Lie 2-algebras and Higher Gauge Theory

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Higher Gauge Theory

Ordinary gauge theory describes how point particles change as they move along paths in spacetime.

In a mathematical formulation this involves the study of principal bundles with some structure group $G$, and connections on these.

Higher gauge theory is a categorification of this:

- groups are promoted to 2-groups,
- bundles are promoted to 2-bundles,
- connections are promoted to 2-connections

This talk will be about trying to understand 2-connections from a more algebraic point of view.
2-groups

A (strict) 2-group $\mathcal{G}$ is a category internal to the category $\text{Grp}$ of groups. So $\mathcal{G}$ consists of

- a group of objects $G_0$, and
- a group of morphisms $G_1$

such that all operations are homomorphisms of groups.

If $G$ is a group then the associated groupoid $G[1]$ is a 2-group if and only if $G$ is abelian. Higher gauge theory for the 2-group $U(1)[1]$ has been extensively studied in the subject of $U(1)$-gerbes. This is a mathematical formulation of ‘higher dimensional electromagnetism’.

2-groups are the same thing as crossed modules. These are homomorphisms $t: H \to G$ together with an action of $G$ on $H$ satisfying some identities.

To understand these, its useful to think of the 2-group $\mathcal{G}$ as a 2-category with one object.
There is a subgroup $H$ consisting of all 2-morphisms of the form

\[
\begin{array}{c}
\bullet \\
1 \Downarrow h \\
\downarrow g \\
\bullet
\end{array}
\]

The target homomorphism restricts to a homomorphism $t$ from $H$ to the group $G = G_0$ of objects of the 2-group $\mathcal{G}$ which sends each $h \in H$ to its target:

\[
\begin{array}{c}
\bullet \\
1 \Downarrow h \\
\downarrow t(h) \\
\bullet
\end{array}
\]

The action of $G$ on $H$ can be described as follows:

\[
\begin{array}{c}
\bullet \\
1 \Downarrow h \\
\downarrow t(h) \\
\bullet
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
1 \Downarrow 1 \\
\downarrow g \quad \downarrow g^{-1} \\
\bullet
\end{array}
\]
Transition functions

Ordinary principal bundles have transition functions: these are maps

\[ g_{ij} : U_i \cap U_j \to G \]

satisfying the cocycle condition

\[ g_{ij}(x) g_{jk}(x) = g_{ik}(x). \]

A good way to think of this is as a diagram in the groupoid \( G[1] \):

\[
\begin{array}{c}
\ast \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
\end{array}
\]

In a categorification of the notion of a principal bundle, these diagrams should be weakened:
The morphism $h_{ijk}(x)$ should satisfy a coherence law of its own — namely the following tetrahedron should commute:

This amounts to a pair of maps

$$g_{ij}: U_i \cap U_j \to G$$
$$h_{ijk}: U_i \cap U_j \cap U_k \to H$$

which satisfy the pair of equations

$$g_{ij} g_{jk} = g_{ik} t(h_{ijk})$$
$$h_{ijl} \alpha(g_{ij})(h_{jkl}) = h_{ikl} h_{i,j,k}$$

What kinds of objects have transition functions like these?
Gerbes and 2-bundles

A principal $G$ bundle on a manifold $M$ consists (in part) of another manifold $P$ together with an action of $G$ on $P$:

$$P \times G \to P$$

such that $M$ is the orbit space of this $G$-action.

If $\mathfrak{G}$ is a 2-group we’d like to define a notion of a ‘principal $\mathfrak{G}$ bundle’.

Categorify the definition of principal bundle:

manifolds $\to$ smooth groupoids

smooth maps $\to$ smooth functors

We would then want to have a smooth functor

$$P \times \mathfrak{G} \to P$$

between smooth groupoids satisfying some conditions.

There are several ways to do this categorification; one way is described in the thesis of Toby Bartels. Whichever way we choose, we want to find lots of examples!!
Connections

Classically, a connection on a principal bundle

\[ G \to P \to M \]

is a preferred way of lifting paths — this is the notion of parallel transport.

Another point of view is that a connection on \( P \) is an invariant choice of horizontal subspace \( H_u \subset T_u P \) for all \( u \in P \).

This horizontal subspace defines a splitting of the short exact sequence

\[
0 \to V_u \to T_u P \xrightarrow{d\pi} T_{\pi(u)} M \to 0
\]

where the vertical subspace \( V_u \) is the kernel of \( d\pi \).
The Atiyah Sequence

If \( P \rightarrow M \) is a principal \( G \)-bundle then there is a canonical exact sequence of Lie algebras of vector fields:

\[
\left\{ \text{inv. vert. vector fields on } P \right\} \rightarrow \left\{ \text{inv. vector fields on } P \right\} \rightarrow \left\{ \text{vec. fields on } M \right\}
\]

A connection on \( P \) is a splitting \( A \) of this short exact sequence.

Is \( A \) a homomorphism of Lie algebras??

In general the answer is no. The curvature \( F_A \) of the connection measures the failure of \( A \) to be a homomorphism of Lie algebras:

\[
F_A(X, Y) = [A(X), A(Y)] - A[X, Y]
\]

\( F_A \) satisfies an equation involving three vector fields \( X, Y \) and \( Z \) — the Bianchi identity.

Can we think of \( F_A \) as a Lie algebra 2-cocycle?
Classification of Lie algebra extensions

Consider an abstract extension of Lie algebras:

\[ 0 \to J \overset{i}{\to} K \overset{p}{\to} L \to 0 \]

Let \( A: L \to K \) be a linear splitting of \( p \). We can measure the ‘curvature’ of \( A \) through the skew linear map \( F_A: L \otimes L \to J \) defined by

\[ F_A(x, y) = [A(x), A(y)] - A([x, y]) \]

We can also define a linear map

\[ \nabla_A: L \to \text{Der}(J) \]

by

\[ \nabla_A(x)(\xi) = [A(x), \xi] \text{ for } \xi \in J \]

\( \nabla_A \) is not a Lie algebra homomorphism; instead we have

\[ \nabla_A([x, y]) = [\nabla_A(x), \nabla_A(y)] - \text{ad}(F_A(x, y)) \]

\( F_A \) satisfies an analogue of the ‘Bianchi identity’.
Interlude: Lie 2-algebras

Lie 2-algebras were introduced by Baez and Crans in HDA6. A (strict) Lie 2-algebra is a category $L$ internal to $\text{LieAlg}$. Thus $L$ consists of

- a Lie algebra of objects $L_0$, and
- a Lie algebra of morphisms $L_1$

such that all structure maps are Lie algebra homomorphisms.

Baez and Crans go further and consider the notion of a semi-strict Lie 2-algebra. This is a 2-vector space $L$ equipped with a skew bilinear functor

$$[\ ,\ ] : L \times L \to L$$

together an isomorphism (the Jacobiator)

$$J_{x,y,z} : [x, [y, z]] \to [[x, y], z] + [y, [x, z]]$$

natural in $x$, $y$ and $z$. $J_{x,y,z}$ is required to satisfy the Jacobiator identity.
A good example of a Lie 2-algebra is the following. Let $J$ be a Lie algebra. Recall that we have the adjoint homomorphism

$$\text{ad}: J \rightarrow \text{Der}(J)$$

which we can think of as a chain complex. Define a Lie 2-algebra $\text{DER}(J)$ with

- objects = $\text{Der}(J)$, the Lie algebra of derivations of $J$, and
- morphisms = $\text{Der}(J) \rtimes J$, the semi-direct product Lie algebra with bracket defined as usual by

$$[(f, x), (g, y)] = ([f, g], [x, y] + f(y) - g(x))$$

The source and target homomorphisms are defined by

$$s(f, x) = f, \quad t(f, x) = f + \text{ad}(x)$$
We can think of the pair \((\nabla A, F_A)\) as giving a homomorphism of Lie 2-algebras 
\[ F : L \to \text{DER}(J), \]
where \(L\) is thought of as a discrete Lie 2-algebra.

For an object \(x\) of \(L\), we set
\[ F(x) = \nabla A(x) \in \text{Der}(J). \]

The ‘curvature’ \(F_A(x, y)\) can be interpreted as a morphism
\[ F([x, y]) \xrightarrow{F_2(x, y)} [F(x), F(y)] \]
in \(\text{DER}(J)\). The ‘Bianchi identity’ for \(F_A(x, y)\) is the statement that the following diagram commutes:

We should think of the homomorphism \(F\) as a ‘non-abelian Lie algebra cocycle’.
Here is a classification theorem for extensions of Lie algebras. It must be well known.

**Theorem.** Let $L$ and $J$ be Lie algebras. Then there is a bijection

$$\text{Ext}(L, J) \approx [L, \text{DER}(J)]$$

between the set of equivalence classes of extensions of $L$ by $J$ and the set of homotopy classes of homomorphisms $L \rightarrow \text{DER}(J)$.

It reduces to the usual description of $\text{Ext}(L, J)$ when $J$ is central in $L$. 
Connections on gerbes

Several authors (Breen-Messing, Baez-Schreiber, and Jurčo et al) have developed a theory of connections on gerbes/2-bundles.

In particular Baez and Schreiber have shown how under certain conditions a connection on a gerbe can be used to define a notion of parallel transport over surfaces.

Connections on gerbes are difficult to understand; whereas a connection on a principal bundle can be described in terms of local gauge potentials

\[ A_i, \]

a connection on a gerbe involves several gauge fields

\[ A_i, \alpha_{ij}, B_i, \nu_i \]

satisfying lots of complicated equations.

We would like to have a description of connections on gerbes analogous to the description of connections on principal bundles in terms of Lie 2-algebras.
Derivations of $L_\infty$ algebras

We need to understand derivations of Lie 2-algebras. To do this it is easiest to work with chain complexes.

Recall that in HDA6 Baez and Crans define the notion of a 2-term $L_\infty$ algebra. This is a chain complex $L$, concentrated in degrees 0: and 1:

$$L_1 \xrightarrow{d} L_0$$

together with skew linear maps

$$\ell_2 : L^\otimes 2 \to L, \deg(\ell_2) = 0$$
$$\ell_3 : L^\otimes 3 \to L, \deg(\ell_3) = 1$$

satisfying some conditions.

They prove the following result:

**Theorem** (Baez-Crans). *There is a 2-equivalence of 2-categories*

$$2\text{Term } L_\infty \simeq \text{Lie2Alg}$$

We define instead a notion of derivation of a 2-term $L_\infty$ algebra.
Let $L$ be any $L_\infty$ algebra. Then we can prove the following results.

**Theorem.** There is a differential graded (DG) Lie algebra $\text{Der}(L)$.

One should think of these as derivations up to chain homotopy.

**Theorem.** There is an $L_\infty$ morphism $\text{ad}: L \to \text{Der}(L)$.

This is a generalisation of the usual adjoint homomorphism $\text{ad}: L \to \text{Der}(L)$ for Lie algebras $L$.

**Theorem.** There is a DG Lie algebra $\text{DER}(L)$ whose underlying graded vector space is

$$\text{Der}(L) \oplus sL \ (s = \text{ suspension}).$$

If $L$ is a 2-term $L_\infty$ algebra then $\text{DER}(L)$ is concentrated in degrees 0, 1 and 2:

$$\text{Der}_0(L) \leftarrow \text{Der}_1(L) \oplus L_0 \leftarrow L_1$$

The underlying idea behind $\text{DER}(L)$ is that it is supposed to be an infinitesimal version of the automorphism 3-group $\text{AUT}(G)$ associated to a (weak) 2-group $G$. 
Extensions of Lie 2-algebras

Let $f : L \to L'$ be an $L_\infty$ morphism between 2-term $L_\infty$ algebras. What do we mean by the kernel of $f$?

The correct notion is the homotopy fibre $H(f)$ of $f$; this has a natural structure of a 2-term $L_\infty$ algebra such that the natural map $H(f) \to L$ extends to an $L_\infty$ morphism.

Let $L$ and $J$ be 2-term $L_\infty$ algebras. An extension of $L$ by $J$ consists of a 2-term $L_\infty$ algebra $K$ and an $L_\infty$ morphism $p : K \to L$ such that the underlying chain map is surjective, together with an $L_\infty$ quasi-isomorphism $J \xrightarrow{\sim} H(p)$.

The analogous notion for 2-groups was first considered by Breen in “Théorie de Schreier Supérieure”.

Denote by $\text{Ext}(L, J)$ the set of equivalence classes of extensions of $L$ by $J$. 

18
Classification Theorem

**Theorem.** Let $L$ and $J$ be 2-term $L_\infty$ algebras. Then there is a bijection

$$\text{Ext}(L, J) \cong [L, \text{DER}(J)]$$

To any gerbe $P$ on $M$, one can associate an analogue of the Atiyah short exact sequence.

$$0 \to \Gamma(\text{ad} P) \to \Gamma(TP/G) \to \text{Vect}(M) \to 0$$

This is now an exact sequence of Lie 2-algebras, where the Lie algebra $\text{Vect}(M)$ of vector fields on $M$ is considered as a discrete Lie 2-algebra.

We can think of this as an extension of $\text{Vect}(M)$ by $\Gamma(\text{ad} P)$. By the theorem, this extension is classified by a ‘3-cocycle’

$$F: \text{Vect}(M) \to \text{DER}(\Gamma(\text{ad}(P))).$$

The condition that this map is a homomorphism of $L_\infty$ algebras neatly encodes the complicated equations for a connection on the gerbe.