}}$ face. We will number vertices as in the ambient Δ^{m+3}. For every subset $S \subseteq [m]$ of size $k+1$, let the simplex with vertices $S \cup \{m+1, m+3\}$ be the $(k+1)^{\text{th}}$ degeneracy of the simplex of λ with vertices $S \cup \{m+3\}$. Moreover, for every subset $S \subseteq [m]$ of size k, let the simplex with vertices $S \cup \{m+1, m+2, m+3\}$ be the $(k+1)^{\text{th}}$ degeneracy of the simplex of λ with vertices $S \cup \{m+2, m+3\}$. Note that since by assumption this last simplex was a k^{th} degeneracy, we get the same result by taking the k^{th} degeneracy of this simplex. Therefore, these two sets of simplices are compatible with one another (they are clearly compatible amongst themselves).

First, let us fill in the $(m+2)^{\text{th}}$ face of $\widetilde{\lambda}$. Inductively assume we have filled in all r-simplices of this face which contain the vertex $m+3$; we may base the induction at $r = k+2$ (as λ gives all such faces which contain $m+3$ but not $m+1$). The only unfilled $(r+1)$-simplices containing the vertex $m+3$ will be those with vertices $S \cup \{m+1, m+3\}$, where $S \subseteq [m]$ has r elements. Each such is a Λ^{r+1}_{r+1} with the property that for each subset $T \subseteq S$ of size $k+1$, the simplex with vertices $T \cup \{m+1, m+3\}$ is a $(k+1)^{\text{th}}$ degeneracy. But we know that $r + 1 \leq m + 2$, so

$$(r-1) - (k+1) \leq m - k - 1 < m - k,$$

and so by the inductive hypothesis we may fill this horn in. We thus have filled in the $(m+2)^{\text{th}}$ face of $\widetilde{\lambda}$.

Next, we fill in all remaining simplices of $\widetilde{\lambda}$. It will be enough to produce a filling such that for all $S \subsetneq [m]$, the simplex with vertices $S \cup \{m+1, m+2, m+3\}$ is filled. We fill all such by inducting on $|S|$, noting that we have covered the case $|S| = k$ by assumption, and in any case we will have $|S| \leq m$. Let $|S| = r$. If we have filled in all such simplices of dimension less than $r+2$, then of the simplex with vertices $S \cup \{m+1, m+2, m+3\}$ we will have a Λ^{r+2}_{r+2} filled in (the $(m+1)^{\text{th}}$ face comes from λ, the $(m+2)^{\text{th}}$

from the preceding paragraph). Moreover, this horn has the property that for every subset $T \subseteq S \cup \{m+1\}$ of size $k+1$, the simplex with vertices $T \cup \{m+2, m+3\}$ is a $(k+1)^{\text{th}}$ degeneracy of a $(k+1)$-simplex. Therefore, as

$$r - (k+1) \leq m - k - 1 < m - k,$$

we can fill this horn by the induction hypothesis, which completes both inductions, and the proof. □

In the following theorem, we denote by S^∞ the (nerve of the) groupoid with two objects 0 and 1 and one isomorphism (in either direction) between them. The name comes from the fact that the geometric realization of this simplicial set is the infinite-dimensional sphere.

Theorem 1.5.24. *Let X be a quasi-category and $f : x \to y$ a morphism in X. The following are equivalent:*

(i) The morphism f is a quasi-isomorphism.

(ii) There is a quasi-functor $S^\infty \to X$ taking $0 \to 1$ to f.

(iii) The quasi-functor $X \to \Pi_1(X)$ maps f to an isomorphism in the (1-quasi-)category $\Pi_1(X)$.

(iv) There is a morphism $g : y \to x$, a 2-simplex with zeroth face f, first face 1_y, and second face g, and another 2-simplex with zeroth face g, first face 1_x, and second face f.

Proof. We first prove that (i)⇒(ii). Assume that f is a quasi-isomorphism. An n-simplex of S^∞ is just a sequence of n composable morphisms, which is characterized by a length-$n+1$ sequence of objects in S^∞, i.e. binary sequence. The nondegenerate simplices are precisely those with no two consecutive digits the same, i.e. the alternating sequences, of which there are two in each dimension. Let σ_n be the nondegenerate simplex with target object (last digit) 0, and τ_n the nondegenerate simplex with target object 1. We would like to find a quasi-functor $S^\infty \to X$ which takes σ_0 to x, τ_0 to y, and τ_1 to f.

We will inductively prove that given a partial quasi-functor $S^\infty \to X$ defined for σ_m with $m < n$ and τ_m with $m \leq n$, we may extend it to σ_n and τ_{n+1}. But of τ_{n+1} we have filled in a horn, specifically a Λ^{n+1}_{n+1}, because the inner faces are all degenerate and the zeroth face is tau_n. But by assumption this horn has a target morphism of f. Since f is a quasi-isomorphism and $X/f \to X/y$ is very surjective, we can fill in the horn, producing our desired τ_{n+1} and σ_n.

To show (ii)⇒(iii), notice that the image of f under $X \to \Pi_1(X)$ is the image of the isomorphism $0 \to 1$ under $S^\infty \to X \to \Pi_1(X)$, and so is an isomorphism.

For (iii)⇒(iv), let $\bar{f} : x \to y$ denote the image of f under $X \to \Pi_1(X)$. We know that $\bar{f}$ is an isomorphism, so there is a morphism $\bar{g}$ in $\Pi_1(X)$ such that $\bar{f}\bar{g} = 1_y$ and $\bar{g}\bar{f} = 1_x$ in $\Pi_1(X)$. By Corollary 1.5.12, we can lift $\bar{g}$ to a morphism $g : y \to x$ in X, and then the fact that $\bar{f}$ and $\bar{g}$ are inverse lifts to the desired pair of 2-simplices, again by the same corollary.

Finally, we will show (iv)⇒(i). To do this, we need to prove that for every $k \geq 0$ and every k-shell $\partial\Delta^k \to X/f$ with a filling of its image $\Delta^k \to X/y$, there is a lift of this filler to X/f. Actually, we will prove by induction on n that this filler lifting property is true for both $X/f \to X/y$ and $X/g \to X/x$ for every $k \leq m$.

First, let us prove this for $m = 0$. By symmetry, it is enough to consider $X/f \to X/y$. We are tasked with filling a Λ^2_2 whose target morphism is f. To this end, form a $\Lambda^3_{1.3}$ whose zeroth face is the given 2-simplex with sides f, 1_y, and g, and whose second face is the first degeneracy of the first face of the given horn; this can be filled by Lemma 1.2.2, and the first face of the filler will be a filler of the given horn.

Now assume that the filler lifting property is true for $X/f \to X/y$ and $X/g \to X/x$ for each $k < m$. Again, by symmetry it is enough to consider $X/f \to X/y$. We desire to fill an arbitrary Λ^{m+2}_{m+2} whose target morphism is f. To wit, we will form a $\Lambda^{m+3}_{m+1,m+3}$ whose partial $(m+1)^{\text{th}}$ face is the given horn; this will fill in X and give us the filler we want. We start with only this partial $(m+1)^{\text{th}}$ face filled in. We start by filling in all simplices whose vertices are of the form $S \cup \{m+1, m+2, m+3\}$, $S \subsetneq [m]$, inducting on $|S|$.

To begin, we put in the given 2-simplex with sides f, 1_y, and g as the simplex with vertices $\{m+1, m+2, m+3\}$. Now let $|S| = r \leq m$, and suppose that we have filled in all such simplices of dimension less than

$r+2$. Of the simplex with vertices $S \cup \{m+1, m+2, m+3\}$, we have filled in its zeroth through $(r-1)^{\text{th}}$ faces, and also its r^{th} face as this is contained in the Λ^{m+2}_{m+2} we started with. We can fill in its $(r+2)^{\text{th}}$ face as at the moment this is partially filled as a Λ^{r+1}_{r+1} with target morphism g, and this can be filled by the induction hypothesis (as $r+1 < m+2$). Thus we complete the induction on $|S|$.

We have now filled in the zeroth through m^{th} faces of the desired horn. We need only fill in the $(m+2)^{\text{th}}$ face. But as is easily verified, what we have already filled in comprises a Λ^{m+2}_{m+2} in this face, and moreover the face has a target morphism of 1_y. Therefore, by Lemma 1.5.23, we can fill in this face, producing our desired $\Lambda^{m+3}_{m+1,m+3}$, and finishing the proof. ☐

Note that if X were the 2-quasi-category associated to the sub-2-category of Cat consisting of all categories, all functors, and all natural isomorphisms, condition (ii) in the theorem can be rephrased "f is an adjoint equivalence of categories," and condition (iv) "f is an equivalence of categories," so that we recover the classical theorem that equivalences of categories are adjoint equivalences.

Corollary 1.5.25. *Let f be a morphism in a quasi-category X. Then f is a quasi-isomorphism if and only if the image of f in X^{op} is a quasi-isomorphism.*

Proof. In Theorem 1.5.24, condition (iv) (say) is symmetric with respect to the operation of taking the opposite quasi-category. ☐

Corollary 1.5.26. *Let X be an n-quasi-category. Then X is a n-quasi-groupoid if and only if all its morphisms are quasi-isomorphisms (i.e., if and only if $\Pi_1(X)$ is a groupoid). In particular, an n-quasi-category which is a quasi-groupoid is an n-quasi-groupoid, and moreover any quasi-category which has fillers of all horns of the form Λ^m_m (or all horns of the form Λ^m_0) is a quasi-groupoid.*

Proof. Suppose that in X, every morphism is a quasi-isomorphism. By symmetry, it is enough to show that we can fill every Λ^m_m, and whenever $m > n$ such horns fill uniquely. But the data of such a Λ^m_m is the same as an $(m-2)$-shell in X/f and a filling of its image in X/y, where $f : x \to y$ is the target morphism in the horn. By very surjectivity of $X/f \to X/y$ and the fact that X/f is an n-quasi-category, morphism $X/f \to X/y$ is actually n-very surjective (after Proposition 1.5.3). But then the horn fills in any case, and fills uniquely if $m-1 \geq n$, i.e. $m > n$, as desired.

For the other direction, we can fill all horns of the form Λ^m_m in a quasi-groupoid, which clearly shows that for every morphism $f : x \to y$, $X/f \to X/y$ is very surjective. ☐

Definition 1.5.27. Let X be a quasi-category. We say that X is a loose n-quasi-groupoid if X is equivalent to an n-quasi-groupoid.

Corollary 1.5.28. *Let X be a quasi-category. Then X is a loose n-quasi-groupoid if and only if X is a quasi-groupoid and X is a loose n-quasi-category, and in this case $\Pi_n(X) = \pi_n(X)$ is an n-quasi-groupoid.*

Proof. If X is a loose n-quasi-groupoid, then certainly X is both a quasi-groupoid and a loose n-quasi-category. Conversely, if X is a quasi-groupoid and a loose n-quasi-category, then as $X \to \pi_n(X)$ is very surjective, every morphism in $\pi_n(X)$ is a quasi-isomorphism, so that $\pi_n(X)$ is actually an n-quasi-groupoid equivalent to X. ☐

Remark 1.5.29. We can thus extend this definition to $n < 1$ as we did for quasi-categories. It is immediate from this that a loose n-quasi-groupoid is equivalent to a point if $n \leq -2$, equivalent to a point or the empty simplicial set if $n = -1$, and equivalent to a (discrete) set if $n = 0$.

Corollary 1.5.30. *Let f, g, h be morphisms in a quasi-category X, and let h be "a composition" of f and g, in the sense that there is a 2-simplex in X whose zeroth, first, and second faces are g, h, and f, respectively. If any two of these morphisms is a quasi-isomorphism then so is the third. In particular, "compositions of quasi-isomorphisms are quasi-isomorphisms," and when h is an identity morphism, "left or right inverses of quasi-isomorphisms are quasi-isomorphisms."*

Proof. Project f, g, and h down to $\Pi_1(X)$, so that $\bar{h} = \bar{g}\bar{f}$. $\square$

Corollary 1.5.31. *Let f be a morphism in a quasi-category X with the property that f has left and right inverses, in the sense that there are two 2-simplices, one with f as its zeroth face, the other with f as its second face, and both with degenerate first faces. Then f is a quasi-isomorphism.*

Proof. Project down to $\Pi_1(X)$. $\square$

Corollary 1.5.32. *Let $F : X \to Y$ be a quasi-functor between quasi-categories, and f a quasi-isomorphism in X. Then $F(f)$ is a quasi-isomorphism in Y.*

Proof. Given a morphism $S^\infty \to X$ taking $0 \to 1$ to f, the composition $S^\infty \to X \to Y$ takes $0 \to 1$ to $F(f)$. $\square$

1.5.7 $\underline{\text{Fill}}$ Spaces and $\underline{\text{Hom}}$ Spaces

We now return to equivalences between quasi-categories.

Definition 1.5.33. Let X and Y be simplicial sets. A morphism $F : X \to Y$ will be called an *n-equivalence* if there is a simplicial set P and n-very surjective morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$ such that there is a section $s : X \to P$ with $F = \pi_Y \circ s$. As usual, an ∞-equivalence will also be known simply as an equivalence.

Notice that as very surjective morphisms always admit sections, simplicial sets X and Y are n-quasi-equivalent if and only if there is a morphism $F : X \to Y$ which is an n-equivalence.

In the next section (Theorem 1.5.44) we prove that there is a characterization of equivalences between quasi-categories which is almost identical to the usual definition for categories (and shows that the usual notion is a special case of this one).

Definition 1.5.34. Let $F : X \to Y$ be a quasi-functor between quasi-categories. We say that F is *essentially surjective* if for every objects y of Y there is an object x of X such that $F(x)$ is quasi-isomorphic to y.

Definition 1.5.35. Let X be a simplicial set, $\Sigma \hookrightarrow \Theta$ a monomorphism of simplicial sets, and $\sigma : \Sigma \to X$ a morphism. Define a simplicial set $\underline{\text{Fill}}(\sigma, \Theta)$ to be the simplicial set of all fillers of σ to Θ. More precisely, if σ is represented by $1 \to [\Sigma, X]$, then

$$\underline{\text{Fill}}(\sigma, \Theta) = 1 \times_{[\Sigma,X]} [\Theta, X].$$

In the case that X is a quasi-category and $\Sigma \hookrightarrow \Theta$ is $\{0, \ldots, n\} \hookrightarrow \Delta^n$, the morphism σ consists of an ordered n-tuple of objects $(x_0, \ldots, x_n)$, and so we define

$$\underline{\text{Hom}}_X(x_0, \ldots, x_n) = \underline{\text{Fill}}(\sigma, \Delta^n).$$

Keep in mind the case $n = 1$.

Notice that if X is an n-quasi-category, it is immediate that $\underline{\text{Fill}}(\sigma, \Theta)$ is an n-quasi-category.

Proposition 1.5.36. *Let X be a loose n-quasi-category (n a possibly nonpositive integer), $k \geq 0$ an integer, and $\Sigma \hookrightarrow \Theta$ a monomorphism of simplicial sets which is an isomorphism on k-skeleta. Then for any $\sigma : \Sigma \to X$, the n-quasi-category $\underline{\text{Fill}}(\sigma, \Theta)$ is a loose $(n - k - 1)$-quasi-groupoid (where we define $\infty - k - 1 = \infty$).*

Proof. First we prove that $\underline{\text{Fill}}(\sigma, \Theta)$ is a quasi-groupoid. Given a horn $\Lambda^m_m \to \underline{\text{Fill}}(\sigma, \Theta)$, we would like to show that this horn fills. By definition, the horn is the same data as a morphism $\Lambda^m_m \times \Theta \to X$, and in order for our filler to land in $\underline{\text{Fill}}(\sigma, \Theta)$, we ought to have that $\Delta^m \times \Sigma \to X$ factors through σ. Therefore, we are given a morphism

$$\Lambda^m_m \times \Theta \cup \Delta^m \times \Sigma \to X,$$

and we wish to fill this to $\Delta^m \times \Theta$. Notice that as the 0-simplices of Θ are all in Σ, all 1-simplices in $\Delta^m \times \Theta$ of the form $\Delta^1 \times \{x\}$ are in the above data and are identity morphisms in X.

As usual, to prove this by induction it will be enough to consider the case $\Sigma \hookrightarrow \Theta = \partial\Delta^s \hookrightarrow \Delta^s$. According to the proof of Lemma 1.2.6, we see that in order to fill this diagram we only need to fill one non-inner horn, which will be a Λ^{m+s}_{m+s} whose target morphism is an identity morphism. Therefore, as identity morphisms are quasi-isomorphisms, the horn fills, and we have proven that $\underline{\mathrm{Fill}}(\sigma, \Theta)$ is a quasi-groupoid.

To show that $\underline{\mathrm{Fill}}(\sigma, \Theta)$ is a loose $(n-k-1)$-quasi-groupoid, consider an r-shell $\partial\Delta^r \to \underline{\mathrm{Fill}}(\sigma, \Theta)$ where $r \geq n-k+1$. This data is the same as a morphism $\partial\Delta^r \times \Theta \to X$ which we must fill to $\Delta^r \times \Theta$ providing that $\Delta^r \times \Sigma \to X$ factors through σ. Progressively filling the skeleta of Θ, we reduce to the problem of filling a given

$$\Delta^r \times \partial\Delta^s \cup \partial\Delta^r \times \Delta^s \to X$$

to $\Delta^r \times \Delta^s$, where $s \geq k+1$. Applying Lemma 1.2.5 will fill all of this by inner horn filling except the last step, where we will have the shell of an $r+s$-simplex. But

$$r + s \geq (n-k+1) + (k+1) = n+2,$$

so this last shell fills because X is a loose n-quasi-category. The proof is complete. □

Remark 1.5.37. Notice the special case that in a loose n-quasi-category X, if x and y are objects of X then $\underline{\mathrm{Hom}}(x, y)$ is a loose $(n-1)$-quasi-groupoid. We thus recover that a category has $\underline{\mathrm{Hom}}$-objects which are sets and that a poset has $\underline{\mathrm{Hom}}$-objects which are singletons or empty. Moreover, a 2-quasi-category has $\underline{\mathrm{Hom}}$-objects which are (equivalent to) groupoids, as was discussed in §1.3. We will show later (in Proposition 2.4.3) that these notions are compatible with one another.

Remark 1.5.38. One might ask why we assumed that X is a loose n-quasi-category in the above proposition, instead of an n-quasi-category. In the latter case, the result as stated cannot be strengthened, as is easy to see upon recalling Lemma 1.2.6, where we had to fill horns of dimension less than that at which point we know fillers to be unique. The result is not so bad though; the filler spaces are certainly equivalent in a canonical way to their truncations to the appropriate level, and so one might just use the truncations as filler and Hom-spaces (as we did implicitly in the case of 2-quasi-categories).

1.5.8 Equivalences of Quasi-Categories

Definition 1.5.39. Let $F : X \to Y$ be a quasi-functor between quasi-categories. We say that F is a *homotopy equivalence* if there is a quasi-functor $G : Y \to X$ and quasi-isomrphisms $GF \to 1_X$ and $FG \to 1_Y$ in $[X, X]$ and $[Y, Y]$ respectively. In this case, G is called a *homotopy inverse* to F.

Remark 1.5.40. The terminology is lifted from topology because in the special case that X and Y are quasi-groupoids, X and Y are the same thing as Kan complexes, and this is the right notion of homotopy equivalence in that context. In the sequel, we will occasionally refer to the morphisms $GF \to 1$ and $FG \to 1$ as "homotopies."

Lemma 1.5.41. *Let $F : X \to Y$ and $G : Y \to X$ be quasi-functors between quasi-categories which are homotopy inverse to one another, say via homotopies $\varphi' : FG \to 1$ and $\psi : GF \to 1$. Then we can find a quasi-isomorphism $\varphi : FG \to 1$ such that there is a 2-simplex in $[X, Y]$ whose zeroth, first, and second faces are 1_F, φF and $F\psi$, respectively.*

Proof. Consider the (ordinary) categories $\Pi_1([X, X])$, $\Pi_1([X, Y])$, and $\Pi_1([Y, Y])$. The functors $[X, X] \to [X, Y]$ and $[Y, Y] \to [X, Y]$ induced from composition by F then descend to functors $\bar{\lambda}_F : \Pi_1([X, X]) \to \Pi_1([X, Y])$ and $\bar{\rho}_F : \Pi_1([Y, Y]) \to \Pi_1([X, Y])$. I claim that $\bar{\lambda}_F$ is an equivalence of categories. Indeed, the functor of left composition $\bar{\lambda}_G : \Pi_1([X, Y]) \to \Pi_1([X, X])$ is an inverse, because the quasi-isomorphisms $GF \to 1$ and $FG \to 1$ induce natural isomorphisms $\bar{\lambda}_{GF} = \bar{\lambda}_G \bar{\lambda}_F \to 1_{[X,X]}$ and $\bar{\lambda}_{FG} = \bar{\lambda}_F \bar{\lambda}_G \to 1_{[X,Y]}$. But then $\bar{\lambda}_F$ is fully faithful, so we may find a unique isomorphism $\bar{\varphi}$ which maps to $\bar{\rho}_F(\bar{\psi})$ under $\bar{\lambda}_F$.

According to Corollary 1.5.12, we may lift $\bar{\varphi}$ to a morphism $\varphi : fg \to 1$, and by Theorem 1.5.24 this morphism will be a quasi-isomorphism. Again by Corollary 1.5.12, we may then lift the fact that $\bar{\lambda}_F(\bar{\varphi}) = \bar{\rho}_F(\bar{\psi})$ to a 2-simplex in $[X, Y]$ whose faces are 1_F, φF and $F\psi$, as desired. $\square$

Lemma 1.5.42. *Let $F : X \to Y$ be a quasi-functor between quasi-categories, and suppose that for any objects $x, x' \in X$ the induced morphism $\underline{\mathrm{Hom}}_X(x, x') \to \underline{\mathrm{Hom}}_Y(F(x), F(x'))$ is a homotopy equivalence. Then for every $m > 0$ and every morphism $\sigma : \partial\Delta^m \to X$, the morphism $\underline{\mathrm{Fill}}(\sigma, \Delta^m) \to \underline{\mathrm{Fill}}(F(\sigma), \Delta^m)$ is surjective on connected components.*

Proof. Fix $\sigma : \partial\Delta^m \to X$, and let τ be an object of $\underline{\mathrm{Fill}}(F(\sigma), \Delta^m)$, so that τ is simply a filler of $F(\sigma)$ to Δ^m. We wish to show that there is a filler of σ so that there is a morphism $\Delta^m \times \Delta^1 \to Y$ where $\Delta^m \times \{0\} = \tau$, $\Delta^m \times \{1\} = \bar{\sigma}$, and $\partial\Delta^m \times \Delta^1 \to Y$ is the projection onto $\partial\Delta^m$ followed by σ.

First, consider the case where $\partial_0\sigma$ is the degeneracy of say x'; let x be vertex 0 of σ. Therefore, $\partial_0\tau$ is the degeneracy of $F(x')$ and vertex 0 of τ is $F(x')$. Consider now the $(m-1)$-fold zeroth degeneracy of τ. This $(2m-1)$-simplex contains within it a $\Delta^1 \times \Delta^{m-1}$, obtained by sending $(i, j) \in [1] \times [m-1]$ to $mi + j$, and this subcomplex has as source and target $(m-1)$-simplices the $(m-1)$-fold degeneracies of $F(x)$ and $F(x')$, respectively. Moreover, τ is contained in this. Call τ' the corresponding $(m-1)$-simplex of $\underline{\mathrm{Hom}}(F(x), F(x'))$. As the boundary of τ' contains only degeneracies of simplices of σ, there is a morphism $\sigma' : \partial\Delta^{m-1} \to \underline{\mathrm{Hom}}(x, x')$ such that the boundary of τ' is $F(\sigma')$.

Let $g : \underline{\mathrm{Hom}}_Y(F(x), F(x')) \to \underline{\mathrm{Hom}}_X(x, x')$ be a homotopy inverse to the map

$$f : \underline{\mathrm{Hom}}_X(x, x') \to \underline{\mathrm{Hom}}_Y(F(x), F(x'))$$

induced from F, and choose $\varphi : fg \to 1$ and $\psi : gf \to 1$ so there is a 2-simplex as in Lemma 1.5.41. Then $g(\tau')$ is an $(m-1)$-simplex of $\underline{\mathrm{Hom}}_X(x, x')$. Notice that the boundary of $g(\tau')$ is just $g(f(\sigma'))$. Using ψ, we can assemble a morphism

$$\Delta^1 \times \partial\Delta^{m-1} \cup \{0\} \times \Delta^{m-1} \to \underline{\mathrm{Hom}}_X(x, x')$$

where the restriction to $\{0\} \times \Delta^{m-1}$ is $g(\tau')$ and the restriction to $\{1\} \times \partial\Delta^{m-1}$ is σ'. This diagram fills (after say Lemma 1.2.6), and so gives a morphism $\Delta^1 \times \Delta^{m-1} \to \underline{\mathrm{Hom}}_X(x, x')$.

Composing this last morphism with f, we obtain a morphism

$$\Delta^1 \times \Delta^{m-1} \to \underline{\mathrm{Hom}}_Y(F(x), F(x'))$$

where the restriction to $\Delta^1 \times \partial\Delta^{m-1}$ is induced from $f\psi$. Moreover, φ induces a morphism $\Delta^1 \to [\Delta^{m-1}, \underline{\mathrm{Hom}}_Y(F(x), F(x'))]$ whose source is $f(g(\tau')$ and whose target is τ'. These two morphisms in $[\Delta^{m-1}, \underline{\mathrm{Hom}}_Y(F(x), F(x'))]$ match at their respective sources, and so make a morphism $\Lambda^2_0 \times \Delta^{m-1} \to \underline{\mathrm{Hom}}_Y(F(x), F(x'))$ (with the morphism induced from φ as the first face). Moreover, the restriction of this to $\Lambda^2_0 \times \partial\Delta^{m-1}$ matches with $f\psi$ along one leg and φf along the other, so the 2-simplex we have from Lemma 1.5.41 allows us to fill this in to $\Delta^2 \times \partial\Delta^{m-1}$ in such a way that the zeroth face of the 2-simplex factors as the projection to $\partial\Delta^{m-1}$ followed by $F(\sigma')$.

We therefore have a morphism

$$\Lambda^2_0 \times \Delta^{m-1} \cup \Delta^2 \times \partial\Delta^{m-1} \to \underline{\mathrm{Hom}}(F(x), F(x')).$$

By Lemma 1.2.6 and the fact that $\underline{\mathrm{Hom}}(F(x), F(x'))$ is a quasi-groupoid, this morphism fills to a morphism $\Delta^2 \times \Delta^{m-1} \to \underline{\mathrm{Hom}}(F(x), F(x'))$. Consider the zeroth face of the 2-simplex, which is a morphism $\Delta^1 \times \Delta^{m-1} \to \underline{\mathrm{Hom}}(F(x,), F(x'))$ whose restriction to $\Delta^1 \times \partial\Delta^{m-1}$ factors as the projection followed by $F(\sigma')$, and where the target of the 1-simplex factors through F. If we look at this as a morphism $\Delta^1 \times \Delta^{m-1} \times [1] \to Y$, we can restrict it to $\Delta^1 \times \Delta^m \to Y$ (where $\Delta^m \to \Delta^{m-1} \times \Delta^1$ is such that $\Delta^m \to \Delta^{m-1} \times \Delta^1 \to Y$ is τ). But then the source of this 1-simplex is τ, the target is in the image of F, and $\Delta^1 \times \partial\Delta^m$ clearly factors as the projection followed by σ.

Now, for the general case, let k be the largest integer such that the target k-simplex of σ is the k-fold degeneracy of an object. We will prove the statement by induction on $m-k$, and notice that we have treated above the case $m-k \leq 1$, so that we may assume $m \geq 2$. We start by assembling a partial $(m+1)$-simplex from σ, calling it μ. The (partial) m^{th} face of μ will be σ, and for $0 \leq r \leq m-1$, $r \neq 1$, the r^{th} face will be the $(m-1)^{\text{th}}$ degeneracy of the r^{th} face of σ. The $(m+1)^{\text{th}}$ face will be a filling of the first horn in σ. Notice now that if we succeed in filling the first face of μ, this becomes an inner horn whose filled face will fill σ.

Mapping this forward to Y, we see that we can fill in the m^{th} face of $F(\mu)$ with τ itself, and so this becomes a Λ_1^{m+1} which we proceed to fill to an $(m+1)$-simplex τ'. The first face of τ' has a boundary which is the image of the first face of σ', and moreover this face of σ' has a target simplex of dimension $k+1$ which is the degeneracy of an object. Therefore, by the induction hypothesis, we can find a filling of the first face of σ' and moreover we can find a morphism $\Delta^1 \times \Delta^m \to Y$ connecting the first face of τ' with the image of the first face of σ' while keeping the boundary constant.

Fill out the rest of σ', as we now have a Λ_m^{m+1}. We can fill out the homotopy above to $\Delta^1 \times \Lambda_m^{m+1} \cup \partial\Delta^1 \times \Delta^{m+1}$ by letting the r^{th} face in the first component be constantly the degeneracy $(0 \leq r < m, r \neq 1)$, the first face be as defined above, and the $(m+1)^{\text{th}}$ face be constantly the image of the $(m+1)^{\text{th}}$ face of σ'. The source in the second component is τ', the target the image of σ'. This then fills according to Lemma 1.2.6 to $\Delta^1 \times \Delta^{m+1}$, and we extract from this a homotopy from τ to the image of the m^{th} face of σ' which is constant along the boundary. We are done. □

Lemma 1.5.43. *Let S be a simplicial set, X a quasi-category, and f a morphism in $[S, X]$. Then f is a quasi-isomorphism if and only if for every 0-simplex $s \in S$, the restriction of f to $[s, X] \simeq X$ is a quasi-isomorphism.*

Proof. Because there are restriction quasi-functors $[S, X] \to [s, X]$ for each point $s \in S$, if f is a quasi-isomorphism then all restrictions will be quasi-isomorphisms. Conversely, it will be enough to show that f has left and right inverses. First, we will show that f has a left inverse. Moreover, by induction it is enough to show this in the case $S = \Delta^m$ and we already know that $f|_{\partial\Delta^m}$ has a left inverse. This means that we have filled in part of a map $\Delta^m \times \Delta^2 \to X$, specifically we have an inverse diagram on $\partial\Delta^m \times \Delta^2 \to X$ and have filled in $\Delta^m \times \partial_1\Delta^2$ with the degeneracy of the source of f, and $\Delta^m \times \partial_2\Delta^2$ with f itself. We wind up with a morphism $\Delta^m \times \Lambda_0^2 \cup \partial\Delta^m \times \Delta^2 \to X$ which we would like to fill to a morphism $\Delta^m \times \Delta^2 \to X$.

Notice that this is precisely the situation of the key step of the proof of Lemma 1.2.6, and that in that proof only inner horns were filled until the very last step. We thus can fill everything except possibly the Λ_0^{m+2} at the end. But this Λ_0^{m+2} is the zeroth horn of the simplex (in the notation of the lemma)

$$\begin{array}{c|ccccccc} 0 & 1 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & \cdots & m-2 & m-1 & m \end{array},$$

whose source morphism is the restriction of f to the vertex 0 of Δ^m. By assumption, this morphism is a quasi-isomorphism, and so the horn can be filled.

This completes the (inductive) proof that f has a left inverse, and by symmetry we conclude that f has a right inverse as well. Thus f is a quasi-isomorphism, as desired. □

Theorem 1.5.44. *Let $F : X \to Y$ be a quasi-functor between quasi-categories. The following are equivalent:*

(i) The quasi-functor F is an equivalence.

(ii) There is a quasi-functor $G : Y \to X$ such that $GF \simeq 1_X$ in $[X, X]$ and $FG \simeq 1_Y$ in $[Y, Y]$.

(iii) The quasi-functor F is essentially surjective and for every two objects $x, x' \in X$, the induced morphism $\underline{\mathrm{Hom}}_X(x, x') \to \underline{\mathrm{Hom}}_Y(F(x), F(x'))$ is a homotopy equivalence.

(iv) The quasi-functor F is essentially surjective and for every $m > 0$ and every shell $\sigma : \partial\Delta^m \to X$, the induced morphism $\underline{\mathrm{Fill}}_X(\sigma, \Delta^m) \to \underline{\mathrm{Fill}}_Y(F(\sigma), \Delta^m)$ is surjective on connected components.

Proof. (i)⇒(ii). Suppose that F is an equivalence. By definition, we may find a quasi-category P and very surjective morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$ as well as a section $s : X \to P$ such that $F = \pi_Y \circ s$. Let $t : Y \to P$ be any section, and let $G = \pi_X \circ t$. I claim that FG is quasi-isomorphic to 1_X in $[X, X]$, and FG is quasi-isomorphic to 1_Y in $[Y, Y]$. By symmetry, it is enough to consider GF.

To show that 1_X and GF are quasi-isomorphic, let us first define a quasi-functor $\Delta^1 \to [X, X]$ whose source and target are 1_X and GF. Indeed, we are given a morphism $\partial\Delta^1 \to X$ (the vertex 0 goes to 1_X, vertex 1 to GF) and moreover we have a good choice of lift of this to $[X, P]$, namely we lift 1_X to s and GF to $t \circ \pi_Y \circ s$. Projecting this to $[X, Y]$, both morphisms are F, so that we may fill the projected 1-shell with the degeneracy of F. Because $[X, P] \to [X, Y]$ is very surjective (by Proposition 1.5.3), this filler lifts, and so projects to a 1-simplex in $[X, X]$ as desired. A similar procedure produces a 1-simplex where 0 maps to GF and 1 maps to 1_X.

But then we have a $\partial\Delta^2 \to [X, X]$ whose zeroth and second faces are these last 1-simplices and whose first face is 1_{GF}. By construction, we can lift this to a morphism $\partial\Delta^2 \to [X, P]$ whose projection to $[X, Y]$ is the boundary of a degenerate 2-simplex. Filling this with the degeneracy, lifting the filler back to $[X, P]$ and projecting to $[X, X]$, we obtain a 2-simplex which witnesse that $1_X \to GF$ and $GF \to 1_X$ are inverse in one direction. A similar argument shows they are inverse on the other side. By Theorem 1.5.24, we are done with this implication.

(ii)⇒(iii). If y is an object of Y, then $F(G(y))$ is quasi-isomorphic to y, so that F is indeed essentially surjective.

Fix quasi-isomorphisms $GF \to 1_X$ and $FG \to 1_Y$. Let x and x' be objects in X. Let us first produce a map $g : \underline{\mathrm{Hom}}_Y(F(x), F(x')) \to \underline{\mathrm{Hom}}_X(x, x')$ which will be a homotopy inverse to the map $f : \underline{\mathrm{Hom}}_X(x, x') \to \underline{\mathrm{Hom}}_Y(F(x), F(x'))$ induced from F. Certainly we have an induced map $G : \underline{\mathrm{Hom}}_Y(F(x), F(x')) \to \underline{\mathrm{Hom}}_X(GF(x), GF(x'))$.

The given quasi-isomorphism $GF \to 1_X$ inductively defines a morphism

$$\Delta^1 \times GF(\underline{\mathrm{Hom}}_X(x, x')) \to [\Delta^1, X]$$

whose target consists only of simplices in $\underline{\mathrm{Hom}}_X(x, x')$. But then inductively (as $GF(\underline{\mathrm{Hom}}_X(x, x'))$ is a subcomplex of $\underline{\mathrm{Hom}}_X(GF(x), GF(x')))$ we can extend the morphism above to

$$\Delta^1 \times \underline{\mathrm{Hom}}_X(GF(x), GF(x'))$$

in the following manner.

On 0-simplices of $\underline{\mathrm{Hom}}_X(GF(x), GF(x')$ whose image is not already defined, consider the $\partial\Delta^1 \times \Delta^1 \cup \Delta^1 \times \Lambda^1_0$ in X which is the morphisms $GF(x) \to x$ and $GF(x') \to x'$ on the first component and is the desired 0-simplex of $\underline{\mathrm{Hom}}_X(x, x')$ on the other. Recalling that $GF \to 1$ is a quasi-isomorphism, we can fill this, and we obtain a 0-simplex of $\underline{\mathrm{Hom}}(x, x')$.

But then given a morphism defined on the k-skeleton of $\underline{\mathrm{Hom}}_X(GF(x), GF(x'))$, we may extend it to an unfilled $(k+1)$-simplex in the same fashion, only this time with a

$$(\partial\Delta^{k+1} \times \Delta^1) \times \Delta^1 \cup (\Delta^{k+1} \times \Delta^1) \times \Lambda^1_0 :$$

the first compoenent is the inductively defined construction, the second the given $(k+1)$-simplex. This is a diagram of the shape

$$\partial\Delta^{k+1} \times \Delta^1 \cup \Delta^{k+1} \times \Lambda^1_0$$

in $[\Delta^1, X]$ where all the morphisms $\{j\} \times \Delta^1$ are quasi-isomorphisms, and so can be filled. Thus is a map $\underline{\mathrm{Hom}}_X(GF(x), GF(x')) \to \underline{\mathrm{Hom}}_X(x, x')$ defined. Notice that in addition to this we have a "mapping cone" $\Delta^1 \times \underline{\mathrm{Hom}}_X(GF(x), GF(x')) \to [\Delta^1, X]$ which extends the mapping cone induced from $GF \to 1$. We let g the composition of this newly constructed map with G above.

Now consider the composition gf. By construction, gf agrees with GF followed by the map constructed above. Extend $GF \to 1$ to $S^\infty \to [X, X]$, and let $1 \to GF$ be the resulting inverse morphism. We inductively define a morphism $\Delta^2 \times \underline{\mathrm{Hom}}_X(x, x') \to [\Delta^1, X]$ whose first face will factor through $\underline{\mathrm{Hom}}_X(x, x')$ and in fact be the desired morphism $1 \to gf$. We start by letting the map on $\partial_2\Delta^2 \times \underline{\mathrm{Hom}}_X(x,x')$ be induced from $1 \to GF$ above. Let the map on $\partial_0\Delta^2 \times \underline{\mathrm{Hom}}(x,x')$ be induced from the mapping cone defined in the previous paragraph, so that in fact it is induced from $GF \to 1$ by construction. Therefore, the restriction to $\{0\} \times \underline{\mathrm{Hom}}_X(x,x')$ is the identity on $\underline{\mathrm{Hom}}(x,x')$ and the restriction to $\{2\} \times \underline{\mathrm{Hom}}_X(x,x')$ is gf.

To complete the desired filling, recast the data we have as a morphism

$$\Lambda^2_1 \times \Delta^1 \to [\underline{\mathrm{Hom}}_X(x,x'), X],$$

so that the restrictions to the two boundary vertices of Δ^1 are the constant maps x and x', respectively. We can thus use the given 2-simplex witnessing $GF \to 1$ and $1 \to GF$ being inverses to add 2-simplices to this diagram, specifically extending it to

$$\Delta^2 \times \partial\Delta^1 \cup \Lambda^2_1 \times \Delta^1 \to [\underline{\mathrm{Hom}}_X(x,x'), X].$$

But we can fill this according to Lemma 1.2.6, and so we obtain a morphism

$$\Delta^2 \times \underline{\mathrm{Hom}}_X(x,x') \to [\Delta^1, X]$$

whose first face factors through $\underline{\mathrm{Hom}}_X(x,x')$. We have produced the desired homotopy $1 \to gf$.

To finish the implication, we are left with producing a homotopy $1 \to fg$. According to Lemma 1.5.41, we may assume that the quasi-isomorphism $1 \to FG$ is such that F applied to this on the right and F applied to $1 \to GF$ on the left comprise the Λ^2_1 of a 2-simplex whose first face is 1_F. The construction of g was by means of a morphism $\Delta^1 \times \underline{\mathrm{Hom}}_X(GF(x), GF(x')) \to [\Delta^1, X]$; composing with F on one side and G on the other, we obtain a morphism

$$\Delta^1 \times \underline{\mathrm{Hom}}_Y(F(x), F(x')) \to [\Delta^1, Y]$$

whose restriction to $\{0\} \times \underline{\mathrm{Hom}}_Y(F(x), F(x')$ is the morphism

$$\underline{\mathrm{Hom}}_Y(F(x), F(x')) \to \underline{\mathrm{Hom}}(FGF(x), FGF(x'))$$

induced from FG, and whose restriction to $\{1\} \times \underline{\mathrm{Hom}}_Y(F(x), F(x'))$ is just fg.

Consider this last datum as a 1-simplex in $[\underline{\mathrm{Hom}}_Y(F(x), F(x')), [\Delta^1, Y]]$. The morphism $1 \to FG$ induces another 1-simplex in this simplicial set whose target is now the induced morphism

$$\underline{\mathrm{Hom}}_Y(F(x), F(x')) \to \underline{\mathrm{Hom}}(FGF(x), FGF(x')).$$

We thus have a Λ^2_1 in this simplicial set. Consider it as a morphism

$$\Lambda^2_1 \times \Delta^1 \to [\underline{\mathrm{Hom}}_Y(F(x), F(x')), Y];$$

we can extend this to a morphism from

$$\Lambda^2_1 \times \Delta^1 \cup \Delta^2 \times \partial\Delta^1$$

by using the 2-simplices $F \to FGF \to F$ which exist (and fit) by our choice of $1 \to FG$. Filling this according to Lemma 1.2.6, we get a morphism

$$\Delta^2 \times \Delta^1 \to [\underline{\mathrm{Hom}}_Y(F(x), F(x')), Y]$$

which is constantly $F(x)$ and $F(x')$ along $\partial_1\Delta^2 \times \partial\Delta^1$. Therefore, if we consider the corresponding 2-simplex in $[\underline{\mathrm{Hom}}_Y(F(x), F(x')), [\Delta^1, Y]]$, its first face is a 1-morphism in

$$[\underline{\mathrm{Hom}}_Y(F(x), F(x')), \underline{\mathrm{Hom}}_Y(F(x), F(x'))]$$

whose source is 1 and whose target is fg. We are done with this implication.

(iii)⇒(iv). This just follows from Lemma 1.5.42.

(iv)⇒(i). Let P be the simplicial set whose m-simplices are ordered pairs (x, y) of an m-simplex in X and an m-simplex in Y together with a quasi-isomorphism $F(x) \to y$ in $[\Delta^m, Y]$. I claim that the natural projections $P \to X$ and $P \to Y$ are very surjective.

Indeed, first we show that the map $P \to X$ is very surjective. An m-simplex X in X together with a lift of its boundary amounts to the data of x together with a morphism $\partial\Delta^m \times \Delta^1 \cup \Delta^m \times \{0\} \to Y$ where the first component is the quasi-isomorphism part of the data of the lift and the second component is just $F(x)$. We can extend this to $\Delta^m \times \Delta^1 \to Y$ (because of the quasi-isomorphism assumption), and so can extract $\Delta^m \times \{1\} \to Y$ to be y with the quasi-isomorphism as given by the full extension.

Secondly, we must show that $P \to Y$ is very surjective. First off, 0-simplices lift because of essential surjectivity. For $m > 0$, an m-simplex y in Y together with a lift of its boundary consists of the data of a morphism $\partial\Delta^m \times \Delta^1 \cup \Delta^m \times \{1\} \to Y$, which, again because of the quasi-isomorphism condition, we may fill to a morphism $\Delta^m \times \Delta^1 \to Y$. Let x' denote the m-simplex of Y which is the image of $\Delta^m \times \{0\}$ in this morphism. The simplex x' has a boundary which is in the image of F (as it was a lift to begin with), but its interior may not be.

By assumption, we know there is a morphism $\Delta^m \times \Delta^1 \to Y$ where $\Delta^m \times \{0\} \to Y$ is in the image of F, $\Delta^m \times \{1\} \to Y$ is x', and $\partial\Delta^m \times \Delta^1 \to Y$ is constantly the boundary of x'. Attaching this to the quasi-isomorphism from before, we obtain a morphism $\Delta^m \times \Lambda^2_1 \to Y$. By construction, we can add degeneracies of the morphism $\partial x' \to \partial y$ to obtain a morphism

$$\Delta^m \times \Lambda^2_1 \cup \partial\Delta^m \times \Delta^2 \to Y,$$

which then fills according to Lemma 1.2.6. The first face of the resulting morphism $\Delta^2 \to [\Delta^m, Y]$ will then agree with the given data along the restriction to $[\partial\Delta^m, Y]$, but also have the property that its source is in the image of F, say it is $F(x)$. Then (x, y) with the first face examined above is our desired lift.

To finish, we need only show that F factors as a section of $P \to X$ followed by the projection to Y. But we can produce a section $X \to P$ by sending an m-simplex x of X to $(x, F(x))$, with the morphism $F(x) \to F(x)$ being the identity. This section composed with the projection to Y is evidently F, and so the proof is complete. □

Remark 1.5.45. The theorem in particular demonstrates that equivalence of 1-quasi-categories is the same thing as the ordinary notion of equivalence of categories.

Corollary 1.5.46. *Let $F : X \to Y$ be an equivalence between quasi-categories, and let $P(F)$ be the simplicial set whose m-simplices are ordered triples (ξ, τ, f) where ξ and τ are m-simplices of X and Y respectively, and $f : F(\xi) \to \tau$ is a quasi-isomorphism. Then the projections $P(F) \to X$ and $P(F) \to Y$ are very surjective. Let $G : Y \to X$ be a quasi-functor. Then G is quasi-inverse to F if and only if G factors as a section of $P(F) \to Y$ followed by $P(F) \to X$.*

Proof. The first statement follows from the proof of (iv)⇒(i) in Theorem 1.5.44. For the second, one direction is proven in (i)→(ii) of Theorem 1.5.44. For the other direction, if we fix a quasi-isomorphism $\varphi : FG \to 1_Y$, we can define the section we want by taking an m-simplex τ of Y to $(G(\tau), \tau, \varphi(\tau)) \in P(F)$. □

Remark 1.5.47. Thus the equivalence witnesses $P(F)$ are particularly nice. Not all equivalence witnesses are so nice. Consider for example the identity quasi-functor $1_X : X \to X$. That this quasi-functor is an equivalence is witnessed by the very surjective morphisms $X \to X$ and $X \to X$. But the identity quasi-functor has more quasi-inverses than just itself! In this case, $P(1_X)$ is the "path space" consisting of the quasi-isomorphisms of X.

Corollary 1.5.48. *Let $F : X \to Y$ and $G : Y \to X$ be equivalences of quasi-categories such that F and G factor through the same simplicial set P very surjective onto both X and Y. Then F and G are quasi-inverse to one another.*

Proof. This follows immediately from the proof of (i)⇒(ii) in Theorem 1.5.44. □

Corollary 1.5.49. *Quasi-inverses of equivalences between quasi-categories are equivalences.*

Proof. Immediate from (ii) in Theorem 1.5.44. ☐

Corollary 1.5.50. *Let F, G, and G' be quasi-functors between quasi-categories with $F : X \to Y$ and $G, G' : Y \to X$. Then the following are equivalent:*

(i) The quasi-functors G and G' are respectively left and right quasi-inverses to F.

(ii) The quasi-functor F is an equivalence, $G \simeq G'$, and both are quasi-inverse to F.

(iii) The quasi-functor F is an equivalence, $G \simeq G'$, and G is a quasi-inverse to F.

Proof. For (i)⇒(ii), we have that

$$G \simeq GFG' \simeq G'.$$

Therefore

$$FG \simeq FG' \simeq 1_Y$$

and

$$G'F \simeq GF \simeq 1_X.$$

Clearly (ii)⇒(iii), and for (iii)⇒(i) we have that

$$FG' \simeq FG \simeq 1_Y.$$

☐

Corollary 1.5.51. *Let $F : X \to Y$ be an equivalence of quasi-categories and G be a left (right) quasi-inverse to F. Then G is a quasi-inverse to F.*

Proof. Let G' be any right (left) quasi-inverse to F and apply the previous corollary. ☐

Corollary 1.5.52. *Let $F, F' : X \to Y$ be two quasi-functors between quasi-categories with F being an equivalence, and suppose that $F \simeq F'$ in $[X, Y]$. Then F' is an equivalence.*

Proof. Let F have a quasi-inverse G. Then $1_X \simeq GF \simeq GF'$ and $1_Y \simeq FG \simeq F'G$, so G is also a quasi-inverse of F', showing that F' is an equivalence. ☐

Corollary 1.5.53. *Compositions of equivalences between quasi-categories are equivalences.*

Proof. Let $F : X \to Y$ and $F' : Y \to Z$ be equivalences of quasi-categories with quasi-inverses F' and G' respectively. Then $1_Z \simeq F'G' \simeq F'FGG'$ and $1_X \simeq GF \simeq GG'F'F$, so GG' is a quasi-inverse of $F'F$. ☐

Corollary 1.5.54. *Let $F : X \to Y$ and $G : Y \to Z$ be quasi-functors between quasi-categories. If any two of F, G, and GF are equivalences, then so is the third.*

Proof. As we have already treated the case where F and G are equivalences, let F and GF be equivalences, with H a quasi-inverse of F. Then H is an equivalence, so GFH is an equivalence, but then $G \simeq GFH$ is an equivalence as well. The case of G and GF is handled similarly. ☐

Definition 1.5.55. Let X be a quasi-category, and let Y_0 be a set of objects in X. The *full sub-quasi-cateogry of X with objects* Y_0 is the simplicial subset Y of X consisting of all simplices of X whose vertices lie in Y_0.

Definition 1.5.56. Let $F : X \to Y$ be a quasi-functor between quasi-categories X and Y. We say that F is *fully faithful* if for every $x, x' \in X$ the induced morphisms $F : \underline{\mathrm{Hom}}_X(x, x') \to \underline{\mathrm{Hom}}_Y(F(x), F(x'))$ are homotopy equivalences.

Corollary 1.5.57. *Let $F : X \to Y$ be a quasi-functor between quasi-functors, and let $F(X) \subseteq Y$ denotes the full sub-quasi-category of Y containing the objects that objects in X map to. Then the following are equivalent:*

(i) The quasi-functor F is fully faithful;

(ii) For any $m \geq 1$ and any $\sigma : \partial\Delta^m \to X$, the induced morphism

$$\underline{\mathrm{Fill}}_X(\sigma, \Delta^m) \to \underline{\mathrm{Fill}}_Y(F(\sigma), \Delta^m)$$

is surjective on connected components.

(iii) The induced quasi-functor $F : X \to F(X)$ is an equivalence.

Proof. To begin, (i)⇒(ii) is a special case of Lemma 1.5.42. Secondly, (ii)⇒(iii) follows from Theorem 1.5.44 and the fact that F is essentially surjective onto $F(X)$. Finally, for (iii)⇒(i), if $F : X \to F(X)$ is an equivalence, then as $F(X)$ is a full sub-quasi-category of Y, $\underline{\mathrm{Hom}}_X(x, x') \to \underline{\mathrm{Hom}}_{F(X)}(F(x), F(x')) = \underline{\mathrm{Hom}}_Y(F(x), F(x'))$ is a homotopy equivalence. □

1.6 Limits and Colimits

We will need a good theory of limits (and colimits) in quasi-categories in order to perform many of the constructions we will want to use.

1.6.1 Terminal and Coterminal Objects

First we will treat terminal and coterminal (initial) objects, later generalizing to limits and colimits of diagrams.

Definition 1.6.1. Let X be a quasi-category, $x \in X$ an object. We say that x is a *terminal object* if the natural morphism $X/x \to X$ is very surjective, and say that x is *initial* or *coterminal* if $x\backslash X \to X$ is very surjective.

The main thing to say about terminal and coterminal objects is that they are unique up to "canonical" quasi-isomorphism.

Proposition 1.6.2. *Let X be a quasi-category, and T the full sub-quasi-category containing all terminal (resp. coterminal) objects, which is to say T consists of all simplices of X all of whose objects are terminal (or coterminal). Then T is a loose (-1)-quasi-category. In particular, if X has a terminal (resp. coterminal) object, then T is a loose (-2)-quasi-category, and every morphism between two terminal (coterminal) objects is an quasi-isomorphism.*

Proof. As the statement for terminal objects is dual to that for coterminal objects, it will be enough to consider terminal objects. Therefore, suppose that $\partial\Delta^m \to T$ is an m-shell in T with $m \geq -1 + 2 = 1$. The target object of this m-shell is a terminal object, say $x \in X$. But then the m-shell consists of the data of an $(m-1)$-shell in X/x together with a filling of its projection to an $(m-1)$-shell in X. As $X/x \to X$ is very surjective, we find that the filler lifts, which is to say the m-shell we started with has a filler. We thus have shown that T is a loose (-1)-quasi-category.

If X has a terminal object, then T is nonempty, so can fill a 0-shell and is a loose (-2)-quasi-category. If $x \to y$ is a morphism between terminal objects, then it is a morphism in T, and as T is a quasi-groupoid, the morphism is an quasi-isomorphism. □

Remark 1.6.3. We should say a word or two about the use of "canonical" in this context. In the situation of a quasi-category, it is too much to hope for a limit (or a terminal object) to be well defined up to canonical isomorphism, as two isomorphisms might be related (canonically) without being equal. The next best possiblity is that the quasi-category of all solutions to a particular problem be (empty or) a "contractible" quasi-groupoid, but this is just what it means for a quasi-category to be a loose (-1)-quasi-category.

Proposition 1.6.4. *Let X be a quasi-category, and let x be an object which is quasi-isomorphic to a terminal object y. Then x is itself terminal.*

Proof. As x is quasi-isomorphic to y, there is a morphism $f : x \to y$ such that $X/f \to X/x$ and $X/f \to X/y$ are both very surjective, as well as being the same upon composition with $X/x \to X$ and $X/y \to X$. Given a monomorphism $\Sigma \hookrightarrow \Theta$ and a morphism $\Theta \to X$ with a lift $\Sigma \to X/x$, lift Σ to X/f and project down to X/y, so that $\Sigma \to X/y$ is in fact a lift of the original $\Theta \to X$. But then we can lift $\Theta \to X$ to X/y, whence to X/x. Projecting this last back down to X/x, we obtain our desired lift of Θ. $\square$

Therefore if X has a terminal object x then the full sub-quasi-category of all terminal objects is the full sub-quasi-category of all objects quasi-isomorphic to x.

Proposition 1.6.5. *Let $F : X \to Y$ be an equivalence of quasi-categories. Then for every object $x \in X$, $F(x)$ is a terminal object if and only if x is.*

Proof. Suppose that $x \in X$ is terminal, so that $X/x \to X$ is very surjective. We know that we can factor F as a section s of a very surjective morphism $\pi_X : P \to X$ followed by a very surjective morphism $\pi_Y : P \to Y$. If we are given a morphism $\Delta^m \to Y$ and a lift $\partial\Delta^m \to Y/F(x)$, this is the same data as a morphism $\partial\Delta^{m+1} \to Y$ whose target vertex is $F(x)$. We can thus lift this to a morphism $\partial\Delta^{m+1} \to P$ whose target vertex is $s(x)$, and then the projection of this to X fills there (as x is terminal), so that the filler lifts and projects back down to Y, giving the filler we were after.

Conversely, let G be a homotopy inverse to F. If $F(x)$ is terminal, then by what we have already proven $G(F(x))$ is also terminal. As x is quasi-isomorphic to $G(F(x))$, we conclude that x is terminal as well. $\square$

1.6.2 Diagram Quasi-Categories

We now lay the groundwork for a theory of limits and colimits in a quasi-category.

Definition 1.6.6. Let X be a quasi-category, D a simplicial set, and $\rho : D \to X$ a morphism (a "diagram"). We define the quasi-category X/ρ to be the fibre product

$$X/\rho = X \times_{[D,X]} [D, X]/\rho,$$

where the morphism $X \to [D, X]$ is the diagonal, also known as the morphism correpsonding to the projection $X \times D \to X$.

We intend X/ρ to be interpreted as the quasi-category of objects of X over ρ. There is, however, another simplicial set which could reasonably be said to represent this idea.

Definition 1.6.7. Let X be a quasi-category, $\rho : D \to X$ a diagram in X. We define the simplicial set $(X/\rho)'$ to be the simplicial set whose m-simplices are pairs (x, α) where x is an m-simplex of X and α is a lift of ρ to $x\backslash X$ over $x\backslash X \to X$. The boundary and degeneracy maps are induced by those in X, and thus we have a natural morphism of simplicial sets $(X/\rho)' \to X$.

Luckily, these two simplicial sets X/ρ and $(X/\rho)'$ are actually equivalent to one another.

Proposition 1.6.8. *Let X be a quasi-category and $\rho : D \to X$ a diagram. Then there is a simplicial set P which admits very surjective maps $P \to X/\rho$ and $P \to (X/\rho)'$, and such that moreover composing these two morphisms with the corresponding projections to X produces the same projection $P \to X$.*

Proof. Form a simplicial set D^* whose simplices are defined as follows. For every m-simplex α in D and integer $r \geq 0$, there is a unique $(m+r)$-simplex α_r in D^* whose first r vertices are ν and whose target m-simplex is α. For $i \geq r$, we take $\partial_i(\alpha_r) = (\partial_{i-r}\alpha)_r$ and $\sigma_i(\alpha_r) = (\sigma_{i-r}\alpha)_r$, and for $i < r$, we take $\partial_i(\alpha_r) = \alpha_{r-1}$ and $\sigma_i\alpha_r = \alpha_{r+1}$. We then have a natural inclusion $D \hookrightarrow D^*$ taking α to α_0. One can think of D^* as "D together with an initial object."

For every integer $m \geq 0$, we may consider the simplicial set D^*_m obtained by removing from $D^* \times \Delta^{m+1}$ the vertex $(\nu, m+1)$ and all incident simplices. There are natural inclusions $D^*_{m-1} \hookrightarrow D^*_m$ corresponding to the faces of $\partial_{m+1}\Delta^{m+1}$, and natural degeneracy maps $D^*_{m+1} \to D^*_m$ corresponding to the degeneracies of $\partial_{m+1}\Delta^{m+1}$. Moreover, we have inclusions $D^* \times \Delta^m \hookrightarrow D^*_m$ and $D \hookrightarrow D^*_m$, the latter by taking D to the target vertex of Δ^{m+1}, so that both are compatible (in the obvious way) with boundary and degeneracy maps. Therefore we may define a simplicial set whose m-simplices are the morphisms $D^*_m \to X$ such that the restriction to $D^* \times \Delta^m$ factors through the projection $D^* \times \Delta^m \to \Delta^m$ and such that the restriction to D is ρ. Call this simplicial set P.

We immediately see that P admits natural projections $P \to X/\rho$ and $P \to (X/rho)'$. The first is defined by composing $D \times \Delta^{m+1} \to D^*_m \to X$. The second is defined by observing that for every r-simplex τ in D, there is a unique $(m+r+1)$-simplex in D^*_m whose target r-simplex is $(\tau, m+1)$ and whose first $(m+1)$ vertices are (ν, i), $i = 0, \ldots, m$. Mapping this to X via a given m-simplex in P, the m-simplex with only ν's maps to an m-simplex x in X, and thus the $(m+r+1)$-simplex as a whole maps to an r-simplex in $x\backslash X$, which we take to be the destination of τ in a lift $D \to x\backslash X$. All together, this defines a lift $D \to x\backslash X$ which we take to be the image of the m-simplex in P. It immediately follows that $P \to X/\rho \to X = P \to (X/\rho)' \to X$.

We are left only with showing that both maps $P \to X/\rho$ and $P \to (X/\rho)'$ are very surjective. Let us treat $P \to X/\rho$ first. Suppose we are given an m-simplex $\Delta^m \to X/\rho$ together with a lift of its boundary $\partial\Delta^m \to P$. This consists of the data of a morphism $D \times \Delta^{m+1} \to X$ and morphisms $D^*_{m-1} \to X$ for each boundary simplex of $\partial_{m+1}\Delta^{m+1}$; we also know that all simplices which do not involve the target copy of D are given to us by virtue of our assumptions (they are degeneracies recoverable from other data). We are therefore missing precisely all those simplices which contain a vertex involving ν and whose projection to Δ^{m+1} is all of Δ^{m+1}. Let us hereafter refer to vertex (ν, i) as the "i^{th} top vertex."

Suppose that a simplex of D^*_m surjects onto Δ^{m+1} and has vertices

$$(x_0, i_0), (x_1, i_1), \ldots, (x_r, i_r).$$

Then the sequence $i_0, i_1, \ldots, i_r$ must be nondecreasing and comprise all of $[m+1]$. Moreover, the x_j must all be ν for all j less than or equal to some $k \leq m$ (say $k = -1$ if there are no such), after which point the x_j come exclusively from D. Therefore, for all $j \leq k$, $x_j = \nu$ and $i_j = j$, and for $j > k$, $x_j \in D$ and $i_j \leq i_{j-1} + 1$. Finally, let h denote the number of vertices over $k+1$ in Δ^{m+1}. Given a simplex of D^*_m which surjects onto Δ^{m+1}, we have thus produced a triple (k, r, h), which we shall refer to as its *signature*.

Given an unfilled simplex with signature (k, r, h), there is a unique index ℓ such that $i_\ell = k$ and $i_{\ell+1} = k+1$, and so we can ask the question of whether or not the simplex is an ℓ^{th} degeneracy when projected to D^*. Moreover, there is a bijection between simplices with such a degeneracy and of signature $(k, r+1, h)$ and simplices without such a degeneracy and of signature (k, r, h) given by omitting vertex ℓ in one direction (this will not have a degeneracy by our construction of k) and producing ℓ^{th} degeneracy in the other. We will construct by lexicographic induction on (k, r, h) a filling of all simplices without such a degeneracy and signature (k, r, h) and all simplices with such a degeneracy and signature $(k, r+1, h)$ by filling the horn implicit in the bijection above. The base $k = 0$ is given.

Indeed, consider a simplex of signature $(k, r+1, h)$ which has a degenerate projection as above, and let ℓ be as above. The zeroth through $(k-1)^{\text{th}}$ faces of the simplex are contained in the data we started with, and its k^{th} face has signature $(k-1, r+1, h)$, so is already filled. The $(k+1)^{\text{th}}$ through $(\ell-1)^{\text{th}}$ faces have signature (k, r, h) and the degeneracy, so have been filled. The $(\ell+1)^{\text{th}}$ face does not have a degeneracy, but its signature is $(k, r, h-1)$, and so is filled. Finally, the j^{th} face for $\ell + 1 < j \leq r$ has the degeneracy and is of signature $(k, r, h-1)$ or (k, r, h) and so is filled. As the ℓ^{th} face of the simplex has signature (k, r, h) and no degeneracy, by construction we have no filled yet and so we may fill this ℓ^{th} (inner) $(r+1)$-horn. Therefore we have filled out our data to proven that $P \to X/\rho$ is very surjective.

We are left with showing that $P \to (X/\rho)'$ is very surjective. Given an m-simplex $\Delta^m \to (X/\rho)'$ and a lift of its boundary $\partial\Delta^m \to P$, we have the data of a morphism defined on part of D^*_m taking values in X; specifically, we are given a morphism defined on all faces D^*_{m-1} as well as all faces which contain the vertices (ν, i) for $0 \le i \le m$ followed only by vertices in $D \times \{m+1\} \hookrightarrow D^*_m$. The form of our desired filling thus forces a particular restriction to $\partial_{m+1}\Delta^{m+1} \times D^*$, the degeneracy induced from the projection to X. We conclude that the simplices we are missing are precisely those which surject onto Δ^{m+1} and which contain a vertex of the form (x, i), where $i \le m$ and $x \in D$.

We will again use an induction on signature, only this time the induction will be downward in k (but still upward in r and in h), and we use a different bijection. Notice that there is a bijection between unfilled simplices with $i_{k+1} = k+1$ and simplices with $i_{k+1} = k$, by adding (or eliminating, in the other direction) $(\nu, k+1)$ at index $(k+1)$. This is well-defined by how D^* was constructed from D. Thus do we again find a collection of horns; for simplicity, we will call the signature of such a horn the triple (k, r, h) corresponding to the omitted face. We prove by lexicographic induction on (k, r, h) (again, downward in k) that we can construct a filling of all horns with this signature.

Indeed, the base is clear again ($k = m$ was the datum of the filling in $(X/\rho)'$). For the induction step, consider a horn with signature (k, r, h) (so that it is a Λ^{r+1}_{k+1}). The zeroth through k^{th} faces have been filled by the data we started with. We know that in the horn $i_{k+2} = k+1$. If $i_{k+3} = k+2$, then the $(k+2)^{\text{th}}$ face is the filled face of a horn with signature $(k+1, r, h')$ for some h', and so has already been filled. If, as is the other case, $i_{k+3} = k+1$, then the $(k+2)^{\text{th}}$ face is the interior of a filled horn of dimension $r-1$ and the same k-value, and so has been filled already. In any case, the j^{th} faces for $j > k+2$ are interiors of filled horns of dimension $r-1$ and the same k-value (or in the given data), and so have been filled already. We conclude that we do indeed have a Λ^{r+1}_{k+1}.

This horn certainly fills, even if $k+1 = 0$, because in this case the morphism from the zeroth vertex to the first vertex is an identity morphism in X (being as $i_0 = i_1 = 0$). Therefore, the induction and the proof are complete. □

Corollary 1.6.9. *Let X be a quasi-category and $\rho : D \to X$ a diagram. Then $(X/\rho)'$ is a quasi-category and there is an equivalence of quasi-categories $F : X/\rho \to (X/\rho)'$ such that*

$$X/\rho \to (X/\rho)' \to X = X/\rho \to X,$$

where $X/\rho \to X$ and $(X/\rho)' \to X$ are the structural projections.

Proof. The simplicial set $(X/\rho)'$ is a quasi-category because it is equivalent to the quasi-cateogry X/ρ. If $P \to (X/\rho)'$ and $P \to X/\rho$ are morphisms as in the statement of the proposition, then as the compositions with the projections to X are the same, any equivalence $F : X/\rho \to (X/\rho)'$ produced from a section of $P \to X/\rho$ will be what we want. □

Of course, we may dualize this entire discussion, defining $\rho\backslash X$, $(\rho\backslash X)'$, and proving that these two quasi-categories are also equivalent to one another.

1.6.3 Limits and Colimits

The preceding discussion motivates the following definition:

Definition 1.6.10. Let X be a quasi-category and $\rho : D \to X$ a diagram. We say that an object in X/ρ is a *limit (of the first type) of ρ* if it is terminal in X/ρ. At times we will also use the term limit to refer to the image of this object under the natural quasi-functor $X/\rho \to X$, and the object itself will then be referred to as a *limit diagram (of the first type)*. A *limit (diagram) of the second type* is by definition a terminal object in $(X/\rho)'$, and of course here the term limit applies to the underlying object of X as well. We dually define *colimits* and *colimit diagrams.*

We observe that a limit (considered as an object of X) can be just as easily thought of as an object of X/ρ as it can be thought of as an object of $(X/\rho)'$. In practice, when (e.g.) we will be looking at fibre products in a quasi-category, the presentation as an object of $(X/\rho)'$ will be a bit more manageable. Nonetheless, X/ρ is a formally simpler construction, and so should come in handy. Of course, all these remarks apply to colimits as well.

One should also note that as there is no preferred composition law in a quasi-category, these notions of limit and colimit are properly seen as notions *homotopy limit* and *homotopy colimit*. That the homotopy-weak notions are forced upon us in this context is one appealing feature of quasi-categories. For example, the fibre products of stacks that we care about are the "2-fibre products," which are the homotopy fibre products in Cat.

We will return to limits and colimits when we discuss adjoint quasi-functors in the next chapter.

Chapter 2

Fibrations

2.1 Fibrations

Before we can discuss more notions which generalize ordinary category theory, we need the notion of right fibration.

Definition 2.1.1. Let $f : X \to Y$ be a morphism of simplicial sets, and $0 < n \leq \infty$ as usual. We say that this morphism is a *left (resp. right) n-fibration* if the morphism

$$[\Delta^1, X] \to X \times_S [\Delta^1, S]$$

is n-very surjective, where the map $[\Delta^1, X] \to X$ is induced from the inclusion of the target (resp. source) object $\Delta^0 \hookrightarrow \Delta^1$, the map $[\Delta^1, X] \to [\Delta^1, S]$ is obtained by composition, and the map $[\Delta^1, S] \to S$ is restriction to the target (resp. source). We sometimes omit the n in the case of $n = \infty$ as usual, and will also refer to right n-fibrations being *fibred in n-quasi-groupoids.*

The meaning of the definition of fibred in n-quasi-groupoids is that a morphism in X is to be "essentially the same thing" as a morphism in S together with a lift of its target to X. Another way of saying this is that "pullbacks along morphisms in S exist and are canonical." This should remind the reader of the familiar notion of a category fibred in groupoids, and in fact we will show that this notion generalizes the familiar one. Another way to phrase this is that "the space of pullbacks of an object in X along a morphism in S is contractible."

The terms right and left fibration are intended to evoke Joyal's definitions in [Joy02]. We will show in the next proposition that our notion is in fact the same as his. In fact, his notion of mid fibration is also expressible in terms similar to the above, only the horn in that context is not Λ^1_1 or Λ^1_0 but Λ^2_1. There seems, however, to be no so easily expressible notion of "n-fibration." As Joyal's definition is relatively easy to check, the proposition will also serve for us as a computational tool.

Proposition 2.1.2. *Let $f : X \to S$ be a morphism of simplicial sets. Then f is a (left, right) n-fibration if and only if for every (left, right) horn $\tau : \Lambda^m_k \to X$ upstairs and every filler $\sigma : \Delta^m \to S$ of $f \circ \tau$ in S, there is a lift of σ to a filler of τ, and moreover this lift is unique whenever $m > n$.*

Proof. First, assume that f possesses the horn-filling condition for right n-fibrations described in the proposition statement, and let $\Delta^m \to X \times_S [\Delta^1, S]$ be an m-simplex with a lift $\partial\Delta^m \to [\Delta^1, X]$ of its boundary upstairs. We thus have a morphism

$$\partial\Delta^m \times \Delta^1 \cup \Delta^m \times \Lambda^1_1 \to X$$

together with a filling of its projection in S to a map $\Delta^m \times \Delta^1 \to S$. But as we may fill $del\Delta^m \times \Delta^1 \cup \Delta^m \times \Lambda^1_1$ to $\Delta^m \times \Delta^1$ by a succession of inner horn fillings followed by a filling of a Λ^{m+1}_{m+1} (by Lemma 1.2.6), we may successively lift the filling in S to the desired filling in X. If $m \geq n$, then these lifts will all be unique as we are only filling $(m+1)$-horns.

The same argument applies for left n-fibrations.

Conversely, let $\Lambda^m_k \to X$ be a horn with $0 < k \leq m$ in X and Δ^m a filling of this horn in S. Construct a shape $\Sigma = \Lambda^m_k \times \Delta^1 \cup \Delta^m \times \{1\} \to X$ by using the morphism $\Sigma \to \Lambda^m_k$ which takes (i, ϵ) to k if $\epsilon = 1$ and $i < k$; otherwise map the point to i. This is well-defined because $k > 0$ and so $\Delta^m \times \{1\}$ maps to a degeneracy of a simplex in Λ^m_k. The morphism $\Sigma \to \Lambda^m_k$ admits a section, as we can take Λ^m_k to $\Lambda^m_k \times \{0\}$. Moreover, we can fill Σ to $\Theta = \Delta^m \times \Delta^1$ in S by using the given filler of the horn. Therefore, we have a map $\Lambda^m_k \to [\Delta^1, X]$ together with a filling of this to $\Delta^m \to X \times_S [\Delta^1, S]$. As $[\Delta^1, X] \to X \times_S [\Delta^1, S]$ is n-very surjective, the filler lifts (giving a filler of the horn via the section) and uniquely if $m > n$. But in this last case, any other lifted filler of Λ^m_k in X produces a different lifting in $[\Delta^1, X]$, and so if $m > n$ fillers of Λ^m_k over S are unique.

Again, the left n-fibration case is analogous. $\square$

Notice that after this last proposition it is automatic that if a simplicial set X is fibred in n-quasi-groupoids over an n-quasi-category S, then X is itself an n-quasi-category (this of course remains true if S is an m-quasi-category for $m \leq n$). Moreover, the condition of being fibred in n-quasi-groupoids is stable under pullback of simplicial sets (by a proof similar to the proof of stability of n-very-surjective maps, using the condition in Proposition 2.1.2).

We will want a loosening of the above notion of fibration for certain purposes.

Definition 2.1.3. Let $X \to S$ be a right (left) fibration. We say that this morphism is a loose right (left) n-fibration if for every $k \geq n+2$ and every $\Delta^k \to S$ with a lift of its boundary $\partial\Delta^k \to X$, there is an extension of this lift across the interior of the k-simplex. We will also refer to loose right n-fibrations as morphisms fibred in loose n-quasi-groupoids.

Proposition 2.1.4. *Let $X \to S$ be a (loose) right n-fibration such that S is a (loose) n-quasi-category (resp. n-quasi-groupoid). Then X is a (loose) n-quasi-category (resp. n-quasi-groupoid). Similarly for left fibrations.*

Proof. As usual, we treat right fibrations and observe that the same arguments work for left fibrations.

First let us treat the n-quasi-category case. If $\Lambda^m_k \to X$ is an inner horn, we may project it down to S and fill it there to $\Delta^m \to S$; this filler lifts to a filler of the horn in X as $X \to S$ is a right fibration. If $m > n$ and there are two fillers of an m-horn in X, these two must map to a single filler in S as S is an n-quasi-category. But as $X \to S$ is a right n-fibration, this filler can lift to only one filler in X, so that the two fillers in X that we started with must have been equal all along.

For the n-quasi-groupoid case, notice that the same arguments as above can be made for right horns, so that right m-horns fill in X and uniquely if $m > n$, and moreover by the above argument X is an n-quasi-category. But then by Proposition 2.1.2 X must be an n-quasi-groupoid, as we wanted.

Finally, the loose cases are treated by observing that the definition of loose right n-fibration permits a similar argument, only this time lifting fillers of k-shells for $k \geq n+2$. $\square$

In particular, the fibre over a 0-simplex of S is always a (loose) n-quasi-groupoid (as Δ^0 is a 0-quasi-groupoid and a loose (-2)-quasi-groupoid), somewhat justifying the term "fibred in (loose) n-quasi-groupoids."

Proposition 2.1.5. *Let $X \to S$ be a right (left) fibration. Then $X \to S$ is a loose right (left) n-fibration if and only if the morphism*

$$[\Delta^{n+2}, X] \to [\partial\Delta^{n+2}, X] \times_{[\partial\Delta^{n+2}, S]} [\Delta^{n+2}, S]$$

is very surjective.

Proof. We may as well prove this only for the case of a right fibration.

First consider the case $n = -2$. Then we are to prove that $X \to S$ is very surjective (i.e. fibred in loose (-2)-quasi-groupoids, by definition) if and only if

$$X = [\Delta^0, X] \to [\partial\Delta^0, X] \times_{[\partial\Delta^0, S]} [\Delta^0, S] = * \times_* S = S,$$

is very surjective, which is tautological. For the remainder of the proof, assume $n > -2$.

If $X \to S$ is a loose right n-fibration, then a morphism

$$\Delta^k \to [\partial\Delta^{n+2}, X] \times_{[\partial\Delta^{n+2}, S]} [\Delta^{n+2}, S]$$

with a lift of its boundary to $[\Delta^{n+2}, X]$ is a morphism

$$\Delta^{n+2} \times \partial\Delta^k \cup \partial\Delta^{n+2} \times \Delta^k \to X$$

together with a filling of this morphism to $\Delta^{n+2} \times \Delta^k$ in S. But all the missing simplices here in the lift are of dimension $n + 2$ and higher, and so can be lifted by the definition of loose right n-fibration.

Conversely, let $k \geq n + 2$ and let $\Delta^k \to S$ be a morphism with a lift of its boundary upstairs $\partial\Delta^k \to X$. Form a shape

$$\Sigma = \Delta^{k-n-2} \times \partial\Delta^{n+2} \cup \partial\Delta^{k-n-2} \times \Delta^{n+2},$$

so that there is a projection map $\Sigma \to \partial\Delta^k$ given by taking a vertex (i, j) to the vertex $i + j$ of $\partial\Delta^k$ if $i = 0$ or $j = n + 2$, and taking (i, j) to $i + j + 1$ otherwise. As Σ contains no nondegenerate k-simplices, this map is in fact well-defined. We thus produce a morphism $\Sigma \to X'$ with a filling to $\Delta^{k-n-2} \times \Delta^{n+2} \to \Delta^k$ in the base.

According to the fact of the assumed very surjectivity of

$$[\Delta^{n+2}, X] \to [\partial\Delta^{n+2}, X] \times_{[\partial\Delta^{n+2}, S]} [\Delta^{n+2}, S],$$

we can lift the filler in S to a morphism $\Delta^{k-n-2} \times \Delta^{n+2} \to X$. If $k = n + 2$, this filler is precisely the filler we are looking for.

Otherwise, let $X' = X \times_S \Delta^k$, so that X' is a right fibration over a category and so is a quasi-category itself. Moreover, the morphism $\Delta^{k-n-2} \times \Delta^{n+2} \to X$ clearly factors through X'. Consider the k-simplex in $\Delta^{k-n-2} \times \Delta^{n+2}$ with vertices

$$(0, 0), (0, 1), \ldots, (0, n + 2), (1, n + 2), \ldots, (k - n - 2, n + 2).$$

By assumption, this simplex agrees with the $\partial\Delta^k$ we started with except possibly at the $(n + 2)^{\text{th}}$ face. I claim that nonetheless this $(n+2)^{\text{th}}$ face is equivalent in the quasi-category X' to the $(n+2)^{\text{th}}$ face of $\partial\Delta^k$ (in the sense of Lemma 1.5.7).

To see the equivalence, note that the $(k - 1)$-simplex in question has vertices

$$(0, 0), (0, 1), \ldots, (0, n + 1), (1, n + 2), \ldots, (k - n - 2, n + 2).$$

As $1 + (n + 1) + 1 = 1 + (n + 2)$, the k-simplex with vertices

$$(0, 0), (0, 1), \ldots, (0, n), (0, n + 1), (1, n + 1), (1, n + 2), \ldots, (k - n - 2, n + 2)$$

has degenerate j^{th} faces for $j \neq n+2, n+3$ in the appropriate way to witness that the face we are interested in is equivalent to the face with vertices

$$(0, 0), (0, 1), \ldots, (0, n), (0, n + 1), (1, n + 1), (2, n + 2), \ldots, (k - n - 2, n + 2).$$

Continuing inductively in the obvious fashion, we can show that for each i with $0 \leq i < k - n - 2$, the above simplices are equivalent to the simplex with vertices

$$(0, 0), (0, 1), \ldots, (0, n), (0, n + 1), (1, n + 1), \ldots, (i, n + 1), (i + 1, n + 2), \ldots, (k - n - 2, n + 2).$$

But then using the k-simplex with vertices

$$(0,0),(0,1),\ldots,(0,n),(0,n+1),(1,n+1),\ldots,(k-n-3,n+1),(k-n-2,n+1),(k-n-2,n+2),$$

all of whose faces save the $(k-1)^{\text{th}}$ and k^{th} are degeneracies in the appropriate way, we see that the simplex in question is equivalent to the simplex with vertices

$$(0,0),(0,1),\ldots,(0,n),(0,n+1),(1,n+1),\ldots,(k-n-3,n+1),(k-n-2,n+1).$$

As this last simplex is contained in Σ and maps to the $(n+2)^{\text{th}}$ face of our original shell, we have proven the claim.

But now by Lemma 1.5.8, we conclude that there is a k-simplex in X' whose boundary is the the given k-shell. But this k-simplex must map to Δ^k downstairs, and so composing with the projection $X' \to X$ we obtain a filler of the original k-shell which lives over the k-simplex in S that we started with. We are done. □

Proposition 2.1.6. *Let $X \to S$ be fibred in (loose) n-quasi-groupoids and Σ a simplicial set. Then $[\Sigma, X] \to [\Sigma, S]$ and $\Sigma \times X \to \Sigma \times S$ are fibred in (loose) n-quasi-groupoids. Similarly for left fibrations.*

Proof. As usual, we treat only the right fibration situation. For the case of being fibred in n-quasi-groupoids, we need to check that

$$[\Delta^1, [\Sigma, X]] \to [\Delta^1, [\Sigma, S]] \times_{[\Sigma,S]} [\Sigma, X]$$

is n-very surjective. But by formal nonsense, this morphism is just the induced morphism

$$[\Sigma, [\Delta^1, X]] \to [\Sigma, [\Delta^1, S] \times_S X],$$

so what we want follows from Proposition 1.5.3. Similarly,

$$[\Delta^1, \Sigma \times X] \to [\Delta^1, \Sigma \times S] \times_{\Sigma \times S} (\Sigma \times X)$$

is formally the same morphism as

$$[\Delta^1, \Sigma] \times [\Delta^1, X] \to [\Delta^1, \Sigma] \times \left([\Delta^1, S] \times_S X\right),$$

which is also immediately very surjective.

Being fibred in loose n-quasi-groupoids is also expressible by an axiom of this form (by the previous proposition) and so the same formal argument finishes the proof. □

Proposition 2.1.7. *Let $X \to S$ and $Y \to S$ be morphisms of simplicial sets such that $Y \to S$ is fibred in (loose) n-quasi-groupoids, and let $[X, Y]_S$ denote the fibre product*

$$[X, Y]_S = [X, Y] \times_{[X,S]} 1$$

where $1 \to [X, S]$ represents the morphism $X \to S$. Then $[X, Y]_S$ is a (loose) n-quasi-groupoid.

Proof. By the preceding proposition, $[X, Y] \to [X, S]$ is fibred in (loose) n-quasi-groupoids, so taking the fibre over the object $1 \to [X, S]$ we obtain that $[X, Y]_S$ is a (loose) n-quasi-groupoid. □

Remark 2.1.8. A special case of the last proposition which one should keep in mind is that where $X \to S$ is also fibred in (loose) n-quasi-groupoids, so that Hom objects between simplicial sets fibred in (loose) n-quasi-groupoids are actually themselves (loose) n-quasi-groupoids. In the sequel, we will compare this to an "internal" $\underline{\text{Hom}}$.

We will usually only use the notion of right fibration in the case where the base S is an ordinary category. The more general definition is nonetheless pedagogically useful in the proof of the main theorem of this section.

2.2 The Quasi-Categories $n\mathrm{Fib}/S$ and $\mathrm{L}n\mathrm{Fib}/S$

Definition 2.2.1. Let S be a simplicial set. Define the (large) simplicial set $n\mathrm{Fib}/S$ by letting the m-simplices be the simplicial sets fibred in n-quasi-groupoids over $(\Delta^m)^{\mathrm{op}} \times S$ (the op is to account for the fact that the quasi-groupoid-valued "functors" corresponding to fibred categories are contravariant), together with simplicial sets isomorphic to pullbacks under all nonidentity morphisms $\Delta^k \to \Delta^m$. Boundary and degeneracy maps are given by rearranging (or truncating) the data in the obvious fashion. As usual, let Fib/S denote $\infty\mathrm{Fib}/S$.

Similarly define $\mathrm{L}n\mathrm{Fib}/S$ by replacing the words "fibred in n-quasi-groupoids" with "fibred in loose n-quasi-groupoids."

The choice of pullbacks, while a bit ugly, seems necessary unless we want to regard the sets of simplices of $n\mathrm{Fib}/S$ and $\mathrm{L}n\mathrm{Fib}$ not as sets but as groupoids (or categories). This sort of concern will fade into the background once we begin to regard $n\mathrm{Fib}/S$ and $\mathrm{L}n\mathrm{Fib}/S$ not as simplicial sets so much as quasi-categories.

To prove the main theorem about $n\mathrm{Fib}/S$ and $\mathrm{L}n\mathrm{Fib}/S$, we need a few lemmas.

Lemma 2.2.2. *Let S and T be simplicial sets, n an integer. Then there is are isomorphisms of (large) simplicial sets $[S^{\mathrm{op}}, n\mathrm{Fib}/T] \to n\mathrm{Fib}/(S \times T)$ and $[S^{\mathrm{op}}, \mathrm{L}n\mathrm{Fib}/T] \to \mathrm{L}n\mathrm{Fib}/(S \times T)$.*

Proof. This follows immediately upon observing that an m-simplex of $[S^{\mathrm{op}}, n\mathrm{Fib}/T]$ is just a morphism $\Delta^m \times S^{\mathrm{op}} \to n\mathrm{Fib}/T$. But an r-simplex in $n\mathrm{Fib}/T$ is by definition a right n-fibration $X \to T \times (\Delta^r)^{\mathrm{op}}$, so a morphism $\Delta^m \times S^{\mathrm{op}} \to n\mathrm{Fib}/T$ is literally the same thing as a right n-fibration $X \to (\Delta^m)^{\mathrm{op}} \times S \times T$, which of course is the same thing as an m-simplex of $n\mathrm{Fib}/(S \times T)$. A similar argument holds for the loose case. □

Remark 2.2.3. After this lemma, it makes sense to identify $n\mathrm{Fib}/(S\times T)$ and $\mathrm{L}n\mathrm{Fib}/(S\times T)$ with $[S^{\mathrm{op}}, \mathrm{Fib}/T]$ and $[S^{\mathrm{op}}, \mathrm{L}n\mathrm{Fib}/T]$ respectively, and we will do so freely. In particular, take note of the case of T being a point, so that e.g. $n\mathrm{Fib}/S = [S, n\mathrm{Fib}/*]$.

Lemma 2.2.4. *Suppose $X \to S$ is a right fibration. Then $X \to S$ is fibred in loose n-quasi-groupoids if and only if all fibres over 0-simplices in S are loose n-quasi-groupoids.*

Proof. One direction follows from the fact that loose right n-fibrations are stable under fibre product.

For the other direction, assume that all fibres of X over 0-simplices in S are loose n-quasi-groupoids. Let $m \geq n+2$ and let $\partial\Delta^m \to X$ be an m-shell together with a filling $\Delta^m \to S$ in S. We may as well pull back along this filler and assume from here on that $S = \Delta^m$. We will prove by descending induction on r that any m-shell in X the image of which in S is contained in an m-simplex whose source r-simplex is the degeneracy of 0 can be filled in X (over this simplex).

If $r = m$, this follows from the fact that fibres over 0-simplices in S are loose n-quasi-groupoids. Assuming the induction hypothesis, let $\partial\Delta^m \to X$ be an m-shell in X the source $(m-r)$-simplex of whose image in S is the degeneracy of 0. Make this shell a part of the $(r+1)^{\mathrm{th}}$ face of an $(m+1)$-simplex. First, pull back a 1-simplex from the $(m+1)^{\mathrm{th}}$ vertex to the $(r+1)^{\mathrm{th}}$ vertex over the 1-simplex from 0 to the image of the target 0-simplex of the shell. Now, inducting on the size of $S \subsetneq [m] - \{r+1\}$ we fill in the simplex with vertices $S \cup \{r+1, m+1\}$ as this will have a $\Lambda^{|S|+1}_{|S|+1}$ already filled in and then we can use the fact that $X \to S$ is a right fibration.

But now we are left with a $\Lambda^{m+1}_{r+1,m+1}$ together with the $(m-1)$-simplex with vertices $[m] - \{r+1\}$ (which was in our original data). Consider the $(m+1)^{\mathrm{th}}$ face of this, which is an m-shell whose source $(r+1)$-simplex maps to the degeneracy of 0 in S, and so fills by the induction hypothesis. We are left, finally, with a Λ^{m+1}_{r+1} which of course fills as $X \to S$ is a right fibration. The induction, whence the proof of the proposition, is complete. □

Lemma 2.2.5. *Let $Z \to S$ be fibred in loose n-quasi-groupoids over an n-quasi-category S. Then $\Pi_n(Z) = \pi_n(Z) \to S$ is fibred in n-quasi-groupoids.*

Proof. As $Z \to \pi_n(Z)$ is very surjective, a right horn in $\pi_n(Z)$ will lift to Z, and so the data of a filler of the horn in S will give a filler of the horn in Z, which projects down toa filler of the horn in $\pi_n(Z)$. Therefore, $\pi_n(Z)$ is a right fibration.

For the uniqueness condition, let $\Lambda^m_k \to \pi_n(Z)$ be a right horn in $\pi_n(Z)$ with $m > n$, together with a filler of its image in S to Δ^m. If $k < m$, then the filler exists and is unique in $\pi_n(Z)$ so that because fillers of such horns are unique in S, the filler is a lift, and so the unique lift.

If $k = m$, suppose that we have two fillings of a single horn $\Lambda^m_m \to Z$. I claim that these two fillings have the same m^{th} face. If $m > n+1$, the m^{th} faces have the same n-skeleton, and so must be equal because $\pi_n(Z)$ is an n-quasi-category. If $m = n+1$, we can form a Λ^{m+1}_{m+1} by letting its zeroth face be one filler, its first face the other, and its j^{th} face for $1 < j \leq m$ be the zeroth degeneracy of the $(j-1)^{\text{th}}$ face of the horn (or either filler). This horn projects into the zeroth degeneracy of the filler in S, and so the filler lifts. Extracting the $(m+1)^{\text{th}}$ face of the filled horn upstairs, we see that the two fillers have equivalent m^{th} faces. As these faces have dimension n, they must be equal in $\pi_n(Z)$, proving the claim.

But now both fillers have the same boundary, which is of dimension at least n. As $\pi_n(Z)$ is an n-quasi-category, the two fillers themselves must therefore be equal. We are done. □

Lemma 2.2.6. *Let $X \to S$ be fibred in loose n-quasi-groupoids over an arbitrary simplicial set S. Then there is a right n-fibration $\pi_n(X/S) \to S$ which is initial in the full subcategory of $X\backslash\mathrm{SSets}/S$ consisting of the right n-fibrations.*

Proof. We define an equivalence relation on m-simplices of X by saying that two m-simplices are equivalent if and only if they lie over the same m-simplex in S and are equivalent in the sense of π_n for quasi-categories. This is clearly an equivalence relation, and we define $\pi_n(X/S)$ to be the simplicial set obtained by taking the quotient of the m-simplices of X by this relation; there is of course a natural map $\pi_n(X/S) \to S$, and it is easy to see that for any m-simplex $\Delta^m \to S$ we have a natural isomorphism $\pi_n(X/S) \times_S \Delta^m \simeq \pi_n(X \times_S \Delta^m)$. But $\pi_n(X \times_S \Delta^m)$ is fibred in n-quasi-groupoids over Δ^m by Lemma 2.2.5, and so $\pi_n(X/S) \times_S \Delta^m$ is as well. But then if $\Lambda^m_k \to \pi_n(X/S)$ is a right horn with a filling to $\Delta^m \to S$ downstairs, any lift of this filler will factor uniquely through $\pi_n(X/S) \times_S \Delta^m$, and so this filler will lift and uniquely if $m > n$, which shows that $\pi_n(X/S) \to S$ is a right n-fibration.

But then if $Y \to S$ is any other right n-fibration over S and under X, then for any $\Delta^m \to S$ the map $\Delta^m \times_S X \to \Delta^m \times_S Y$ factors uniquely through $\pi_n(X/S) \times_S \Delta^m = \pi_n(X \times_S \Delta^m)$ over Δ^m, which shows that $X \to Y$ factors through $\pi_n(X/S)$, as desired. □

While it won't be of use to us just yet, we will put here a relativization of the Π_n functors from the last chapter.

Proposition 2.2.7. *Let $X \to S$ be a right fibration. Then there is a right n-fibration $\Pi_n(X/S) \to S$ which is initial in the full subcategory of $X\backslash\mathrm{SSets}/S$ consisting of the right n-fibrations.*

Proof. Let $X^n_0 = X$. Given X^n_i, form a simplicial set X^n_{i+1} together with a morphism $X^n_{i+1} \to S$ by inductively adjoining an m-simplex whenever $m \geq n+2$ and there is an m-shell $\partial\Delta^m \to X^n_i$ together with a filling $\Delta^m \to S$ downstairs which cannot be lifted. Taking X^n to be the union of the X^n_i, we see that X^n is fibred in loose n-quasi-groupoids over S as right horns $\Lambda^q_p \to X^n$ with fillers $\Delta^q \to S$ can be filled by a lift as long as $q \leq n+2$ (as we added no new q-horns), and if $q > n+2$ then we can fill the missing face by lifting the $(q-1)$-shell filler downstairs, and then fill the interior similarly.

Let $\Pi_n(X/S) = \pi_n(X^n/S)$. Any right n-fibration Y over S with a map from X over S will admit a unique extension to $X^n \to Y$ over S as fillers downstairs of q-shells upstairs in Y for $q \geq n+2$ lift uniquely in Y. Therefore, we get a unique map $\Pi_n(X/S) \to Y$, as desired. □

Lemma 2.2.8. *Let Y be a simplicial set fibred in (resp. loose) n-quasi-groupoids over the inner horn Λ^m_k. Then there is a simplicial set Z fibred in (resp. loose) n-quasi-groupoids over Δ^m whose restriction to Λ^m_k is Y.*

Proof. We define simplicial sets and morphisms $Z_r \to \Delta^m$ inductively so that Y is the fibre over Λ^m_k. Let $Z_{-1} = Y$. To form Z_r, we consider all unfilled inner d-horns α in Z_{r-1} whose "tip" (the vertex whose opposite face is omitted) lies over the vertex k of Δ^m. We add simplices $s(\alpha)$ and $t(\alpha)$ to Z_{r-1} of dimensions d and $d-1$, respectively, so that $s(\alpha)$ agrees with α along its j^{th} horn, $\partial_j s(\alpha) = t(\alpha)$, and for any i $0 \leq i \leq d-1$, $\partial_i t(\alpha) = \partial_i \partial_j \alpha$; these simplices clearly have uniquely defined places to map in Δ^m.

Let Z be the union of all the Z_r, so that Z agrees with Y along Λ^m_k. First we show that $Z \to \Delta^m$ is a right fibration. For this it will be enough to show (as Δ^m is a 1-quasi-category) that every right horn in Z whose projection fills in Δ^m can be filled in Z. Notice that by construction, every inner horn whose tip is over k can be filled.

Let β be a right horn in Z whose projection to Δ^m can be filled. We will prove the statement by induction on the number of vertices of β which are *not* over k.

Suppose (as the base of the induction) that β has at most $m-1$ vertices not over k. Then the image of β is contained in Λ^m_k, so that β is in Y and can be filled by the original hypothesis.

Now for the induction step, assume the statement is true for all right horns with fewer non-k vertices than β, and that β has at least m non-k vertices. If the image of β in Δ^m does not contain some vertex $i \neq k$, then the image of β is contained in $\partial_i \Delta^m$ and is thus fillable in Y; we may thus assume that the filling of the image of β contains $\partial_k \Delta^m$. Let β be a Λ^q_p. If vertex p of β lies over $k \in \Delta^m$, then β is an inner horn whose tip is over k, and so is fillable by construction; we may thus assume that p does not lie over k.

Extend $[q]$ to a linearly ordered set $U = [q] \cup \{v\}$ where v lies over k. We think of *beta* as being a subcomplex of Δ^U, where by Δ^U we mean the standard $(q+1)$-simplex whose vertices are indexed by U.

We will fill out all of Δ^U in four steps, a fortiori filling its face v, which fills β as we originally wanted. In the following, we identify subsets S of U with simplices of Δ^U; note that the simplices we have already filled are those simplices $S \subseteq U - \{v\}$ such that $S \not\supseteq U - \{v, p\}$.

Step 1. We first fill all unfilled $S \subseteq U$ such that $S \not\supseteq U - \{v, p, q\}$. If $v \notin S$, then as $S \not\supseteq U - \{v, p\}$ the simplex S has already been filled; we conclude that $v \in S$. There is thus a bijection between such simplices which contain q and those which do not. We fill in Λ^S_q by induction on $|S|$ over all $S \ni q$. We can do this because all such are right horns with fewer vertices not over k than $[q]$, so we can fill them by the induction hypothesis on β.

Step 2. Now, fill in all unfilled $S \subseteq U$ such that $S \not\supseteq U - \{v, p\}$ (this step is vacuous if $p = q$). Again, this forces $v \in S$; also, all such must have $S \supseteq U - \{v, p, q\}$ by step 1. We conclude that there are only two such (if $p < q$), namely $U - \{p, q\}$ and $U - \{q\}$. Thus we are tasked with filling $\Lambda^{U-\{q\}}_p$. As $p > 0$ and $0 \in U - \{q\}$, this is a right horn. As $U - \{q\}$ has fewer vertices not over k than $[q]$ we may fill it by the induction hypothesis on β.

Step 3. Fill in all unfilled $S \subseteq U$ such that $S \not\supseteq U - \{v\}$. As $S \supseteq U - \{v, p\}$ by step 2, there are only two such, namely $U - \{v, p\}$ and $U - \{p\}$, and filling these is the same as filling $\Lambda^{U-\{p\}}_v$. But this is a right horn, and $U - \{p\}$ has fewer vertices not over k than $[q]$. We can thus fill it and proceed.

Step 4. Finally, we fill in the two remaining simplices, $U - \{v\}$ and U. But our previous work gives us the inner horn Λ^U_v with v over k. We can fill this by the original construction.

This concludes the proof that $Z \to \Delta^m$ is a right fibration.

Now let us consider the case where $Y \to \Lambda^m_k$ is fibred in loose n-quasi-groupoids. The same Z as constructed above has all fibres over 0-simplices in Δ^m being loose n-quasi-groupoids, and so Z itself is fibred in loose n-quasi-groupoids by Lemma 2.2.4, finishing the argument in this case.

Finally, suppose that $Y \to \Lambda^m_k$ is fibred in n-quasi-groupoids. We know already that $Z \to \Delta^m$ is fibred in loose n-quasi-groupoids. Let $Z' = \pi_n(Z)$, so that as $\pi_n(Y) = Y$, the restriction of Z' to Λ^m_k is still Y. But then by Lemma 2.2.6 (using that Δ^m is an n-quasi-category), the simplicial set Z' is fibred in n-quasi-groupoids over Δ^m. We are done. $\square$

Lemma 2.2.9. *Let Y be fibred in (loose) n-quasi-groupoids over $\partial\Delta^m$ for some $m \geq n+3$. Then there is a simplicial set Z fibred in (loose) n-quasi-groupoids over Δ^m whose restriction to $\partial\Delta^m$ is Y.*

Proof. First let us consider the loose case. Let $Z_0 = Y$ and let Z_i be obtained from Z_{i-1} by adding one new r-simplex for every unfilled r-shell in Z_{i-1} whose image in Δ^m is filled by a degeneracy of Δ^m. Clearly this leaves the fibre over $\partial\Delta^m$ unchanged, so we may let Z be the union of these, and its fibre over $\partial\Delta^m$ will be Y.

I claim that Z is fibred in loose n-quasi-groupoids over Δ^m. Indeed, let $\partial\Delta^q \to Z$ be a q-shell in Z together with a filler to $\Delta^q \to \Delta^m$ downstairs, with $q \geq n+2$. If $\Delta^q \to \Delta^m$ is surjective, then the shell fills by construction. If $\Delta^q \to \Delta^m$ is not surjective then it factors through $\partial\Delta^m$ so that the q-shell factors through Y and the filler lifts by assumption on Y.

Now let $\Lambda^q_p \to Z$ be a right horn together with a filler of its image $\Delta^q \to \Delta^m$. If $q < m$, then $\Delta^q \to \Delta^m$ factors through (some face of) $\partial\Delta^m$, and so the horn filler lifts by assumption on Y. If $q \geq m \geq n+3$, the missing face of the horn has dimension at least $n+2$ and fills as we can lift fillers of shells of dimension at least $n+2$. But then what remains is a shell of dimension $q > n+2$, and so this shell filler lifts as well. We are done with the loose case.

For the case of being fibred in n-quasi-groupoids, let $Z' = \pi_n(Z)$, so that Z' is fibred in n-quasi-groupoids over Δ^m, and agrees with Y when restricted to $\partial\Delta^m$. We are done. ☐

Theorem 2.2.10. *Let S be a simplicial set. Then $n\mathrm{Fib}/S$ and $\mathrm{L}n\mathrm{Fib}/S$ are (large) loose $(n+1)$-quasi-categories.*

Proof. First, notice that from Lemmas 2.2.8 and 2.2.9 it follows immediately that $n\mathrm{Fib}/*$ and $\mathrm{L}n\mathrm{Fib}/*$ are loose $(n+1)$-quasi-categories. But then $n\mathrm{Fib}/S = [S, n\mathrm{Fib}/*]$ and $\mathrm{L}n\mathrm{Fib}/S = [S, \mathrm{L}n\mathrm{Fib}/*]$ are loose $(n+1)$-quasi-categories as well. ☐

One corollary of this theorem is that $n\mathrm{Fib}/*$ and $\mathrm{L}n\mathrm{Fib}/*$ are loose $(n+1)$-quasi-categories. Considering that these quasi-category have as objects all (loose) n-quasi-groupoids (a (loose) right n-fibration over a point is a (loose) n-quasi-groupoid), it makes sense to give these quasi-categories special names. In fact, for us this quasi-category will serve an analogous purpose to the category of sets in sheaf theory as well as the weak 2-category of groupoids in the classical theory of stacks.

Definition 2.2.11. The quasi-categories $n\underline{\mathrm{QGpd}}$ and $\mathrm{L}n\underline{\mathrm{QGpd}}$ are defined to be the loose $(n+1)$-quasi-categories $n\mathrm{Fib}/*$ and $\mathrm{L}n\mathrm{Fib}/*$ of (resp. loose) right n-fibrations over a point.

2.3 Morphisms in $n\mathrm{Fib}/S$ and $\mathrm{L}n\mathrm{Fib}/S$

We now have two competing notions of morphism between simplicial sets X and Y fibred over S. We can look at the "external" quasi-groupoid $[X, Y]_S$ of morphisms over S, and we can also look at the "internal" quasi-groupoid $\underline{\mathrm{Hom}}_{\mathrm{Fib}/S}(X, Y)$. These two quasi-groupoids are in fact homotopy equivalent to one another in a natural way, as we will show in §2.3.2.

2.3.1 Equivalence of Right Fibrations

Our first step is to establish a natural notion of equivalence of right fibrations (in Theorem 2.3.4), analyzing certain $\underline{\mathrm{Hom}}$ spaces occurring in right fibrations along the way.

Lemma 2.3.1. *Let $X \to \Delta^1$ be a right fibration (so that X is a quasi-category), let x and y be objects of X over 0 and 1 respectively, and let $z \xrightarrow{f} y$ be a 1-simplex over $0 \longrightarrow 1$. Consider $\{0\} \cup \partial_0\Delta^2$ as a subcomplex of Δ^2 and let $\sigma : \{0\} \cup \partial_0\Delta^2 \to X$ be defined by taking $\{0\}$ to x and $\partial_0\Delta^2$ to f. Then the natural morphisms of simplicial sets*

$$\begin{tikzcd} & \underline{\mathrm{Fill}}(\sigma, \Delta^2) \arrow[dl] \arrow[dr] & \\ \underline{\mathrm{Hom}}(x, y) & & \underline{\mathrm{Hom}}(x, z) \end{tikzcd}$$

(corresponding to the first and second faces of Δ^2) are very surjective. In particular, there is a homotopy equivalence of quasi-groupoids $\underline{\mathrm{Hom}}_X(x,y) \to \underline{\mathrm{Hom}}_X(x,z)$.

Proof. To show very surjectivity of $\underline{\mathrm{Fill}}(\sigma,\Delta^2) \to \underline{\mathrm{Hom}}(x,z)$, notice that an m-shell of $\underline{\mathrm{Fill}}(\sigma,\Delta^2)$ together with a filler of its image in $\underline{\mathrm{Hom}}(x,z)$ amounts to a morphism

$$(\Lambda^2_1 \times \Delta^m) \cup (\Delta^2 \times \partial\Delta^m) \longrightarrow X,$$

as $\partial_0\Delta^2 \times \Delta^m \to X$ is forced by the fact that the filler ought to land in the $\underline{\mathrm{Fill}}$ space. But this shape can be filled by inner horn fillings by Lemma 1.2.6, and so as we have a filler in Δ^1 (the appropriate degeneracy of the nondegenerate 1-simplex in Δ^1), we can lift the filler to fill the shape in X, giving a morphism $\Delta^m \times \Delta^2 \to X$ agreeing with σ on $\Delta^m \times (\{0\} \cup \partial_0\Delta^2)$, i.e. an m-simplex in $\underline{\mathrm{Fill}}(\sigma,\Delta^2)$, as we wanted.

For very surjectivity of $\underline{\mathrm{Fill}}(\sigma,\Delta^2) \to \underline{\mathrm{Hom}}(x,y)$, we proceed similarly, only this time the partially filled shape will be a morphism

$$(\Lambda^2_2 \times \Delta^m) \cup (\Delta^2 \times \partial\Delta^m) \longrightarrow X.$$

As this shape can be filled by a succession of inner horn fillers followed by a filling of a right horn (again by Lemma 1.2.6), we can lift the given filler in Δ^1 to X, obtaining the m-simplex in $\underline{\mathrm{Fill}}(\sigma,\Delta^2)$ that we want. $\square$

Lemma 2.3.2. *Let $X \to \Delta^1$ and $Y \to \Delta^1$ be right fibrations (so that X and Y are quasi-categories), and let $F : X \to Y$ be a morphism over Δ^1. If the restrictions of F to the fibres of X and Y over 0 and 1 are homotopy equivalences of quasi-groupoids, then F is an equivalence of quasi-categories.*

Proof. By Theorem 1.5.44, it will be enough to show that F is essentially surjective and a homotopy equivalence on $\underline{\mathrm{Hom}}$ spaces. Essential surjectivity is immediate as F is essentially surjective on fibres over objects in Δ^1, and every object in X lies in one such fibre.

Now let x and x' be objects in X. We will argue that

$$\underline{\mathrm{Hom}}_X(x,x') \longrightarrow \underline{\mathrm{Hom}}_Y(F(x),F(x'))$$

is a homotopy equivalence, completing the proof.

If both x and x' lie over the same 0-simplex ϵ in Δ^1, then all 1-simplices from one to the other will lie over the degeneracy of ϵ, from which it follows that

$$\underline{\mathrm{Hom}}_X(x,x') = \underline{\mathrm{Hom}}_{X_\epsilon}(x,x')$$

and $\underline{\mathrm{Hom}}_Y(F(x),F(x')) = \underline{\mathrm{Hom}}_{Y_\epsilon}(F(x),F(x'))$. But then as $X_\epsilon \longrightarrow Y_\epsilon$ is a homotopy equivalence,

$$\underline{\mathrm{Hom}}_{X_\epsilon}(x,x') \longrightarrow \underline{\mathrm{Hom}}_{Y_\epsilon}(F(x),F(x'))$$

is a homotopy equivalence (by say Theorem 1.5.44), whence

$$\underline{\mathrm{Hom}}_X(x,x') \longrightarrow \underline{\mathrm{Hom}}_Y(F(x),F(x'))$$

is a homotopy equivalence as well.

If x lies over 1 and x' lies over 0, then $\underline{\mathrm{Hom}}_X(x,x') = \underline{\mathrm{Hom}}_Y(x,x') = \emptyset$ and the statement is vacuous.

Finally, suppose that x lies over 0 and x' lies over 1. Let $x'' \xrightarrow{f} x'$ lie over $0 \longrightarrow 1$, and define the corresponding $\sigma : \{0\} \cup \partial_0\Delta^2 \to X$ as in Lemma 2.3.1. Then we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_X(x,x'') \arrow[d] & \underline{\mathrm{Fill}}_X(\sigma,\Delta^2) \arrow[l] \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_X(x,x') \arrow[d] \\
\underline{\mathrm{Hom}}_Y(F(x),F(x'')) & \underline{\mathrm{Fill}}_Y(F(\sigma),\Delta^2) \arrow[l] \arrow[r] & \underline{\mathrm{Hom}}_Y(F(x),F(x'))
\end{tikzcd}$$

with all horizontal arrows homotopy equivalences (in fact, very surjective maps) by Lemma 2.3.1. Since the leftmost vertical arrow is also a homotopy equivalence, we conclude that the rightmost is a homotopy equivalence, as desired. □

Definition 2.3.3. Let X and Y be right fibrations over a simplicial set S, and let $F : X \to Y$ be a morphism of simplicial sets over S (i.e. such that

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutes). Then we say that F is an *equivalence of right fibrations over S* if there is a simplicial set P and very surjective morphisms $P \to X$ and $P \to Y$ such that F factors as a section of $P \to X$ followed by $P \to Y$ and such that

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutes.

Theorem 2.3.4. *Let X and Y be right fibrations over a simplicial set S, and let $F : X \to Y$ be a morphism of simplicial sets over S. Then the following are equivalent:*

(i) The morphism F is an equivalence of right fibrations over S.

(ii) There is a morphism $G : Y \to X$ over S and 1-simplices $1_X \to GF$ and $1_Y \to FG$ in the quasi-groupoids $[X, X]_S$ and $[Y, Y]_S$ (so that the 1-simplices are quasi-isomorphisms).

(iii) The morphism F induces homotopy equivalences on all fibres over 0-simplices in S.

Proof. That (i)⇒(ii) follows by the same argument as in Theorem 1.5.44, after we notice that $[X, P]_S$ is very surjective onto both $[X, Y]_S$ and $[X, X]_S$ (and symmetrically for Y), since all three are obtained by changing base along the morphism $1 \to [X, S]$ corresponding to the projection $X \to S$.

For (ii)⇒(iii), notice that for a 0-simplex s of S, we have restricted morphisms $F_s : X_s \to Y_s$ and $G_s : Y_s \to X_s$, and moreover as the homotopies $1_X \to GF$ and $1_Y \to FG$ are in $[X, X]_S$ and $[Y, Y]_S$ respectively, they also restrict to homotopies $1_{X_s} \to G_sF_s$ and $1_{Y_s} \to F_sG_s$, completing the proof that F is a homotopy equivalence on fibres.

To show that (iii)⇒(i), note that by Lemma 2.3.2 we have that for any 1-simplex $f : \Delta^1 \to S$, the restriction to fibre products $F_f : X_f \to Y_f$ is an equivalence of quasi-categories. It follows from this that for any 0-simplices x and x' in X lying over the source and target of a 1-simplex f in S, the induced morphism

$$\underline{\mathrm{Hom}}_{X_f}(x, x') \to \underline{\mathrm{Hom}}_{Y_f}(F(x), F(x'))$$

is a homotopy equivalence.

Consider now a morphism $\xi : \Delta^r \to S$. The fibres X_ξ and Y_ξ are quasi-categories, and I claim that $F_\xi : X_\xi \to Y_\xi$ is an equivalence of quasi-categories. Indeed, F_ξ is essentially surjective because every object of Y_ξ lies in a fibre over a 0-simplex s of S, and the morphism $X_s \to Y_s$ is a homotopy equivalence (so essentially surjective).

Let x and x' be objects of X_ξ, and consider the quasi-groupoid $\underline{\mathrm{Hom}}_{X_\xi}(x, x')$. If x and x' map to vertices i and j of Δ^r, then as there is a unique morphism $i \to j$ in Δ^r corresponding to a 1-simplex f in S, the induced morphism

$$\underline{\mathrm{Hom}}_{X_f}(x, x') \to \underline{\mathrm{Hom}}_{X_\xi}(x, x')$$

is in fact an isomorphism of simplicial sets. After similar remarks for Y, we conclude that we have a commutative diagram

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{X_f}(x,x') \arrow[r, "\sim"] \arrow[d] & \underline{\mathrm{Hom}}_{X_\xi}(x,x') \arrow[d] \\ \underline{\mathrm{Hom}}_{Y_f}(F(x),F(x')) \arrow[r, "\sim"] & \underline{\mathrm{Hom}}_{Y_\xi}(F(x),F(x')) \end{tikzcd}$$

demonstrating that

$$\underline{\mathrm{Hom}}_{X_\xi}(x,x') \to \underline{\mathrm{Hom}}_{Y_\xi}(F(x),F(x'))$$

is a homotopy equivalence, whence by Theorem 1.5.44 that F_ξ is an equivalence of quasi-categories.

For every ξ as in the previous paragraph, define the simplicial set P_ξ as in the proof of Theorem 1.5.44 to have m-simplices which are ordered triples (σ, τ, φ) where σ is an m-simplex of X_ξ, τ is an m-simplex of Y_ξ, and $\varphi : F_\xi(\sigma) \to \tau$ is a quasi-isomorphism. As these quasi-isomorphisms must map to quasi-isomorphisms in $[\Delta^m, \Delta^r]$ and the only quasi-isomorphisms in this 1-quasi-category are the identities, we see that the diagram

$$\begin{tikzcd} & P_\xi \arrow[dl] \arrow[dr] & \\ X_\xi \arrow[dr] & & Y_\xi \arrow[dl] \\ & \Delta^r & \end{tikzcd}$$

commutes.

Moreover, if ξ is an r-simplex of S factoring through an s-simplex ζ of S, then it is clear that the natural morphism $P_\xi \to P_\zeta$ is a morphism of simplicial sets such that

$$\begin{tikzcd} & & P_\xi \arrow[d] \arrow[dll] \arrow[drr] & & \\ & & P_\zeta \arrow[dl] \arrow[dr] & & \\ X_\xi \arrow[r] \arrow[ddrr] & X_\zeta \arrow[dr] & & Y_\zeta \arrow[dl] & Y_\xi \arrow[l] \arrow[ddll] \\ & & \Delta^s & & \\ & & \Delta^r \arrow[u] & & \end{tikzcd}$$

commutes, and that these morphisms $P_\xi \to P_\zeta$ are compatible with one another in the obvious sense. Therefore, we can define P to be the colimit of the P_ξ. We end up with a simplicial set P with morphisms $P \to X$ and $P \to Y$ such that

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutes.

I claim that $P \to X$ and $P \to Y$ are very surjective. Indeed, given the data of an m-simplex in X and a lift of its boundary to P, we can map the simplex down to an m-simplex ξ of S, and then pulling back along ξ, the m-simplex of X and the lift of its boundary to P restrict to an m-simplex of X_ξ and a lift of its boundary to P_ξ. As $P_\xi \to X_\xi$ is very surjective, the interior can be lifted and then mapped back to P, as desired. Similar remarks hold for Y, and so we are done. □

Remark 2.3.5. Note that for formal reasons the corollaries following Theorem 1.5.44 have corresponding versions here.

Remark 2.3.6. This theorem gives one sense in which a simplicial set fibred in quasi-groupoids acts as a quasi-groupoid-valued pseudo-functor: To check that a morphism of right fibrations is an equivalence, it is enough to check the equivalence fibrewise. Note that it is *not* enough that fibres be homotopy equivalent to one another only; there must be a map delivering the homotopy equivalence. This corresponds to the "natural" part of a natural isomorphism of functors.

2.3.2 Relating the Two Notions of Morphism

Lemma 2.3.7. *Let X be a right fibration over S, and suppose $\Lambda^m_k \to X$ is a right horn with a filler $\Delta^m \to S$ downstairs. Then the filled faces of any two lifts of this filler are equivalent (in the sense of Lemma 1.5.7)*

Proof. Form a morphism $\Lambda^{m+1}_{k+1} \to X$ whose j^{th} face for $j < k$ is the $(k-1)^{\text{th}}$ degeneracy of the j^{th} face of the horn, whose k^{th} face is one filler, whose$(k+1)^{\text{th}}$ face is the other, and whose j^{th} face for $j > k+1$ is the k^{th} degeneracy of the $(j-1)^{\text{th}}$ face of the horn. Then this is a right horn with a filler downstairs (which is the k^{th} degeneracy of the original filler) and so fills, providing for the equivalence we were looking for. □

Definition 2.3.8. Let $X_0, \ldots, X_s$ be simplicial sets, and $f_i : X_{i-1} \to X_i$ morphisms of simplicial sets. Define the simplicial set $C(f_1, \ldots, f_s)$ to have m-simplices which are ordered $2s+1$-tuples

$$(r_0, \ldots, r_{s-1}, \alpha_0, \ldots, \alpha_s)$$

such that $-1 \le r_0 \le r_1 \le \cdots \le r_{s-1} \le r_s := m$, for all i, α_i is an r_i-simplex of X_i, and for all $i < s$, $f_{i+1}(\alpha_i)$ is the target r_i-simplex of α_{i+1}. Here for notational simplicity every simplicial set is defined to have a unique (-1)-simplex (think of this simplex as empty). Define

$$\partial_j(r_0, \ldots, r_{s-1}, \alpha_0, \ldots, \alpha_s) = (r'_0, \ldots, r'_{s-1}, \alpha'_0, \ldots, \alpha'_{s-1}, \partial_j \alpha_s),$$

where $r'_i = r_i - 1$ and $\alpha'_i = \partial_{j+r_i-m}\alpha_i$ if $j \ge m - r_i$ and otherwise $r'_i = r_i$ and $\alpha'_i = \alpha_i$; define degeneracies similarly. This clearly comprises a simplicial set.

Notice that $C(f_1, \ldots, f_s)$ comes with a natural map to $(\Delta^s)^{\text{op}}$, namely that which takes an m-simplex (ordered $(2s+1)$-tuple $(r_0, \ldots, r_{s-1}, \alpha_0, \ldots, \alpha_s)$) to the nonincreasing sequence of elements of $[s]$ consisting of $r_i - r_{i-1}$ instances of i (where we define $r_{-1} = -1$ and $r_s = m$). It is immediate that the fibre of this morphism over i is X_i. We will henceforth think of $C(f_1, \ldots, f_s)$ as a simplicial set together with a morphism

$$C(f_1, \ldots, f_s) \to (\Delta^s)^{\text{op}}.$$

Notice moreover that there are natural maps $\partial_i : C(g_1, \ldots, g_{s-1}) \to C(f_1, \ldots, f_s)$, where $(g_1, \ldots, g_{s-1})$ is obtained from $(f_1, \ldots, f_s)$ by either omitting f_1 (this is ∂_0) omitting f_s (this is ∂_s), or by composing f_i with f_{i+1} (this is ∂_i). In fact, these maps are easily seen to be simply the pullbacks

$$\begin{tikzcd} C(g_1, \ldots, g_{s-1}) \arrow[r, "\partial_i"] \arrow[d] & C(f_1, \ldots, f_s) \arrow[d] \\ (\Delta^{s-1})^{\text{op}} \arrow[r, "\partial_i"] & (\Delta^s)^{\text{op}}. \end{tikzcd}$$

Finally, note that if $f : X \to Y$ is a morphism, there is a natural map $X \times (\Delta^1)^{\mathrm{op}} \to C(f)$ given by taking an m-simplex (ξ, ϵ) (where ϵ is a nonincreasing sequence of $(m-r)$ 1's and $(r+1)$ 0's) to $(r, \alpha, f(\xi))$, where α is the target r-simplex of ξ.

Proposition 2.3.9. *Suppose $X_0, \ldots, X_s$ are right n-fibrations over S and $f_i : X_{i-1} \to X_i$ are morphisms over S. Then $C(f_1, \ldots, f_s)$ has a natural map to S and is a right n-fibration over $S \times (\Delta^s)^{\mathrm{op}}$.*

Proof. Define a map to S by taking an m-simplex $(r_0, \ldots, r_{s-1}, \alpha_0, \ldots, \alpha_s)$ to the image of α_s under $X_s \to S$; it follows that the image of α_i is the target r_i-simplex of this simplex.

Now, let $\Lambda^m_k \to C(f_1 \ldots, f_s)$ be a right horn together with a filler downstairs $\Delta^m \to S \times (\Delta^s)^{\mathrm{op}}$. Let i be the greatest element of $[s]$ occurring in the projection of the filler to $(\Delta^s)^{\mathrm{op}}$. The horn has α_j filled in for all $j < i$ because $r_j < r_i$ and so α_j, being as it is the target r_j-simplex of the horn, is already filled. The horn in $C(f_1, \ldots, f_s)$ only defines a right horn in X_i (i.e. not our desired α_i), but we can fill it over S (uniquely if $m > n$) as X_i is a right n-fibration; call this filler α_i. Finally, this filler extends uniquely to a filler of the horn in $C(f_1, \ldots, f_s)$, as we are forced to define $\alpha_j = f_j(f_{j-1}(\cdots f_{i+1}(\alpha_i)\cdots))$ for each $j > i$. We are done. □

Remark 2.3.10. Let $\mathcal{F}_S$ denote the full subcategory of SSets$/S$ consisting of the right fibrations over S. Then the preceding proposition, together with the remarks which precede it, establish that C actually defines a canonical quasi-functor

$$C : \mathcal{F}_S \longrightarrow \mathrm{Fib}/S,$$

because to define a quasi-functor on (the nerve of) a category we need only tell to which s-simplex we send a given chain of composable morphisms $(f_1, \ldots, f_s)$ (and check that these assignments are compatible). Theorem 2.3.12 below gives a sense in which this quasi-functor is "an equivalence of simplicially enriched categories." Of course, Fib$/S$ is not a category, so this statement should be taken "in quotes only."

Let $X_0, \ldots X_s$ be right n-fibrations over a simplicial set S, and let f_i be an m-simplex in $[X_{i-1}, X_i]^{\mathrm{op}}_S$, so that we may think of f_i as a morphism $f : X_{i-1} \times (\Delta^m)^{\mathrm{op}} \to X_i \times (\Delta^m)^{\mathrm{op}}$ over $S \times (\Delta^m)^{\mathrm{op}}$. Then $C(f_1, \ldots, f_s)$ is a right n-fibration over $S \times (\Delta^m)^{\mathrm{op}} \times (\Delta^s)^{\mathrm{op}}$, and so can be identified with a morphism

$$C(f_1, \ldots, f_s) : \Delta^m \times \Delta^s \to n\mathrm{Fib}/S$$

whose restriction to $\Delta^m \times \{i\}$ is constantly X_i for all i. Therefore, $C(f_1, \ldots, f_s)$ identifies an m-simplex in $\underline{\mathrm{Hom}}_{n\mathrm{Fib}/S}(X_0, \ldots, X_s)$. As this mapping $(f_1, \ldots, f_n) \mapsto C(f_1, \ldots, f_n)$ clearly respects boundaries and degeneracies, we conclude that we have a morphism of simplicial sets

$$C : \ [X_0, X_1]^{\mathrm{op}}_S \times \cdots \times [X_{s-1}, X_s]^{\mathrm{op}}_S \longrightarrow \underline{\mathrm{Hom}}_{n\mathrm{Fib}/S}(X_0, X_1, \ldots, X_s).$$

Notice moreover that this morphism is a monomorphism, as for example we may recover $f_i(\alpha)$ for an r-simplex α by looking at the unique simplex of $C(f_1, \ldots, f_s)$ of the form

$$(0, \ldots, 0, r, r, \ldots, \emptyset, \ldots, \emptyset, \alpha, \ldots),$$

where of course the r's begin at index i and α is at the term for simplices in X_i. Here $f_i(\alpha)$ is simply the term following α.

Definition 2.3.11. Let $X_0, \ldots, X_s$ be right fibrations over S, $f_i : X_{i-1} \to X_i$ morphisms over S, and Z an s-simplex with i^{th} object X_i in Fib$/S$, i.e. a right fibration over $S \times (\Delta^s)^{\mathrm{op}}$ with given isomorphisms $Z|_i \simeq X_i$. We define an *embedding of* $(f_1, \ldots, f_s)$ *into* Z to be a morphism

$$\varphi : C(f_1 \ldots, f_s) \to Z$$

over $S \times (\Delta^s)$op compatible with the given isomorphisms over each $i \in [s]$.

Theorem 2.3.12. *Let $X_0 \dots, X_s$ be right n-fibrations over S. Define the simplicial set $P(X_0, \dots, X_s)$ to have m-simplices which are ordered $(s+2)$-tuples $(f_1, \dots, f_s, Z, \varphi)$ of m-simplices f_i in $[X_{i-1}, X_i]_S^{\mathrm{op}}$, an m-simplex Z in $\underline{\mathrm{Hom}}_{n\mathrm{Fib}/S}(X_0, \dots, X_s)$, and a morphism $\varphi : C(f_1, \dots, f_s) \to Z$ over $S \times (\Delta^m)^{\mathrm{op}} \times (\Delta^s)^{\mathrm{op}}$ respecting the given fibres $X_i \times (\Delta^m)^{\mathrm{op}}$ over i. Then the natural projections*

$$\begin{tikzcd} & P(X_0, \dots, X_s) \arrow[dl] \arrow[dr] & \\ {[X_0, X_1]_S^{\mathrm{op}} \times \cdots \times [X_{s-1}, X_s]_S^{\mathrm{op}}} & & \underline{\mathrm{Hom}}_{n\mathrm{Fib}/S}(X_0, \dots, X_s) \end{tikzcd}$$

are both very surjective. In particular, the morphism of quasi-groupoids

$$C : \; [X_0, X_1]_S^{\mathrm{op}} \times \cdots \times [X_{s-1}, X_s]_S^{\mathrm{op}} \longrightarrow \underline{\mathrm{Hom}}_{n\mathrm{Fib}/S}(X_0, \dots, X_s)$$

defined above is a homotopy equivalence.

Proof. Notice that C being a homotopy equivalence follows from the fact that we have a canonical section

$$[X_0, X_1]_S^{\mathrm{op}} \times \cdots \times [X_{s-1}, X_s]_S^{\mathrm{op}} \to P(X_0, \dots, X_s)$$

given by taking an m-simplex $(f_1, \dots, f_s)$ where $f_i : X_{i-1} \times (\Delta^m)^{\mathrm{op}} \to X_i$ to

$$(f_1, \dots, f_s, C(f_1, \dots, f_s), 1_{C(f_1, \dots, f_s)}),$$

and this section clearly factorizes C. It remains to prove the very surjectivity statements.

First we treat the morphism

$$P(X_0, \dots, X_s) \longrightarrow [X_0, X_1]_S^{\mathrm{op}} \times \cdots \times [X_{s-1}, X_s]_S^{\mathrm{op}}.$$

Given an m-simplex $(f_1, \dots, f_s)$ of $[X_0, X_1]_S^{\mathrm{op}} \times \cdots \times [X_{s-1}, X_s]_S^{\mathrm{op}}$ and a lifting of its boundary to a right n-fibration Z' over $S \times (\partial\Delta^m)^{\mathrm{op}} \times (\Delta^s)^{\mathrm{op}}$ together with a morphism $C(f_1', \dots, f_s') \to Z'$, where

$$f_i' = f_i|_{X_{i-1} \times (\partial\Delta^m)^{\mathrm{op}}},$$

we will fill in Z' to a right n-fibration Z admitting an extension of the given morphism to a morphism $C(f_1, \dots, f_s) \to Z$.

If $m = 0$, $Z' = \emptyset$ and we may simply take $Z = C(f_1, \dots, f_s)$. We thus assume that $m > 0$.

Let $p : Z' \to C(f_1', \dots, f_s')$ be a quasi-inverse to the given morphism (which exists by Theorem 2.3.4 and the fact that the morphism is in fact an isomorphism on fibres over 0-simplices). Define a simplicial set Y to have q-simplices which are ordered pairs (α, β) where α is a q-simplex of $C(f_1, \dots, f_s)$ and β is a lift of the simplices of α not lying over the interior of $(\Delta^m)^{\mathrm{op}}$ to Z' along p (so that the lift is compatible with shared boundaries). Then evidently $Z' = Y \times_{\Delta^m} \partial\Delta^m$.

I claim that Y is a right fibration over $S \times (\Delta^m)^{\mathrm{op}} \times (\Delta^s)^{\mathrm{op}}$. Indeed, let $\Lambda_p^q \to Y$ be a right horn with a filling $\Delta^q \to S \times (\Delta^m)^{\mathrm{op}} \times (\Delta^s)^{\mathrm{op}}$. If the filling lies over $(\partial\Delta^m)^{\mathrm{op}}$, then the horn is contained in Z' and can be filled there. If the interior of the horn lies over the interior of $(\Delta^m)^{\mathrm{op}}$, then we fill the image of the horn in $C(f_1, \dots, f_s)$. If the k^{th} face of the filler does not lie over $(\partial\Delta^m)^{\mathrm{op}}$, then we need find no new lifting data.

Otherwise, the filler we just lifted will have a k^{th} face which lives in $C(f_1', \dots, f_s')$ and comes equipped with a lift of its boundary to Z' along p. Taking fibres over this k^{th} face (as a morphism $\Delta^{q-1} \to S \times (\Delta^m)^{\mathrm{op}} \times (\Delta^s)^{\mathrm{op}}$), we see that as p restricted to these fibres is an equivalence of quasi-categories, it must be surjective on connected components of $\underline{\mathrm{Fill}}$ spaces (by Theorem 1.5.44), and so the lift of the boundary must have a filler in Z' mapping into the connected component of the k^{th} face we just found. We conclude that this k^{th} face is equivalent (in the sense of Lemma 1.5.7) to a $(q-1)$-simplex in the image of p, and so (working in the fibre over Δ^q of $C(f_1, \ldots, f_s)$) we may replace the filler of the horn with another which has this $(q-1)$-simplex as its k^{th} face (and this filler will map where we want it to because we are working in the fibre). We thus obtain a filler lift in $C(f_1, \ldots, f_s)$ together with a lift of its k^{th} face to Z', as desired.

As Y has fibres over 0-simplices which are n-quasi-groupoids, we can let $Z = \pi_n(Y/S \times (\Delta^m)^{\text{op}} \times (\Delta^s)^{\text{op}})$, so that Z' is still the fibre of Z over $(\partial\Delta^m)^{\text{op}}$, and now Z is a right n-fibration. We have in addition a commutative diagram of simplicial sets

$$\begin{tikzcd} C(f'_1, \ldots, f'_s) \arrow[d] & Z' \arrow[l, "p"'] \arrow[d] \\ C(f_1, \ldots, f_s) & Z, \arrow[l, "q"'] \end{tikzcd}$$

where q is the projection $Z \to C(f_1, \ldots, f_s)$.

Let P' be a simplicial set with very surjective maps onto $C(f'_1, \ldots, f'_s)$ and Z' such that the given $C(f'_1, \ldots, f'_s) \to Z'$ as well as p factor as sections of the respecitve very surjective maps composed with the other (we know that such exists by the construction of p). By performing the same construction on P' as we did on Z', we obtain a simplicial set P which is very surjective onto both Z and $C(f_1, \ldots, f_s)$, and which comes equipped with a morphism $P' \to P$ compatible with all the projections, so producing a commutative diagram:

$$\begin{tikzcd} C(f'_1, \ldots, f'_s) \arrow[d] & P' \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[d] \\ C(f_1, \ldots, f_s) & P \arrow[l] \arrow[r] & Z. \end{tikzcd}$$

Let $i' : C(f'_1, \ldots, f'_s) \to P'$ be the section which gives rise to the map $C(f'_1, \ldots, f'_s) \to Z'$, so that we can compose to produce a morphism $C(f'_1, \ldots, f'_s) \to P$. Using the facts that $P \to C(f_1, \ldots, f_s)$ is very surjective and $C(f'_1, \ldots, f'_s) \to C(f_1, \ldots, f_s)$ is a monomorphism, we can find a lift $C(f_1, \ldots, f_s) \to P$ making the diagram

$$\begin{tikzcd} C(f'_1, \ldots, f'_s) \arrow[r] \arrow[d] & P \arrow[d] \\ C(f_1, \ldots, f_s) \arrow[r, equal] \arrow[ur] & C(f_1, \ldots, f_s) \end{tikzcd}$$

commute, which by construction is a section of $P \to C(f_1, \ldots, f_s)$. But then if we compose this section with the projection $P \to Z$, we clearly obtain a map $C(f_1, \ldots, f_s) \to Z$ which extends the given map $C(f'_1, \ldots, f'_s) \to Z'$, as we wanted.

Now we treat the morphism

$$P(X_0, \ldots, X_s) \longrightarrow \underline{\text{Hom}}_{n\text{Fib}/S}(X_0, \ldots, X_s)\ ,$$

which will complete the proof. Given an m-simplex of $\underline{\text{Hom}}_{n\text{Fib}/S}(X_0, \ldots, X_s)$ together with a lift of its boundary upstairs, we have a right n-fibration Z over $S \times (\Delta^m)^{\text{op}} \times (\Delta^s)^{\text{op}}$ and morphisms $f'_i : X_{i-1} \times (\partial\Delta^m)^{\text{op}} \to X_i \times (\partial\Delta^m)^{\text{op}}$ over $S \times (\partial\Delta^m)^{\text{op}}$ together with a morphism $C(f'_1, \ldots, f'_s) \to Z$ over $S \times (\Delta^m)^{\text{op}} \times (\Delta^s)^{\text{op}}$. We wish to extend the f'_i to morphisms $f_i : X_{i-1} \times (\Delta^m)^{\text{op}} \to X_i \times (\Delta^m)^{\text{op}}$ over $S \times (\Delta^m)^{\text{op}}$ such that $C(f'_1, \ldots, f'_s) \to Z$ extends to $C(f_1, \ldots, f_s) \to Z$.

Our plan of attack will be to build up the simplicial sets $X_i \times (\Delta^m)^{\text{op}}$ from $X_i \times (\partial\Delta^m)^{\text{op}}$ simplex by simplex, extending the morphisms f'_i as appropriate, so as to allow the desired extension of the map to Z.

More precisely, suppose we are given morphisms $g_i : Y_{i-1} \to Y_i$ where the Y_i are simplicial sets with $X_i \times (\partial\Delta^m)^{\text{op}} \subseteq Y_i \subseteq X_i \times (\Delta^m)^{\text{op}}$ and we are given also an extension of $C(f'_1, \ldots, f'_s) \to Z$ to $C(g_1, \ldots, g_s) \to Z$. Let j be least so that $Y_i = X_i$ for all $i > j$. Let ξ be a q-simplex of X_j such that ξ is not contained in Y_j but $\partial\xi$ is so contained. We will show that we can extend g_{j+1} to h_{j+1} defined on $Y_j \cup \xi$ (with no hypothesis here if $j = s$) so that if $h_i = g_i$ for $i \neq j+1$, we can extend the given map to a map $C(h_1, \dots, h_s) \to Z$. We base the induction on the case $Y_i = X_i \times (\partial\Delta^m)^{\text{op}}$ for all i with $g_i = f_i'$, and then clearly this inductive step will give the desired extension.

First we will define h_{j+1} (if necessary, i.e. if $j < s$). Indeed, as $\partial\xi$ is in Y_j, this boundary defines a morphism

$$\partial\Delta^q \times (\Delta^1)^{\text{op}} \longrightarrow C(g_{j+1}) \longrightarrow C(g_1, \dots, g_s) \longrightarrow Z.$$

Defining the map $\Delta^q \times \{0\} \to Z$ to be $\xi : \Delta^q \to X_j \hookrightarrow Z$ itself, we obtain a shape

$$\Delta^q \times \{0\} \cup \partial\Delta^q \times (\Delta^1)^{\text{op}} \to Z$$

which can be filled in $S \times (\Delta^m)^{\text{op}} \times (\Delta^s)^{\text{op}}$ by the map defined by

$$\xi : \Delta^q \longrightarrow X_j \longrightarrow S \times (\Delta^m)^{\text{op}}$$

and the morphism $(\Delta^1)^{\text{op}} \to (\Delta^s)^{\text{op}}$ giving the 1-simplex from $j+1$ to j. As the shape above can be filled by right horns (by Lemma 1.2.6), we can lift the filler downstairs and obtain a full-blown morphism $\Delta^q \times (\Delta^1)^{\text{op}} \to Z$. We define $h_{j+1}(\xi)$ to be the q-simplex $\Delta^q \times \{1\} \to Z$, which of course factors through $Y_{j+1} = X_{j+1}$. Notice moreover that by construction $h_{j+1}(\xi)$ lies over the same q-simplex in $S \times (\Delta^m)^{\text{op}}$ as ξ, so that h_{j+1} is a morphism over $S \times (\Delta^m)^{\text{op}}$.

We now form an extension of $C(g_1, \dots, g_s) \to Z$ to $C(h_1, \dots, h_s)$. The nondegenerate k-simplices $(r_0, \dots, r_{s-1}, \alpha_0, \dots, \alpha_s)$ which we must fill are characterized by the criterion that α_j is a (possibly nullary) degeneracy of ξ. We map $\xi \in X_j \subset C(h_1, \dots, h_s)$ to $\xi \in X_j \subset Z$ (as we must).

First let us treat the case of unmapped nondegenerate k-simplices $(r_0, \dots, r_{s-1}, \alpha_0, \dots, \alpha_s)$ for which $r_j = k$. As we have already filled in the unique nondegenerate k-simplex of this sort with $r_{j-1} = -1$, we know that $r_{j-1} \geq 0$. Moreover, $k > r_{j-1}$ as no simplex in Y_{j-1} maps to (any degeneracy of) ξ under h_j. Say a k-simplex as above is *of degenerate type* if α_j is an $(k - r_{j-1} - 1)^{\text{th}}$ degeneracy, and *of nondegenerate type* otherwise (a simplex of degenerate type is not necessarily degenerate as a simplex).

I claim that every k-simplex of degenerate type has a $(k - r_{j-1} - 1)^{\text{th}}$ face which is of nondegenerate type, and every $(k-1)$-simplex of nondegenerate type is an $(k - r_{j-1} - 1)^{\text{th}}$ face of a unique k-simplex of degenerate type. Indeed, suppose that a k-simplex of degenerate type has an $(k - r_{j-1} - 1)^{\text{th}}$ face which is of degenerate type; then we would have α_j being an $(k - r_{j-1})^{\text{th}}$ degeneracy as well. This would imply that the simplex as a whole was a $(k - r_{j-1})^{\text{th}}$ degeneracy, contradicting our assumption that the simplex was nondegenerate.

Conversely, given a $(k-1)$-simplex $(r_0', \dots, r_{s-1}', \alpha_0', \dots, \alpha_s')$ of nondegenerate type, any k-simplex of degenerate type whose $(k - r_{j-1} - 1)^{\text{th}}$ face is this simplex must have $\alpha_j = \sigma_{k - r_{j-1} - 1}\alpha_j'$, with all $\alpha_i = \alpha_i'$ for $i < j$; this clearly forces only one k-simplex of degenerate type, and this simplex will actually work because there are nonempty simplices over indices less than j. Thus the claim is proven.

It is clear that in a k-simplex like what we are considering, α_j cannot be an i^{th} degeneracy for any $i < k - r_{j-1} - 1$, for then the whole simplex would be degenerate. Therefore the i^{th} boundaries for any such i must lie in $C(g_1, \dots, g_s)$.

By induction on $k \geq 1$, we now simultanously fill in the map on all k-simplices of degenerate type and all $(k-1)$-simplices of nondegenerate type by filling in the map on corresponding pairs of such. For the base case $k = 1$, there is nothing to do, as here α_j must be the zeroth degeneracy of a 0-simplex which must be ξ, and ξ cannot be mapped to by any 0-simplex in Y_{j-1}.

For the induction step, consider an unmapped k-simplex of degenerate type; its $(k - r_{j-1})^{\text{th}}$ boundary will also be unmapped by construction. Its i^{th} faces for $i < k - r_{j-1} - 1$ will already be mapped as they lie in $C(g_1, \dots, g_s)$, and its i^{th} faces for $i > k - r_{j-1} - 1$ are $(k-1)$-simplices of degenerate type, and so also have already been mapped. We conclude that we have already mapped the $\Lambda^k_{k - r_{j-1} - 1}$ of the k-simplex of degenerate type in question. But $k > r_{j-1} + 1$ (otherwise α_{j-1} would map to a degeneracy of ξ) and we have a filler in $S \times (\Delta^m)^{\text{op}} \times (\Delta^s)^{\text{op}}$ (namely that given by the k-simplex in $C(h_1, \dots, h_s)$), and so we can lift that right horn filler to fill the horn and extend the map. Thus we complete this case.

If $j = s$, this case is in fact everything and so we are done. We thus take $j < s$ for the remainder of the proof.

Consider the map $\Delta^q \times (\Delta^1)^{\mathrm{op}} \to Z$ produced above. Let ϵ_t denote the $(q+1)$-tuple of $(q-t)$ 1's followed by $t+1$ 0's. I claim that for all $t < q$, the image of the q-simplex (Δ^q, ϵ_t) under this map is equivalent (in the sense of Lemma 1.5.7) to the image of the q-simplex

$$\beta_t = (r_0, \ldots, r_{s-1}, \alpha_0, \ldots, \alpha_s) \in C(g_1, \ldots, g_s),$$

where $r_{j-1} = -1$, $r_j = t$, $r_{j+1} = q$, α_j is the target t-simplex of ξ, and α_{j+1} is $h_{j+1}(\xi)$. Indeed, we induct on t.

For $t = -1$ both images are $h(\xi)$. Given the statement for $t-1$, we look at the image of the $(q+1)$-simplex $(\sigma_{q-t}\Delta^q, \sigma_{q-t-1}\epsilon_t)$. All faces of this simplex arise from the morphism $C(g_1, \ldots, g_s) \to Z$ except for its $(q-t)^{\text{th}}$ and $(q-t+1)^{\text{th}}$ faces. As the latter is equivalent to the image of β_{t-1} by the induction hypothesis, we may replace this $(q+1)$-simplex with one whose $(q-t+1)^{\text{th}}$ face also arises from the morphism $C(g_1, \ldots, g_s) \to Z$.

But then the $(q-t)^{\text{th}}$ horn of this new simplex can be filled so that the filled face is the image of (Δ^q, ϵ_t). Moreover, consider the $(q+1)$-simplex $(r_0, \ldots, r_{s-1}, \alpha_0, \ldots, \alpha_s)$ of $C(g_1, \ldots, g_s)$ where $r_{j-1} = -1$, $r_j = t$, $r_{j+1} = q+1$, α_j is the target t-simplex of ξ, and $\alpha_{j+1} = \sigma_{q-t}h_{j+1}(\xi)$. The image of this $(q+1)$-simplex also fills the horn in question, and in this case the missing face is the image of β_t. By Lemma 2.3.7, we obtain that the image of β_t is equivalent to the image of (Δ^q, ϵ_t), as desired.

But now, applying the claim in the case $t = q-1$, we see that the image of β_{q-1} is equivalent to the image of $(\Delta^q, \epsilon_{q-1})$. We conclude that there is a $(q+1)$-simplex in Z which shares all its faces with the image of $(\sigma_0\Delta^q, \epsilon')$, where ϵ' is the $(q+2)$-tuple consisting of one 1 followed by $(q+1)$ 0's, except for its first face, which is the image of β_{q-1}. Finally, map the $(q+1)$-simplex $(r_0, \ldots, r_{s-1}, \alpha_0, \ldots, \alpha_s)$ of $C(h_1, \ldots, h_s)$ with $r_{j-1} = -1$, $r_{j+1} = q+1$, $\alpha_j = \xi$, and $\alpha_{j+1} = \sigma_0\xi$ to this $(q+1)$-simplex we have found. By construction, it agrees with all preceding data.

We now imitate the construction following the mapping of ξ to map all unmapped k-simplices with $r_j = k-1$, $r_{j+1} = k$, and α_{j+1} a zeroth degeneracy. Again, a k-simplex of (non)degenerate type will be a k-simplex as above with α_j (not) a $(k - r_{j-1} - 2)^{\text{th}}$ degeneracy. These fall into pairs as before, and the same induction will fill in all such simplices.

To fill in what remains of $C(h_1, \ldots, h_s)$, we once more define classes of simplices of degenerate and nondegenerate type, only this time the terms shall refer to whether or not α_{j+1} is a $(r_{j+1} - r_j - 1)^{\text{th}}$ degeneracy (in the usual notation). Again, degenerate and nondegenerate simplices are in bijection with one another by means of taking the $(r_{j+1} - r_j - 1)^{\text{th}}$ face, and for similar reasons. Here notice that the simplices we filled in above correspond under this bijection (the two inductions we performed were in fact on the simplices of nondegenerate and degenerate type in the new sense). Thus the bijection is exact on the unmapped simplices. We prove by induction first on the dimension k of a degenerate simplex, then on the dimension r_j of α_j, that we can fill in the degenerate k-simplices and nondegenerate $(k-1)$-simplices (in corresponding pairs, as above).

The base of induction on k is established by observing that every simplex of degenerate type of dimension less than or equal to $q+1$ is already filled in (we know $r_j < r_{j+1} \leq q+1$, and so in order for α_j to be a degeneracy of ξ, it must be in fact ξ, and then $\alpha_{j+1} = \sigma_0 h_{j+1}(\xi)$ and we are in the second inductive filling performed above). The base of induction on r_j is established by noting that if $r_j < q$, then the simplex must already be filled (again because α_j is not a degeneracy of ξ).

But then given a k-simplex of degenerate type, its i^{th} faces for $i < k - r_j - 1$ are $(k-1)$-simplices of degenerate type, so filled, and its i^{th} faces for $i > k - r_j - 1$ have lesser r_j (so that even if they are nondegenerate at least r_j drops), and so are filled as well. We thus have a $\Lambda^k_{k-r_j-1}$ in Z over a filled k-simplex in $S \times (\Delta^m)^{\mathrm{op}} \times (\Delta^s)^{\mathrm{op}}$. But $k > r_j + 1$ because otherwise $r_{j+1} = r_j + 1$ as

$$r_j + 1 = k \geq r_{j+1} > r_j,$$

and then $\alpha_{j+1} = \sigma_0 h_{j+1}(\alpha_j)$ so that we are covered by the second induction above. Thus this is a right horn, and the filler can be lifted, completing the induction and the proof. $\square$

Corollary 2.3.13. *Let X and Y be two right n-fibrations over S, and let $f, g : X \to Y$ over S. Then $f \simeq g$ in $[X, Y]_S$ if and only if $C(f) \simeq C(g)$ in $\underline{\mathrm{Hom}}_{n\mathrm{Fib}/S}(X, Y)$*

Proof. Immediate. ☐

Corollary 2.3.14. *Let X, Y, and Z be right n-fibrations over S, and let $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$ be morphisms over S. Then $gf \simeq h$ in $[X, Z]_S$ if and only if there is a 2-simplex in $n\mathrm{Fib}/S$ with zeroth, first and second faces $C(g)$, $C(h)$, and $C(f)$ respectively.*

Proof. By the preceding corollary, we need only show that $C(gf) \simeq C(h)$ if and only if there is a 2-simplex as in the corollary statement. But $C(gf)$ is the first face of the 2-simplex $C(f, g)$ in $n\mathrm{Fib}/S$. If $C(gf) \simeq C(h)$, then there is a 2-simplex with zeroth and first faces $C(gf)$ and $C(h)$ respectively, with the second face degenerate. We then can form an inner 3-horn whose second face is this 2-simplex, and whose zeroth face is $C(f, g)$ and third face is the zeroth degeneracy of $C(f)$. Filling this, we obtain the desired 2-simplex.

Conversely, use the same 3-simplex, only this time the horn will have the zeroth, first and third faces filled so that we can conclude that the second face also fills, whence $C(gf) \simeq C(h)$, as desired. ☐

Corollary 2.3.15. *Let $f : X \to Y$ be a morphism of right n-fibrations over S. Then f is an equivalence of right fibrations if and only if $C(f)$ is a quasi-isomorphism.*

Proof. By Theorem 2.3.4, f is an equivalence if and only if there is a morphism $g : Y \to X$ over S with $gf \simeq 1_X$ in $[X, X]_S$ and $fg \simeq 1_Y$ in $[Y, Y]_S$. By the preceding corollary, this happens if and only if there is a morphism $g : Y \to X$ over S such that there are two 2-simplices with sides $C(g)$, 1_X, and $C(f)$ and $C(f)$, 1_Y, and $C(g)$, respectively. But this happens if and only if there is a $g : Y \to X$ with $C(g)$ a quasi-inverse to $C(f)$. By Theorem 2.3.12, any quasi-inverse Z to $C(f)$ is quasi-isomorphic to some $C(g)$ which is then also a quasi-inverse to $C(f)$, so we are done. ☐

Remark 2.3.16. After this corollary, we will use the terms "equivalence" and "quasi-isomorphism" interchangeably when referring to morphisms of right fibrations.

Proposition 2.3.17. *Let S be a simplicial set and $n \geq -2$ an integer. Then the quasi-category $\mathrm{L}n\mathrm{Fib}/S$ is a full sub-quasi-category of Fib/S.*

Proof. An m-simplex of Fib/S is a right fibration $X \to S \times (\Delta^m)^{\mathrm{op}}$. If the 0-simplices of this m-simplex are contained in $\mathrm{L}n\mathrm{Fib}/S$, then certainly the fibre of X over each object in $S \times (\Delta^m)^{\mathrm{op}}$ is a loose n-quasi-groupoid. But then X itself is fibred on loose n-quasi-groupoids by Lemma 2.2.4, and so the m-simplex in question is contained in $\mathrm{L}n\mathrm{Fib}/X$, as desired. ☐

Corollary 2.3.18. *The inclusion quasi-functor $n\mathrm{Fib}/S \to \mathrm{L}n\mathrm{Fib}/S$ is an equivalence of quasi-categories, and it has a quasi-inverse given by π_n.*

Proof. The inclusion quasi-functor is essentially surjective, as a loose right n-fibration X over S is clearly equivalent (i.e. quasi-isomorphic) to $\pi_n(X/S)$. The inclusion functor also induces homotopy equivalences on $\underline{\mathrm{Hom}}$ spaces, as for any two right n-fibrations X and Y we have a commutative diagram

$$\begin{tikzcd} {[X,Y]_S} \arrow[r, "C"] \arrow[dr, "C"'] & \underline{\mathrm{Hom}}_{n\mathrm{Fib}/S}(X,Y) \arrow[d] \\ & \underline{\mathrm{Hom}}_{\mathrm{Fib}/S}(X,Y) \end{tikzcd}$$

two of whose arrows are homotopy equivalences by Theorem 2.3.12. We conclude that the inclusion quasi-functor is an equivalence of quasi-categories.

To see that π_n is quasi-inverse to the inclusion quasi-functor, note that π_n is actually a section of the inclusion quasi-functor, and so as it is a left quasi-inverse, it must be a quasi-inverse. ☐

Remark 2.3.19. What these last two corollaries show is that no information is lost by considering a right n-fibration to be in Fib$/S$ as opposed to nFib$/S$, and in fact nFib$/S$ is equivalent to the natural full sub-quasi-category LnFib$/S$.

2.4 Existence of Limits in $n\mathrm{Fib}/S$ and $\mathrm{L}n\mathrm{Fib}/S$

In this section we prove that the quasi-categories nFib$/S$ and LnFib$/S$ have all limits, and in fact that these limits are given quite explicitly.

2.4.1 Slice Quasi-Categories as Right Fibrations

We first prove an easy proposition putting slice quasi-categories into the context we have so far developed in this chapter.

Proposition 2.4.1. *Let X be a (resp. loose) n-quasi-category, $x \in X$ an object. Then the quasi-functor $X/x \to X$ is a (resp. loose) right $(n-1)$-fibration.*

Proof. Let $\Lambda^m_k \to X/x$ be a right horn; it is straightforward to see that this is the same data as a morphism $\Lambda^{m+1}_{k,m+1} \to X$ whose target is x. Filling in the projection of this horn in X/x to X is the same thing as assigning an $(m+1)^{\text{th}}$ face to this last $\Lambda^{m+1}_{k,m+1}$. We conclude that a Λ^m_k upstairs with a filler downstairs is the same thing as a Λ^{m+1}_k in X. As this is an inner horn, it can be filled.

If X is an n-quasi-category and $m+1 > n$, then this inner horn can be filled uniquely. We conclude that in this case $X/x \to X$ is a right $(n-1)$-fibration.

Moreover, an m-shell upstairs together with a filler downstairs is the same thing as an $(m+1)$-shell in X. Therefore, if X is a loose n-quasi-category, then X/x is a loose right $(n-1)$-fibration. □

Definition 2.4.2. Let X be a quasi-category, $x, y \in X$ objects. Define the quasi-groupoid $\underline{\mathrm{Hom}}^{\ell}_X(x,y)$ to be the fibre over x in the right fibration $X/y \to X$.

It is immediate that if X is a (resp. loose) n-quasi-category that $\underline{\mathrm{Hom}}^{\ell}_X(x,y)$ is a (resp. loose) n-quasi-groupoid. In fact, in the case of a 2-quasi-category X this $\underline{\mathrm{Hom}}^{\ell}_X$ is the same as the groupoid $\underline{\mathrm{Hom}}'_X$ defined in §1.3. There is a natural quasi-functor $\alpha : \underline{\mathrm{Hom}}^{\ell}_X(x,y) \to \underline{\mathrm{Hom}}_X(x,y)$, defined by taking an $(m+1)$-simplex τ with target object y and source m-simplex the m-fold degeneracy of x to the $\Delta^1 \times \Delta^m$ inside the $(2m+1)$-simplex $\sigma^m_{m+1}\tau$ (the prism is defined by mapping (i,j) to $(m+1)i+j$).

Proposition 2.4.3. *The quasi-functor α is a homotopy equivalence.*

Proof. Let Σ_m be the subposet of $\Delta^1 \times \Delta^{m+1}$ which omits $(0, m+1)$. We fix embeddings $\Delta^1 \times \Delta^m \to \Sigma_m$ and $\Delta^{m+1} \to \Sigma_m$ where the first takes (i,j) to (i,j) and the second takes i to $(0,i)$ if $i \le m$ and i to $(1, m+1)$ otherwise. There are $m+1$ $(m+2)$-simplices in Σ_m, and so we define ξ_i to be the $(m+2)$-simplex with vertices

$$(0,0,\ldots,(0,i),(1,i),\ldots,(1,m),(1,m+1).$$

Moreover, we identify δ_i, $0 \le i \le m+1$, to be the $(m+1)$-simplex with vertices

$$(0,0),\ldots,(0,i-1),(1,i),\ldots,(1,m+1),$$

so that δ_i is the only m-simplex shared by ξ_i and ξ_{i-1} (for $i \ge 1$), and the $(m+1)$-simplex identified above is δ_{m+1}. Notice also that δ_i is the i^{th} face of both ξ_{i-1} and ξ_i.

We now define a simplicial set $P(x,y)$ whose m-simplices are morphisms $\Sigma_m \to X$ with $\{0\} \times \Delta^m$ being the m-fold degeneracy of x and $\{1\} \times \Delta^{m+1}$ (i.e. δ_0) being the $(m+1)$-fold degeneracy of y. The i^{th} boundary map is defined to be the i^{th} boundary map on $\Delta^1 \times \Delta^{m+1}$ (which clearly restricts to Σ_m); degeneracies are defined similarly.

We then clearly have projections $P(x, y) \to \underline{\mathrm{Hom}}_X(x, y)$ and $P(x, y) \to \underline{\mathrm{Hom}}^{\ell}_X(x, y)$ given by composing with $\Delta^1 \times \Delta^m \to \Sigma_m$ and $\Delta^{m+1} \to \Sigma_m$ respectively. Moreover, α factors as a section of the second projection followed by the first.

We claim that these two projections are very surjective. Indeed, for the first, an m-simplex in $\underline{\mathrm{Hom}}_X(x, y)$ with a lift of its boundary upstairs has all simplices of Σ_m forced except for the δ_i for $i \geq 1$ and the ξ_i for $i \geq 0$ (δ_0 is fixed because we need it to be the degeneracy of y). But then ξ_0 has its first horn filled in, so we can fill in the horn in X, which will fill in the second horn of ξ_1, and so on, until at the last stage we fill in the $(m+1)^{\mathrm{th}}$ horn of ξ_m, which fills in Σ_m.

For the other projection, consider an m-simplex in $\underline{\mathrm{Hom}}^{\ell}_X(x, y)$ with a lift of its boundary upstairs. Given that $\{0\} \times \Delta^m$ and $\{1\} \times \Delta^{m+1}$ must be degeneracies of x and y, respectively, we conclude that of $\Delta^1 \times \Delta^m \to \Sigma_m$ we have filled in $\partial\Delta^1 \times \Delta^m \cup \Delta^1 \times \partial\Delta^m$. By Lemma 1.2.5, we can fill this in to only omit the $(m+1)$-simplex with vertices $(0, 0), (1, 0), \ldots, (m, 0)$. But now it is apparent that we have filled in all simplices of Σ_m except for this last $(m+1)$-simplex, δ_i for $1 \leq i \leq m$, and ξ_i for $0 \leq i \leq m$. We thus have filled in the m^{th} horn of ξ_m, so can fill it in X, which gives us the $(m-1)^{\mathrm{th}}$ horn of ξ_{m-1}, and so forth, until we are left with only ξ_0 and the remaining $(m+1)$-simplex above. But this data comprises the $(m+2)^{\mathrm{th}}$ horn of ξ_0. As the target 1-simplex of ξ_0 is the degeneracy on x (a quasi-isomorphism), we can fill this horn as well, completing the proof. □

2.4.2 (Projective) Diagrams as Right Fibrations

Proposition 2.4.4. *Let X be a quasi-category, $\rho : D \to X$ a diagram. Then $X/\rho \to X$ and $(X/\rho)' \to X$ are quasi-isomorphic right fibrations over X.*

Proof. To see that X/ρ is a right fibration, notice that $[D, X]/\rho$ is a right fibration over $[D, X]$, and X/ρ is obtained by pulling this back along the diagonal $X \to [D, X]$. For the rest, we notice that (according to Proposition 1.6.8) there is a simplicial set P over X with very surjective morphisms to X/ρ over X and to $(X/\rho)'$ over X, whence $(X/\rho)'$ is also a right fibration over X and is quasi-isomorphic to X/ρ by Theorem 2.3.4. □

Lemma 2.4.5. *Let $f : X \to Y$ be a quasi-functor between quasi-categories, and suppose that f is an equivalence and a right fibration. Then f is very surjective.*

Proof. Let $\partial\Delta^m \to X$ be an m-shell with a filler $\Delta^m \to Y$ in Y. As f is an equivalence, we know that there is a filler ξ of the m-shell whose image under f is homotopic (or if $m = 0$, quasi-isomorphic) to the given filler. Let this be witnessed by an $(m+1)$-simplex in Y whose zeroth face is $f(\xi)$, whose first face is the given filler in Y, and whose r^{th} face for $1 < r \leq m+1$ is the zeroth degeneracy of the $(r-1)^{\mathrm{th}}$ face of either filler. Then the first horn of this $(m+1)$-simplex can be lifted to X by lifting the zeroth face to the filler we just found and the r^{th} face for $1 < r \leq m+1$ to the zeroth degeneracy of the $(r-1)^{\mathrm{th}}$ face of the m-shell we started with. As f is a right fibration, we can lift the filler of this Λ^{m+1}_1, and the first face of the lift is the lift we wanted to begin with. □

Lemma 2.4.6. *Let $f : X \to Y$ be a morphism of right fibrations over S which is a right fibration and a quasi-isomorphism. Then f is very surjective.*

Proof. It is clearly enough to check this for $S = \Delta^r$ (after pulling X and Y back along $\Delta^r \to S$). But in this case we know that $X \to Y$ is a right fibration and an equivalence of quasi-categories, so the previous lemma applies. □

Proposition 2.4.7. *Let X be a quasi-category, $\rho : D \to X$ a diagram, and $x \in X/\rho$ an object. Then x is a limit of ρ if and only if there is a quasi-isomorphism $X/x \to X/\rho$. The limit is given by x/ρ (i.e. a morphism $x \to \rho$ in $[D, X]$) if and only if there is such a quasi-isomorphism taking 1_x to x/ρ.*

Proof. Notice that for any $x/\rho \in X/\rho$, the morphism $(X/\rho)/(x/\rho) \to X/x$ is very surjective. This is because $([D, X]/\rho)/(x/\rho) = [D, X]/(x/\rho)$ (where here x/ρ refers to the 1-simplex in $[D, X]$ from the diagonal of x

to ρ), we already know that $[D, X]/(x/\rho) \to [D, X]/x$ is very surjective, and $(X/\rho)/(x/\rho)$ is obtained from $[D, X]/(x/\rho)$ by pulling back along $X/x \to [D, X]/x$.

First suppose that x is a limit of ρ, say by means of a lift of x to $x/\rho \in X/\rho$. Then $(X/\rho)/(x/\rho) \to X/\rho$ is very surjective, so by the previous paragraph (and the fact that both of these very surjective maps are over X) we obtain a quasi-isomorphism $X/x \to X/\rho$. Clearly we may choose this quasi-isomorphism to take 1_x to x/ρ.

Conversely, if $X/x \to X/\rho$ is a quasi-isomorphism, let x/ρ be the image of 1_x under this map. Then $(X/\rho)/(x/\rho) \to X/x$ is very surjective, and $(X/\rho)/(x/\rho) \to X/\rho$ is (at least) a right fibration. I claim that this latter morphism is a quasi-isomorphism. To wit, I will produce a section $X/x \to (X/\rho)/(x/\rho)$ whose composition with the projection will yield the quasi-isomorphism we started with; as very surjective maps are also quasi-isomorphisms, that the projection is a quasi-isomorphism will follow.

Indeed, we induct on the dimension of simplices in X/x; filling in pairs of k-simplices and $(k+1)$-simplices, the second of which is obtained from the first by taking the $(k+1)^{\text{th}}$ degeneracy of the underlying $(k+1)$-simplex in X. Indeed, the base of induction is given by 1_x which already has been mapped to x/ρ and can be lifted to $1_{x/\rho}$. For the induction step, given all such pairs of lesser dimension, we will have a Λ^{k+1}_{k+1} already filled in. This right horn projects from $(X/\rho)/(x/\rho)$ to X/ρ, and fills downstairs according to what we wish the $(k+1)$-simplex to map to. We can lift this upstairs according as $(X/\rho)/(x/\rho) \to X/\rho$ is a right fibration, and this will define the destination of the two simplices in question. Thus the induction, and the claim, is proven.

But now we can apply the preceding lemma to conclude that $(X/\rho)/(x/\rho) \to X/\rho$ is very surjective, and so x/ρ is a limit of ρ as desired. $\square$

Remark 2.4.8. We might say that a limit of ρ exists if and only if the right fibration X/ρ is representable by an object of X.

2.4.3 Construction of Limits of Right Fibrations

Definition 2.4.9. Let D and S be simplicial sets, and let $\rho : D \to \mathrm{Fib}/S$ be a diagram, corresponding to a right fibration Z over $D^{\mathrm{op}} \times S$. We define $L(\rho)$ to be the fibre product

$$\begin{tikzcd} L(\rho) \arrow[r] \arrow[d] & {[D^{\mathrm{op}}, Z]} \arrow[d] \\ S \arrow[r] & {[D^{\mathrm{op}}, D^{\mathrm{op}} \times S],} \end{tikzcd}$$

where the morphism $S \to [D^{\mathrm{op}}, D^{\mathrm{op}} \times S]$ is the adjoint to the identity map $D^{\mathrm{op}} \times S \to D^{\mathrm{op}} \times S$. We think of $L(\rho)$ as being equipped with a morphism $D^{\mathrm{op}} \times L(\rho) \to Z$ over $D^{\mathrm{op}} \times S$ given by the commutative diagram

$$\begin{tikzcd} D^{\mathrm{op}} \times L(\rho) \arrow[r] \arrow[d] & D^{\mathrm{op}} \times [D^{\mathrm{op}}, Z] \arrow[r] \arrow[d] & Z \arrow[d] \\ D^{\mathrm{op}} \times S \arrow[r] & D^{\mathrm{op}} \times [D^{\mathrm{op}}, D^{\mathrm{op}} \times S] \arrow[r] & D^{\mathrm{op}} \times S, \end{tikzcd}$$

where here the morphism $D^{\mathrm{op}} \times [D^{\mathrm{op}}, D^{\mathrm{op}} \times S] \to D^{\mathrm{op}} \times S$ is the evaluation map.

Observe that $L(\rho)$ is automatically a right fibration over S. Notice also that the diagonal $\mathrm{Fib}/S \to \mathrm{Fib}/(D^{\mathrm{op}} \times S)$ simply takes (an m-simplex) X to $D^{\mathrm{op}} \times X$.

Lemma 2.4.10. *Let D and S be simplicial sets, and let $\rho : D \to \mathrm{Fib}/S$ be a diagram corresponding to a* *right fibration Z over $D^{\mathrm{op}} \times S$. Let L' be a right fibration over S, and let α be the composition morphism*

$$\begin{tikzcd} {[L', L(\rho)]_S} \arrow[d, "\alpha"] \\ {[D^{\mathrm{op}} \times L', Z]_{D^{\mathrm{op}} \times S}} \end{tikzcd}$$

given by composing with $D^{\mathrm{op}} \times L(\rho) \to Z$. Then α is an isomorphism of simplicial sets.

Proof. An m-simplex $\Delta^m \times D^{\mathrm{op}} \times L' \to Z$ in $[D^{\mathrm{op}} \times L', Z]_{D^{\mathrm{op}} \times S}$ is the same thing as a morphism $\Delta^m \times L' \to [D^{\mathrm{op}}, Z]$ making the following diagram commute:

$$\begin{tikzcd} \Delta^m \times L' \arrow[r] \arrow[d] & {[D^{\mathrm{op}}, Z]} \arrow[d] \\ S \arrow[r] & {[D^{\mathrm{op}}, S].} \end{tikzcd}$$

But such a morphism making the diagram commute is the same thing as a morphism $\Delta^m \times L' \to L$ over S. $\square$

Lemma 2.4.11. *Let X be a quasi-category, $\Sigma = \{0\} \cup \partial_0 \Delta^2 \subseteq \Delta^2$, and $\sigma : \Sigma \to X$ a morphism taking 0 to x and 1 to y. Then the morphism*

$$\underline{\mathrm{Fill}}_X(\sigma, \Delta^2) \to \underline{\mathrm{Hom}}_X(x, y)$$

is very surjective.

Proof. Given an m-simplex $\Delta^m \times \Delta^1 \to X$ downstairs and a lift of its boundary upstairs to $\partial\Delta^m \times \Delta^2 \to X$, we wish to fill in this last shape so that it is an m-simplex in $\underline{\mathrm{Fill}}_X(\sigma, \Delta^2)$.

To do this, note that of the $\Delta^m \times \Delta^2$ we seek, $\partial\Delta^m \times \Delta^2 \to X$ is forced by the boundary lift, $\Delta^m \times \partial_0\Delta^2 \to X$ is forced by σ, and $\Delta^m \times \partial_2\Delta^2 \to X$ is forced by the filler downstairs; all else we are free to fill in. But then we have a map

$$\partial\Delta^m \times \Delta^2 \cup \Delta^m \times \Lambda^2_1 \to X,$$

and so this can be filled by virtue of Lemma 1.2.6. $\square$

Proposition 2.4.12. *Let D and S be simplicial sets, and let $\rho : D \to \mathrm{Fib}/S$ be a diagram corresponding to a right fibration Z over $D^{\mathrm{op}} \times S$. Then $L(\rho)$, with its attendant morphism $L(\rho) \times D^{\mathrm{op}} \to Z$ over $S \times D^{\mathrm{op}}$, is a limit of ρ.*

Proof. The morphism $L(\rho) \times D^{\mathrm{op}} \to Z$ gives an object $L(\rho)/\rho$ in $(\mathrm{Fib}/S)/\rho$, and so we may assemble a diagram

$$\begin{tikzcd} & ((\mathrm{Fib}/S)/\rho)/(L(\rho)/\rho) \arrow[dl] \arrow[d] \\ (\mathrm{Fib}/S)/L(\rho) & (\mathrm{Fib}/S)/\rho \end{tikzcd}$$

where the left arrow is very surjective and the right arrow is a right fibration. We will be done (according to Proposition 2.4.7) if we can prove that

$$((\mathrm{Fib}/S)/\rho)/(L(\rho)/\rho) \longrightarrow (\mathrm{Fib}/S)/\rho$$

is a quasi-isomorphism.

It will be enough to show this on fibres over a right fibration L' over S, which is to say we would like to show that

$$[((\mathrm{Fib}/S)/\rho)/(L(\rho)/\rho)]_{L'} \longrightarrow \underline{\mathrm{Hom}}^{\ell}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(L'\times D^{\mathrm{op}}, Z)$$

is a quasi-isomorphism. Let $\Sigma = \{0\}\cup \partial_0\Delta^2 \subset \Delta^2$, and let $\sigma : \Sigma \to \mathrm{Fib}/(S\times D^{\mathrm{op}})$ assign 0 to the diagonal of L' and $\partial_0\Delta^2$ to the diagonal of the morphism $L(\rho)\to Z$. Let F fit in the fibre product diagram

$$\begin{tikzcd}
F \arrow[rr] \arrow[d] & & \underline{\mathrm{Fill}}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(\sigma,\Delta^2) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Fib}/S}(L', L(\rho)) \arrow[rr] & & \mathrm{Hom}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(L'\times D^{\mathrm{op}}, L(\rho)\times D^{\mathrm{op}}),
\end{tikzcd}$$

so that F parametrizes 2-simplices in $\mathrm{Fib}/(S\times D^{\mathrm{op}})$ whose target morphism is $L(\rho)\to Z$ and whose source morphism is the diagonal of some $L'\to L(\rho)$. As

$$\underline{\mathrm{Fill}}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(\sigma,\Delta^2) \to \mathrm{Hom}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(L', L(\rho))$$

is very surjective (by Lemma 2.4.11), we see that $F \to \underline{\mathrm{Hom}}_{\mathrm{Fib}/S}(L', L(\rho))$ is also very surjective.

I claim that we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}^{\ell}_{\mathrm{Fib}/S}(L', L(\rho)) \arrow[d] & {[((\mathrm{Fib}/S)/\rho)/(L(\rho)/\rho)]_{L'}} \arrow[l] \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}^{\ell}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(L'\times D^{\mathrm{op}}, Z) \arrow[d] \\
\underline{\mathrm{Hom}}_{\mathrm{Fib}/S}(L', L(\rho)) & F \arrow[l] \arrow[r] & \underline{\mathrm{Hom}}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(L'\times D^{\mathrm{op}}, Z) \\
{[L', L(\rho)]_S} \arrow[u, "C"] \arrow[rr, equal] & & {[L'\times D^{\mathrm{op}}, Z]_{S\times D^{\mathrm{op}}}.} \arrow[u, "C"']
\end{tikzcd}$$

Indeed, we see immediately that $[((\mathrm{Fib}/S)/\rho)/(L(\rho)/\rho)]_{L'}$ is the simplicial set whose m-simplices are $(m+2)$-simplices in $\mathrm{Fib}/(S\times D^{\mathrm{op}})$ whose source $(m+1)$-simplex is the diagonal of an m-simplex in $\underline{\mathrm{Hom}}^{\ell}_{\mathrm{Fib}/S}(L', L(\rho)$ and whose target morphism is $L(\rho)\to Z$. By applying $\sigma^m_{m+1}\sigma^m_{m+2}$ and pulling back along the morphism $\Delta^2\times\Delta^m \to \Delta^{3m+2}$ defined by $(i,j)\mapsto (m+1)i+j$, we visibly obtain a morphism

$$[((\mathrm{Fib}/S)/\rho)/(L(\rho)/\rho)]_{L'} \to F$$

which makes the diagram commute.

Next, consider the morphism

$$[L', L(\rho)]_S \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(L'\times D^{\mathrm{op}}, L(\rho)\times D^{\mathrm{op}}, Z)$$

defined by taking an m-simplex $f : \Delta^m\times L' \to \Delta^m\times L(\rho)$ to the image under C of the ordered pair of the diagonal of f and the given morphism $L(\rho)\to Z$. This visibly factors through F, so that we obtain a morphism $[L', L(\rho)]_S \to F$. Notice that a similar interpretation (using the universal property of $L(\rho)$) gives us this same morphism as $[L', Z]_{S\times D^{\mathrm{op}}} \to F$. Moreover, this morphism is an equivalence by the commutativity of the following diagram:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathrm{Fib}/S}(L', L(\rho)) & F \arrow[l] \\
{[L', L(\rho)]_S} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

But now we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}^{\ell}_{\mathrm{Fib}/S}(L', L(\rho)) \arrow[d] & {[((\mathrm{Fib}/S)/\rho)/(L(\rho)/\rho)]_{L'}} \arrow[l] \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}^{\ell}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(L'\times D^{\mathrm{op}}, Z) \arrow[d] \\
\underline{\mathrm{Hom}}_{\mathrm{Fib}/S}(L', L(\rho)) & F \arrow[l] \arrow[r] & \underline{\mathrm{Hom}}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(L'\times D^{\mathrm{op}}, Z) \\
{[L', L(\rho)]_S} \arrow[u, "C"] \arrow[ur] \arrow[rr, equal] & & {[L'\times D^{\mathrm{op}}, Z]_{S\times D^{\mathrm{op}}}.} \arrow[u, "C"'] \arrow[ul]
\end{tikzcd}$$

We conclude that $F \to \underline{\mathrm{Hom}}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(L'\times D^{\mathrm{op}}, Z)$ is a quasi-isomorphism, whence

$$[((\mathrm{Fib}/S)/\rho)/(L(\rho)/\rho)]_{L'} \to \underline{\mathrm{Hom}}^{\ell}_{\mathrm{Fib}/(S\times D^{\mathrm{op}})}(L'\times D^{\mathrm{op}}, Z)$$

is an quasi-isomorphism, and the proof is complete. □

This immediately gives us:

Theorem 2.4.13. *Let S be a simplicial set. Then $n\mathrm{Fib}/S$ and $\mathrm{L}n\mathrm{Fib}/S$ have all limits, and limits are given by the $L(\rho)$ construction above.*

Proof. We have proven this for $n = \infty$. If $\rho : D \to \mathrm{L}n\mathrm{Fib}/S$ is a diagram, then $L(\rho)$ is a loose right n-fibration by construction, and the statement follows formally from the fact $\mathrm{L}n\mathrm{Fib}/S$ is a full subcategory of Fib/S. Similarly, if $\rho : D \to n\mathrm{Fib}/S$ is a diagram, then $L(\rho)$ is a right n-fibration, and the fact that $n\mathrm{Fib}/S \to \mathrm{Fib}/S$ is fully faithful gives us the result (again, formally). □

2.5 Yoneda Lemmas

We are now in a position to produce Yoneda-type results for quasi-categories, where one should keep in mind the interpretation that a right fibration is a kind of presheaf in quasi-groupoids over the base. First we will develop some theory of slice quasi-categories in the context of right fibrations, then apply it to prove a first Yoneda Lemma. Then for a loose n-quasi-category X we will define a natural Yoneda quasi-functor $X \to \mathrm{L}(n-1)\mathrm{Fib}/X$ and show that it is fully faithful (a second Yoneda Lemma).

2.5.1 First Yoneda Lemma

If X is an n-quasi-category, $x \in X$ an object, the results of the previous section establish X/x as being a right $(n-1)$-fibration over X with fibres equivalent to $\underline{\mathrm{Hom}}_X(x, y)$. We will thus want to think of X/x as the Yoneda image of x in $(n-1)\mathrm{Fib}/X$.

Let $x \in X$ be an object, F a right fibration over X and choose an m-simplex in $[X/x, F]_X$. This m-simplex can be seen as a morphism $\Delta^m \times X/x \to F$. Consider the image of the m-simplex of $\Delta^m \times X/x$ which is the nondegenerate m-simplex of Δ^m in the first factor and $\sigma_0^{m+1}x$ in the second. This is an m-simplex in F over $\sigma_0^m x \in X$, and so is literally an m-simplex in F_x, the fibre of F over x. We obtain a morphism of simplicial sets

$$\varphi : [X/x, X/y]_X \longrightarrow F_x.$$

The following proposition ought to recall [MA71] I.1.4.

Proposition 2.5.1. *Let X be a quasi-category, $x \in X$ an object, and F a right fibration over X. Then the morphism*

$$\varphi : [X/x, F]_X \longrightarrow F_x$$

is very surjective.

Proof. We suppose we are given an m-simplex in F_x together with a lift of the boundary of this simplex to $[X/x, F]_X$. Stated another way, we begin with a partially defined morphism $X/x \times \Delta^m \to F$ over X, defined on the union of $X/x \times \partial\Delta^m$ and $(\sigma_0^{m+1}x, 1_{[m]})$, where of course $\sigma_0^{m+1}x$ (an $(m+1)$-simplex in X) refers to the m-simplex in X/x, and $1_{[m]} : [m] \to [m]$ refers to the unique nondegenerate m-simplex of Δ^m.

We inductively fill in the unmapped k-simplices (ξ, τ) of $X/x \times \Delta^m$, where here we consider ξ to be an $(m+1)$-simplex in X (with target x), and consider τ to be an order-preserving map $\tau : [k] \to [m]$. Let i be the least index such that ξ is equal to σ_i^{k+1-i} applied to its source i-simplex (and make this definition for all simplices degenerate and nondegenerate). Similarly to the proof of Theorem 2.3.12, we say that a nondegenerate (ξ, τ) is *of degenerate type* if $\tau(i-1) = \tau(i)$ (where here $\tau(-1) := -1$, $\tau(k+1) := m+1$) and *of nondegenerate type* otherwise. Notice that if (a nondegenerate) (ξ, τ) is of degenerate type then $1 \le i \le k$ as $\tau(k+1) = m+1 > \tau(k)$ and $\tau(-1) = -1 < \tau(0)$.

Notice that if (ξ, τ) is (nondegenerate and) of degenerate type, then the i-value of $(\partial_i\xi, \partial_i\tau)$ is still i. This is because certainly $\partial_i\xi$ is σ_i^{k-i} applied to the source i-simplex of $\partial_i\xi$, and if $\partial_i\xi$ were also $\sigma_{i-1}^{k-(i-1)}$ applied to its source $(i-1)$-simplex, then $\xi = \sigma_i(\partial_i\xi)$ would also be $\sigma_{i-1}^{k+1-(i-1)}$ applied to its source $(i-1)$-simplex. I claim that there is a bijection here between unmapped $(k+1)$-simplices of degenerate type and unmapped k-simplices of nondegenerate type given by taking (ξ, τ) to $\partial_i(\xi, \tau)$ in one direction, and taking (ξ, τ) to $(\sigma_i\xi, \sigma_{i-1}\tau)$ in the other. By the preceding remarks, this bijection will preserve the indices i.

First we show that the maps are well-defined. If $\partial_i(\xi, \tau)$ were of degenerate type, then $\partial_i\tau(i) = \partial_i\tau(i-1)$, and then $\tau(i+1) = \tau(i-1) = \tau(i)$, so that (ξ, τ) would be an i^{th} degeneracy, a contradiction. Moreover, $(\partial_i\xi, \partial_i\tau)$ cannot be degenerate. Suppose say $(\partial_i\xi, \partial_i\tau) = \sigma_r(\nu, \eta)$. We know that $\xi = \sigma_i(\partial_i\xi)$ and $\tau = \sigma_{i-1}(\partial_i\tau)$, so that if $r \le i-1$ then $(\xi, \tau) = \sigma_r(\sigma_{i-1}\nu, \sigma_{i-2}\eta)$ and if $r \ge i$ then $(\xi, \tau) = \sigma_r(\sigma_i\nu, \sigma_{i-1}\eta)$, in either case a contradiction.

For the other direction, notice that $(\sigma_i\xi, \sigma_{i-1}\tau)$ is a fortiori of degenerate type. If it were degenerate, say equal to $\sigma_r(\nu, \eta)$, then $\sigma_r\nu = \sigma_i\xi$ and $\sigma_r\eta = \sigma_{i-1}\tau$. If $r \ne i-1, i$, then (ξ, τ) would be an r^{th} or $(r-1)^{\text{th}}$ degeneracy according as $r < i-1$ or $r > i$ respectively. If $r = i-1$, then $\sigma_{i-1}\nu = \sigma_i\xi$, so ξ would be an $(i-1)^{\text{th}}$ degeneracy and so would be equal to $\sigma_{i-1}^{k+1-(i-1)}$ applied to its source $(i-1)$-simplex, a contradiction. If $r = i$, then $\sigma_i\eta = \sigma_{i-1}\tau$ and so

$$\tau(i-1) = \sigma_{i-1}\tau(i) = \sigma_i\eta(i) = \sigma_i\eta(i+1) = \sigma_{i-1}\tau(i+1) = \tau(i),$$

showing that (ξ, τ) is of degenerate type, a contradiction.

Now that the two maps are well-defined, we need only show they are inverse to one another. But by our work above, if (ξ, τ) is of degenerate type then $\xi = \sigma_i(\partial_i\xi)$ and $\tau = \sigma_{i-1}(\partial_i\tau)$; the other direction follows immediately from the fact that $\partial_i\sigma_i = \partial_i\sigma_{i-1} = 1_{[k]}$. The claim is proved.

Now finally we extend our partial map $X/x \times \Delta^m \to F$ to a full map. We inductively fill in the k-simplices of degenerate type and the $(k-1)$-simplices of nondegenerate type, inducting within each dimension on the index i defined above. Notice that we have filled in all k-simplices for $k < m$, so we begin at $k = m$. We have also filled in all (degenerate and nondegenerate) simplices with $i = 0$, as these are either defined by the boundary lifting or equal to or degeneracies of the simplex $(\sigma_0^{k+1}x, 1_{[k]})$ we started with.

Now, given a $(k+1)$-simplex (ξ, τ) of degenerate type in $X/x \times \Delta^m$ with index i, let us look at its r^{th} boundary. If $r < i$, then the i-value of this boundary is decremented by 1, and so we have filled in the map here. If $r > i$, then the i-value remains the same but $\partial_r\tau(i) = \partial_r\tau(i-1)$, so this boundary is a k-simplex of degenerate type, and we've filled in the map here as well. We conclude that the i^{th} boundary is the only missing face, so that we have defined the map on precisely the i^{th} horn of (ξ, τ) in $X/x \times \Delta^m$. But then the image in F is also defined on an i^{th} horn in F, in other words a $\Lambda_{i,k+1}^{k+1}$ in X. But the $(k+1)^{\text{th}}$ face is forced by the fact that the morphism should be over X (so that this face must be $\partial_{k+1}\xi$), and so we actually have a Λ_i^{k+1} in X. As $0 < i < k+1$, this is in fact an inner horn, so we fill it in X and define the images of (ξ, τ) and $\partial_i(\xi, \tau)$ accordingly. We thus complete the induction, and the proof. □

Keeping in mind Theorem 2.3.12, we obtain the following form of the Yoneda Lemma.

Corollary 2.5.2. *(First Yoneda Lemma) Let X be a quasi-category and $x, y \in X$ objects. Then there is a* *natural morphism*

$$\varphi : [X/x, X/y]_X \longrightarrow \underline{\mathrm{Hom}}^{\ell}_X(x,y)$$

which is very surjective.

Proof. We apply the preceding proposition in the case $F = X/y$, recalling that the fibre of X/y over x is by definition $\underline{\mathrm{Hom}}^{\ell}(x,y)$. $\square$

2.5.2 The Yoneda Quasi-functor

Definition 2.5.3. Let X be a quasi-category. Define a simplicial set $\mathcal{Y}_X$ to have m-simplices which consist simply of the $(2m+1)$-simplices of X. The r^{th} boundary map takes a $(2m+1)$-simplex ξ to $\partial_r\partial_{2m+1-r}\xi$, and the r^{th} degeneracy takes ξ to $\sigma_r\sigma_{2m+1-r}\xi$.

It is straightforward to check that $\mathcal{Y}_X$ is in fact a simplicial set. Moreover, $\mathcal{Y}_X$ has natural projections to X and X^{op} given by taking $\xi \in X_{2m+1}$ to its source and target m-simplices, respectively.

Proposition 2.5.4. *Let X be an n-quasi-category. Then*

$$\begin{array}{c} \mathcal{Y}_X \\ \big\downarrow \\ X \times X^{\text{op}} \end{array}$$

is a right n-fibration. If X is a category, then $\mathcal{Y}_X \to X \times X^{\text{op}}$ is in fact a right 0-fibration.

Proof. Consider a right horn $\Lambda^m_k \to \mathcal{Y}_X$ together with a filling of its projection to $X \times X^{\text{op}}$. We thus have part of a $(2m+1)$-simplex X, specifically we have the source and target m-simplices as well as all $(2m-1)$-simplices $\partial_r\partial_{2m+1-r}\xi$, $0 \le r \le m$, $r \ne k$ of our desired $(2m+1)$-simplex ξ. As $m \ge 1$, we observe that this gives us all 0-simplices of ξ.

First, we fill in the $(m+1)$-simplices with vertices $\{0,1,\dots,m,2m+1-k\}$ and $\{k,m+1,\dots,2m+1\}$. The first has a Λ^{m+1}_k filled, the second a Λ^{m+1}_{m+1-k}, and so both can be filled in X as $0 < k \le m$. These fillers are unique if $m > n-1$.

Now, let us consider the remaining unmapped simplices. These naturally fall into quadruples consisting of all possibilities of containing or not containing k and $2m+1-k$. Notice moreover that any unfilled simplex which contains neither k nor $2m+1-k$ must nonetheless contain either r or $2m+1-r$ for any $r \ne k$ (otherwise it would be contained in $\partial_r\partial_{2m+1-r}\xi$), and so its dimension must be at least $(m-1)$, but also at least 1 as all 0-simplices have already been defined.

We inductively fill such quadruples, inducting on the dimension s of the simplex containing both k and $2m+1-k$. Indeed, we have already filled all such quadruples for $s \le m$. For an s-simplex containing k and $2m+1-k$ (at indices i and j respectively, say), its p^{th} faces for $p \ne i,j$ have been filled by the induction hypothesis (as they all contain both k and $2m+1-k$ and are of lesser dimension) and so we are left with a $\Lambda^s_{i,j}$ (with the four omitted simplices being the quadruple we are looking at).

I claim that $[s] - \{i,j\}$ is nonconsecutive in $[s]$. Indeed, if it were consecutive then $\{i,j\}$ would have to be $\{0,1\}$, $\{s-1,s\}$ or $\{0,s\}$. The first case is ruled out because if k and $2m+1-k$ are the first two vertices of our s-simplex, then as r or $2m+1-r$ must be present for each $r \ne k$ we must have $k = m$ and the s-simplex having vertices $\{m,\dots,2m+1\}$, but we filled this above. The second case is treated similarly. For the third case, notice that as $k \ne 0$, either 0 or $2m$ must be a vertex in our s-simplex, so k and $2m+1-k$ cannot be the first and last vertices. This shows the claim.

But then by Lemma 1.2.2, we can fill the $\Lambda^s_{i,j}$, and uniquely so if $s-2 \ge n$. We observed above that $s-2 \ge m-1$ and $s-2 \ge 1$, so these fillers are unique if $m > n$ or $n = 1$. We conclude that our lift of the horn filler exists in any case and is unique if $m > n$ or $n = 1$. Therefore if X is a category then $\mathcal{Y}_X$ is a right 0-fibration, and if X is an n-quasi-category then $\mathcal{Y}_X$ is a right n-fibration, as we wanted. $\square$

Remark 2.5.5. Since $\mathcal{Y}_X$ is an object in $\mathrm{Fib}/(X \times X^{\mathrm{op}})$, we can think of it as a quasi-functor $\mathcal{Y}_X : X \to \mathrm{Fib}/X$. In this guise, we say that $\mathcal{Y}_X$ is the *Yoneda quasi-functor*.

We now analyze the fibres of $\mathcal{Y}_X$. For objects $x, y \in X$, there is a morphism $\rho : \underline{\mathrm{Hom}}^{\ell}_X(x,y) \to \mathcal{Y}_X(x,y)$ given by taking an m-simplex in $\underline{\mathrm{Hom}}^{\ell}_X(x,y)$ represented by an $(m+1)$-simplex ξ in X to the m-simplex $\sigma^m_{m+1}\xi$ of $\mathcal{Y}_X$.

Proposition 2.5.6. *The morphism of quasi-groupoids $\rho : \underline{\mathrm{Hom}}^{\ell}_X(x,y) \to \mathcal{Y}_X(x,y)$ is a homotopy equivalence.*

Proof. Define a simplicial set P to have m-simplices which are $(2m+2)$-simplices of X, with the r^{th} boundary map being $\partial_r\partial_{2m+2-r}$ and the r^{th} degeneracy map being $\sigma_r\sigma_{2m+2-r}$, and such that the source m-simplex is the degeneracy of x and the target $(m+1)$-simplex is the degeneracy of y. Then P has natural projections to $\underline{\mathrm{Hom}}^{\ell}_X(x,y)$ and to $\mathcal{Y}_X(x,y)$ (given by taking the source $(m+1)$-simplex and the $(m+1)^{\text{th}}$ face, respectively), and ρ factors as a section of the first (applying σ^{m+1}_{m+1} to an m-simplex) followed by the second. We need only show that these two projections are very surjective.

To wit, first consider $P \to \mathcal{Y}_X(x,y)$. An m-simplex of $\mathcal{Y}_X(x,y)$ together with a lift of its boundary to P consists of the data of the simplices $\partial_r\partial_{2m+2-r}\xi$, $0 \le r \le m$ and $\partial_{m+1}\xi$ of the $(2m+2)$-simplex ξ we wish to fill, along with its target $(m+1)$-simplex (the degeneracy of y; the source m-simplex is accounted for by the filling downstairs). Notice that the simplices which remain to be filled are precisely those which contain $m+1$, have at least one of r and $2m+2-r$ for each $r \in [m]$, and are not the target $(m+1)$-simplex.

We induct on the size $|S|$ of a subset $S \subseteq [m]$ to fill in the simplex $\xi(S)$ with vertices $S \cup \{m+1, \dots, 2m+1\}$; the case $S = \emptyset$ is treated by our assumed filling of the target $(m+1)$-simplex with the degeneracy of y. For the induction step, suppose $|S| > 0$ and set $S' = \{2m+2-s | s \in S\}$; I claim that what is filled of $\xi(S)$ is precisely $\Lambda^{m+1+|S|}_{S'}$. This is because a simplex of $\xi(S)$ is unfilled if and only if it is not contained in any $\xi(S'')$ with $S'' \subset S$, it is contains either r or $2m+2-r$ for all r, and it contains $m+1$. Thus such a simplex must contain all of S and must contain all of $\{m+1, \dots, 2m+1\} - S'$, which is to say that it must contain the complement of S'. Conversely, the complement of S' is clearly not filled at any preceding stage.

To proceed, we notice that the target 1-simplex of our $\Lambda^{m+1+|S|}_{S'}$ is 1_y, and so as $S \cup \{m+1, \dots, 2m+1\} - S'$ is not an initial consecutive segment in $S \cup \{m+1, \dots, 2m+1\}$, we may apply Lemma 1.2.2 and the fact that 1_y is a quasi-isomorphism to fill this shape by means of right horn fillings. The induction, and this half of the argument, is complete.

Now consider $P \to \underline{\mathrm{Hom}}^{\ell}_X(x,y)$. An m-simplex of $\underline{\mathrm{Hom}}^{\ell}_X(x,y)$ together with a lift of its boundary upstairs consists of the data of the simplices $\partial_r\partial_{2m+2-r}\xi$, $0 \le r \le m$ and the source and target $(m+1)$-simplices, which are the filler downstairs and the degeneracy of y respectively. The simplices which remain to be filled are precisely those which contain at least one of r and $2m+2-r$ for each $r \in [m]$ and which are not contained in the source or target $(m+1)$-simplices. Such simplices fall into a bijection between those which contain $m+1$ and those which do not, given by removing $m+1$.

Therefore we induct on the dimension k of a simplex containing $m+1$. The base of the induction is given by the fact that all simplices of dimension less than $m+1$ and containing $m+1$ have been filled. At the inductive step, we have a k-horn as omitting any vertex but $m+1$ gives a face of dimension $k-1$ containing $m+1$, so already filled. The only way we would not be able to fill this is if this were not an inner k-horn. But then the vertices of the simplex would have to be $\{0, \dots, m+1\}$ or $\{m+1, \dots, 2m+1\}$, and thus already filled. This completes the induction, and the proof. □

Remark 2.5.7. This last proposition somewhat justifies the terminology "Yoneda quasi-functor." In the next subsection we will give the terminology better justification.

Corollary 2.5.8. *Let X be a loose n-quasi-category. Then $\mathcal{Y}_X$ is a loose right $(n-1)$-fibration over $X \times X^{\mathrm{op}}$.*

Proof. Let $x, y \in X$ be objects. Because X is a loose n-quasi-category, $\underline{\mathrm{Hom}}^{\ell}_X(x,y)$ is a loose $(n-1)$-quasi-groupoid, so $\mathcal{Y}_X(x,y)$ is a loose $(n-1)$-quasi-groupoid. But then all the fibres of $\mathcal{Y}_X$ are loose $(n-1)$-quasi-groupoids, so by Lemma 2.2.4, $\mathcal{Y}_X$ is a loose right $(n-1)$-fibration over $X \times X^{\mathrm{op}}$, as desired. □

Corollary 2.5.9. *Let X be an n-quasi-category. Then $\mathcal{Y}_X$ is a right n-fibration and a loose right $(n-1)$-fibration over $X \times X^{\mathrm{op}}$.*

Proof. Immediate. $\square$

Corollary 2.5.10. *Define a quasi-functor $X \to X \times X^{\mathrm{op}}$ which is the identity on X and constantly an object $y \in X^{\mathrm{op}}$ (i.e. an object in X) on X^{op}. Then the pullback of $\mathcal{Y}_X$ along this morphism is quasi-isomorphic to X/y as a right fibration over X, with the quasi-isomorphism given by taking an m-simplex in X/y, considered as an $(m+1)$-simplex in X, to its image under σ^m_{m+1}.*

Proof. We need only check this statement on fibres, by Theorem 2.3.4. But fibrewise this is the morphism ρ, and ρ is a homotopy equivalence by the proposition. $\square$

Corollary 2.5.11. *The Yoneda quasi-functor $\mathcal{Y}_X : X \to \mathrm{L}(n-1)\mathrm{Fib}/X$ takes an object $y \in X$ to a right fibration over X quasi-isomorphic to X/y, and in fact there is a natural quasi-isomorphism $\theta : X/y \to \mathcal{Y}_X(y)$.*

Proof. Immediate. $\square$

Lemma 2.5.12. *Let $X \hookrightarrow Y$ be a monomorphism of quasi-categories which is an equivalence. Then this map has a very surjective left (strict) inverse $Y \to X$.*

Proof. Let $P \to X$ and $P \to Y$ witness that the given monomorphism is an equivalence, so that we also have a section $X \to P$ through which the mono factors. Define a section $Y \to P$ by first using the given section $X \to P$ on $X \subseteq Y$ and then extending to all of Y. I claim that the composition of this with the projection to X is the morphism we are looking for.

By construction it is a left inverse. Given an m-simplex of X with a lift of its boundary to Y, we can map this boundary up to P (by means of the section from the previous paragraph), lift the filler from X, and map the filler back down to Y. As $Y \to P$ was a section, this gives a lift of the filler from X, as desired. $\square$

Proposition 2.5.13. *Let $x \in X$ be an object, F a right fibration over X. Define a morphism*

$$[\mathcal{Y}_X(x), F]_X \longrightarrow F_x$$

by taking an m-simplex $f : \Delta^m \times \mathcal{Y}_X(x) \to F$ to $f(\sigma^{2m+1}x)$. Then this morphism is very surjective.

Proof. The morphism we are interested in factorizes as

$$[\mathcal{Y}_X(x), F]_X \longrightarrow [X/x, F]_X \longrightarrow F_x.$$

The second morphism here is very surjective by Proposition 2.5.1; therefore we need only show that the first is also very surjective.

Consider the morphism $\theta : X/x \to \mathcal{Y}_X(x)$ from Corollary 2.5.11; this is visibly a monomorphism. Suppose we are given an m-simplex $g : \Delta^m \times X/x \to F$ together with a lift to a morphism $\partial\Delta^m \times \mathcal{Y}_X(x) \to F$, so that we wish to extend a morphism

$$\Delta^m \times X/x \cup \partial\Delta^m \times \mathcal{Y}_X(x) \to F$$

to all of $\Delta^m \times \mathcal{Y}_X(x)$.

Now, in the usual way we define pairs of unmapped r-simplices of $\Delta^m \times \mathcal{Y}_X(x)$ of degenerate and nondegenerate type. For an r-simplex ξ of $\mathcal{Y}_X(x)$, define $i(\xi)$ to be the smallest i such that ξ is σ_i^{r-i} applied to its source i-simplex. Notice that for $xi \notin X/x$, $i(\xi) > 0$. We then define an unmapped r-simplex (τ, ξ) of $\Delta^m \times \mathcal{Y}_X(x)$ to be of degenerate type if $\tau(i(\xi)-1) = \tau(i(\xi))$, and to be of nondegenerate type otherwise. The correspondence is defined by taking a simplex (τ, ξ) of degenerate type to its $i(\xi)^{\mathrm{th}}$ boundary and taking a simplex (τ, ξ) of nondegenerate type to $(\sigma_{i(\xi)-1}\tau, \sigma_{i(\xi)}\xi)$. Notice that these operations fix $i(\xi)$. I claim that they define a bijection between nondegenerate simplices of degenerate type and nondegenerate simplices of nondegenerate type.

Indeed, if (τ,ξ) is nondegenerate of degenerate type, then $\tau(i(\xi))+1 = t(i(\xi)+1)$ (as $\tau : [r] \to [m]$ is surjective) and so $(\tau,\xi) = (\sigma_{i(\xi)-1}\partial_{i(\xi)}\tau, \sigma_{i(\xi)}\partial_{i(\xi)}\xi)$. Then if $(\partial_{i(\xi)}\tau, \partial_{i(\xi)}\xi) = \sigma_k(\tau',\xi')$, then (τ,ξ) is $\sigma_k(\sigma_{i(\xi)-1}\tau', \sigma_{i(\xi)}\xi')$ if $k < i(\xi)$, $\sigma_k(\sigma_{i(\xi)}\tau', \sigma_{i(\xi)}\xi')$ if $k = i(\xi)$, $\sigma_{k+1}(\sigma_{i(\xi)}\tau', \sigma_{i(\xi)}\xi')$ if $k = i(\xi)+1$, and $\sigma_{k+1}(\sigma_{i(\xi)}\tau', \sigma_{i(\xi)+1}\xi')$ if $k > i(\xi)+1$, contradicting nondegeneracy of (τ,ξ). Finally, $(\partial_{i(\xi)}\tau, \partial_{i(\xi)}\xi)$ is of nondegenerate type because if it were of degenerate type, then $\tau(\xi) = \tau(i(\xi)-1) = \tau(i(\xi)+1) = \tau(i(\xi))+1$, a contradiction.

Conversely, if (τ,ξ) is nondegenerate of nondegenerate type, suppose that $(\sigma_{i(\xi)-1}\tau, \sigma_{i(\xi)}\xi)$ is degenerate, say equal to $(\sigma_k\tau', \sigma_k\xi')$. If $k < i(\xi)-1$ then (τ,ξ) is a k^{th} degeneracy; if $k > i(\xi)$ then (τ,ξ) is a $(k-1)^{\text{th}}$ degeneracy. If $k = i(\xi)-1$ then ξ is of the form $\sigma_{i(\xi)-1}^{r+1-i(\xi)}\xi''$, contradicting minimality of $i(\xi)$. Lastly, if $k = i(\xi)$, then $\tau(i(\xi)-1) = \tau(i(\xi))$, contradicting that (τ,ξ) was of nondegenerate type.

But then $(\sigma_{i(\xi)-1}\tau, \sigma_{i(\xi)}\xi)$ is evidently of degenerate type with $i(\xi)^{\text{th}}$ boundary (τ,ξ), and we have our bijection. Notice that an r-simplex (τ,ξ) of nondegenerate type with ξ (as a $(2r+1)$-simplex in X) the r-fold degeneracy of its source $(r+1)$-simplex (i.e., an r-simplex in the image of θ) also has its corresponding $(r+1)$-simplex of degenerate type in the image of θ. In addition, surjectivity of τ is preserved by the correspondence, so the unmapped simplices really do fall into pairs here.

Inducting on the dimension r of an r-simplex (τ,ξ) of degenerate type, after that on $i(\xi)$, we fill in the map on this simplex and its $i(\xi)^{\text{th}}$ boundary. As usual, this works because the k^{th} boundaries for $k \neq i(\xi), i(\xi)-1$ are of degenerate type and lesser dimension, and the $(i(\xi)-1)^{\text{th}}$ face can have i-value at most $i(\xi)-1$; at this stage we will thus have defined a morphism from a $\Lambda^r_{i(\xi)}$ to F (a right horn as $i(\xi) > 0$), and we can extend it because the filler in X is given by the image of ξ, so we can lift the filler in $F \to X$. We are done. □

Corollary 2.5.14. *Let X be a quasi-category, and let x and y be objects in X. Then the natural morphism $[\mathcal{Y}_X(x), \mathcal{Y}_X(y)]_X \to \mathcal{Y}_X(x,y)$ is very surjective.*

Proof. Immediate upon observing that $\mathcal{Y}_X(x,y)$ is the fibre of $\mathcal{Y}_X(y)$ over x. □

2.5.3 Second Yoneda Lemma

Our main objective here is to prove that the Yoneda quasi-functor $\mathcal{Y}_X : X \to \mathrm{Fib}/X$ is fully faithful.

Theorem 2.5.15. *(Second Yoneda Lemma) Let X be a quasi-category. Then*

$$\mathcal{Y}_X : X \longrightarrow \mathrm{Fib}/X$$

is fully faithful.

Proof. We check criterion (ii) from Lemma 1.5.57. Let $m \geq 1$, $\sigma : \partial\Delta^m \to X$, and fix an m-simplex in Fib/X, which is to say a right fibration $Z \to X \times (\Delta^m)^{\mathrm{op}}$, whose restriction to $X \times (\partial\Delta^m)^{\mathrm{op}}$ is $\mathcal{Y}_X(\sigma)$. We will show that we can extend σ to an m-simplex ξ in X such that there is a morphism $\mathcal{Y}_X(\xi) \to Z$ over $X \times (\Delta^m)^{\mathrm{op}}$ which fixes $\mathcal{Y}_X(\sigma)$. This will be enough because this morphism will give rise to (via C) a 1-simplex in $\mathrm{Fib}/[(\Delta^m)^{\mathrm{op}} \times X]$, i.e. a prism $\Delta^1 \times \Delta^m \to \mathrm{Fib}/X$, such that its restriction to $\Delta^1 \times \partial\Delta^m$ is constantly $\mathcal{Y}_X(\sigma)$, its target $\{1\} \times \Delta^m \to \mathrm{Fib}/X$ is Z, and its source $\{0\} \times \Delta^m \to \mathrm{Fib}/X$ is in the image of $\mathcal{Y}_X$.

The bulk of the work comes in producing the m-simplex ξ. For all non-target faces $\partial_i\sigma$ $(0 < i \leq m)$, let η_i be the $(m-1)$-simplex in $\mathcal{Y}_X(\partial_i\sigma)$ corresponding to the $(2m-1)$-simplex of X $\sigma_0^m\partial_i\sigma$. Then η_i maps to $\sigma_0^{m-1}x$ in X (where x is the source object of σ) and to the i^{th} boundary of $(\Delta^m)^{\mathrm{op}}$ in the other factor. As the $(m-1)$-simplices η_i clearly patch together, we obtain a right horn $\Lambda^m_m \to Z$ with a natural filler downstairs, namely $(\sigma_0^m x, 1_{[m]})$. Lifting this horn filler to an m-simplex η' in Z, the filled face is an $(m-1)$-simplex χ' of $\mathcal{Y}_X(\partial_0\sigma)$, i.e. a $(2m-1)$-simplex of X, whose source $(m-1)$-simplex is $\sigma_0^{m-1}x$ and whose target $(m-1)$-simplex is $\partial_0\sigma$.

We have shown that the morphism $\theta : X^{\mathrm{op}}/x \to \mathcal{Y}_{X^{\mathrm{op}}}(x)$ is a quasi-isomorphism. Pulling back along $(\partial_0\sigma)^{\mathrm{op}} : (\Delta^{m-1})^{\mathrm{op}} \to X^{\mathrm{op}}$, we obtain in particular that χ' is homotopic to a simplex χ in $\mathcal{Y}_X(\partial_0\sigma)$ (over the identity of $\{x\} \times (\Delta^{m-1})^{\mathrm{op}}$) which, as a $(2m-1)$-simplex in X, is σ_0^{m-1} applied to an m-simplex ξ. We may then replace η' with another filler η whose filled face is $\chi = \sigma_0^{m-1}\xi$.

I claim that $\partial\xi = \sigma$. Indeed, the target face of ξ is $\partial_0\sigma$ by construction. If $m = 1$, the first face (source object) is just x as it comes from the degeneracy we filled in downstairs. If $m > 1$, let $0 < i \leq m$ and let $j \neq m-i$ have $0 \leq j \leq m-1$. Then the i^{th} face of ξ is the same as the $(m-1)$-simplex of $\chi = \sigma_0^{m-1}\xi$ with vertices

$$j, m, \ldots, m+i-2, m+i, \ldots, 2m-1,$$

and this coincides with the $(m-1)$-simplex in the $(2m-1)$-simplex η_i with vertices $j, m+1, \ldots, 2m-1$ if $j < m-i$, and $j-1, m+1, \ldots, 2m-1$ otherwise. As $\eta_i = \sigma_0^m \partial_i\sigma$, this last is simply $\partial_i\sigma$.

To finish, we produce the desired extension

$$\begin{tikzcd} \mathcal{Y}_X(\sigma) \arrow[r] \arrow[d] & Z \\ \mathcal{Y}_X(\xi) \arrow[ur, dashed] & \end{tikzcd},$$

starting by sending the m-simplex $\sigma_0^{m+1}\xi$ of $\mathcal{Y}_X(\xi)$ to η (by our work above, this is compatible with the map already defined on $\mathcal{Y}_X(\sigma)$).

We follow the usual pattern for this sort of proof. Let α be an unmapped nondegenerate r-simplex of $\mathcal{Y}_X(\xi)$, and let i_α be the minimal i such that α as a $(2r+1)$-simplex of X is σ_i^{r+1-i} applied to an $(r+i)$-simplex. Notice that as α is unmapped, it cannot be $\sigma_0^{m+1}\xi$ (or any boundary thereof) and so as α is nondegenerate and its target r-simplex is a degeneracy of ξ, i_α must be positive. We say that α is of degenerate type if α is of the form $\sigma_{2r+1-i_\alpha}\beta$, and say that α is of nondegenerate type otherwise.

To an α of degenerate type we associate its i_α^{th} boundary in $\mathcal{Y}_X(\xi)$ (an operation which fixes i_α), and to an α of nondegenerate type we associate $\sigma_{i_\alpha}\sigma_{2r+2-i_\alpha}\alpha$ (in X). The claim, as usual, is that this establishes a bijection.

Indeed, first let α be of degenerate type, so that it is clear that $\alpha = \sigma_{i_\alpha}\sigma_{2r-i_\alpha}\partial_{i_\alpha}\partial_{2r+1-i_\alpha}\alpha$. If $\partial_{i_\alpha}\partial_{2r+1-i_\alpha}\alpha$ were degenerate, say equal to $\sigma_k\sigma_{2r-3-k}\beta$, then we have

$$\alpha = \sigma_{i_\alpha}\sigma_{2r-i_\alpha}\sigma_k\sigma_{2r-3-k}\beta = \begin{cases} \sigma_k\sigma_{2r-1-k}\sigma_{i_\alpha-1}\sigma_{2r-i_\alpha-1}\beta & k < i_\alpha - 1 \\ \sigma_{i_\alpha-1}\sigma_{2r-i_\alpha}\sigma_{i_\alpha-1}\sigma_{2r-i_\alpha-2}\beta & k = i_\alpha - 1 \\ \sigma_{i_\alpha+1}\sigma_{2r-i_\alpha-2}\sigma_{i_\alpha}\sigma_{2r-i_\alpha-2}\beta & k = i_\alpha \\ \sigma_{k+1}\sigma_{2r-2-k}\sigma_{i_\alpha}\sigma_{2r-2-i_\alpha}\beta & k > i_\alpha \end{cases}.$$

As all of these are degenerate r-simplices of $\mathcal{Y}_X(\xi)$, we obtain our contradiction. If $\partial_{i_\alpha}\partial_{2r+1-i_\alpha}\alpha$ were of degenerate type, then it would be of the form $\sigma_{2r-1-i_\alpha}\beta$, and then we would have

$$\alpha = \sigma_{i_\alpha}\sigma_{2r-i_\alpha}\sigma_{2r-1-i_\alpha}\beta = \sigma_{i_\alpha}\sigma_{2r-1-i_\alpha}\sigma_{2r-1-i_\alpha}\beta,$$

so that α would be degenerate, again a contradiction.

For the other direction, let α be of nondegenerate type. Suppose that

$$\sigma_{i_\alpha}\sigma_{2r+2-i_\alpha}\alpha = \sigma_k\sigma_{2r+1-k}\beta.$$

If $k < i_\alpha - 1$, then there would be a β' with $\alpha = \sigma_k\sigma_{2r-1-k}\beta'$, and α would be degenerate. If $k = i_\alpha - 1$, then α would be of the form $\sigma_{i_\alpha-1}^{r-i_\alpha}\gamma$ contradicting the minimality of i_α. If $k = i_\alpha$, then α would be of the form $\sigma_{2r+1-i_\alpha}\gamma$, and would be of degenerate type. Finally, if $k > i_\alpha$, then there would be a β' with $\alpha = \sigma_{k-1}\sigma_{2r-k}\beta'$, and α would be degenerate. We conclude that $\sigma_{i_\alpha}\sigma_{2r+2-i_\alpha}\alpha$ is nondegenerate. It is visibly of degenerate type.

But then as these operations are clearly inverse to one another, we have our bijection. The bijection takes unmapped simplices to unmapped simplices because neither operation will alter surjectivity onto $(\Delta^m)^{\mathrm{op}}$, and the operation is not defined when $i_\alpha = 0$, i.e. when we are considering (a degeneracy of) the one simplex we mapped above.

We now induct on the dimension r of an unmapped nondegenerate r-simplex α of degenerate type, as well as on i_α, filling both it and its corresponding $(r-1)$-simplex of nondegenerate type. All simplices with $r < m$ have been mapped already (they lie over $(\partial\Delta^m)^{\mathrm{op}}$), and $i_\alpha = 0$ has been covered by our base case. But now for an arbitrary α, all faces except the $(i_\alpha - 1)^{\mathrm{th}}$ and i_α^{th} are of degenerate type of lesser dimension, so have been filled. The $(i_\alpha - 1)^{\mathrm{th}}$ face might be of nondegenerate type, but it has lesser i-value and so is also filled already. We are left with a $\Lambda^r_{i_\alpha}$ in Z, and the filler downstairs is forced by the image of α. Lifting the filler, we complete the induction, the map extension, and the proof. $\square$

Corollary 2.5.16. *Let X be a quasi-category. If X is an n-quasi-category, then*

$$\mathcal{Y}_X : X \longrightarrow n\mathrm{Fib}/X$$

is fully faithful. If X is a loose n-quasi-category, then

$$\mathcal{Y}_X : X \longrightarrow \mathrm{L}(n-1)\mathrm{Fib}/X$$

is fully faithful.

Proof. Immediate. $\square$

Bibliography

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces.* Springer-Verlag, Berlin, 1973. Lecture Notes in Mathematics, Vol. 347.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in mathematics.* Birkhäuser Verlag, 1999.

[Joy02] André Joyal. Quasi-categories and kan complexes. *Journal of Pure and Applied Algebra*, 175(1-3):207–222, 2002.

[Lei02] Tom Leinster. A survey of definitions of n-category. *Theory and Applications of Categories (electronic)*, 10:1–70, 2002.

[MA71] J.L. Verdier M. Artin, A. Grothendieck. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III*, volume 269, 270, 305 of *Lecture Notes in Mathematics.* Springer, 1971.

[Mak97] Micheal Makkai. First order logic with dependent sorts, with applications to category theory. preprint, 1997.