Spin Foam Perturbation Theory

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Abstract

We study perturbation theory for spin foam models on triangulated manifolds. Starting with any model of this sort, we consider an arbitrary perturbation of the vertex amplitudes, and write the evolution operators of the perturbed model as convergent power series in the coupling constant governing the perturbation. The terms in the power series can be efficiently computed when the unperturbed model is a topological quantum field theory. Moreover, in this case we can explicitly sum the whole power series in the limit where the number of top-dimensional simplices goes to infinity while the coupling constant is suitably renormalized. This 'dilute gas limit' gives spin foam models that are triangulation-independent but not topological quantum field theories. However, we show that models of this sort are rather trivial except in dimension 2.

1 Introduction

Recent work on loop quantum gravity and topological quantum field theory has focused attention on a class of theories called 'spin foam models' [4]. In a spin foam model, states are described as linear combinations of spin networks, while transition amplitudes are computed as sums over spin foams. A 'spin network' is a graph with edges labelled by representations of some group or quantum group, and with vertices labelled by intertwiners. Similarly, a 'spin foam' is a 2-dimensional complex with polygonal faces labelled by representations and edges labelled by intertwiners. For each spin foam one calculates an amplitude as a product of amplitudes associated to its faces, edges and vertices. The transition amplitude between spin network states is computed by summing the amplitudes of spin foams that go from one spin network to another.

There are various versions of this basic idea. The most radical and perhaps ultimately most promising option is to work with 'abstract' spin networks and spin foams, not embedded in any background manifold [3, 11, 21, 31]. A more conservative approach is to describe states by spin networks embedded in a manifold that represents space, and similarly, to sum over spin foams embedded in a manifold that represents spacetime. This strategy has the advantage of making more rapid contact with familiar physics. However, it has certain problems that do not arise — or at least, not so obviously — in the 'abstract' approach. In this paper we consider one of these, namely, the problem of triangulation-dependence.

When summing over spin foams embedded in a given spacetime manifold, one has a number of choices. One could try to sum, or integrate, over *all* such spin foams. Unfortunately the set of all such spin foams is uncountable, and there is no obvious measure on it. A somewhat more promising alternative is to sum over all diffeomorphism equivalence classes of embedded spin foams. This idea is rather natural if we start with a formula for the Hamiltonian constraint in canonical quantum gravity and try to derive a spin foam model from that [26, 27]. This set is still uncountable if we work with spin foams smoothly embedded in a smooth manifold, since there are moduli spaces of intersections [17], but it becomes countable in the piecewise-linear context. Of course, even with a countable sum one faces the issue of convergence.

A more drastic way to obtain well-defined transition amplitudes is to fix a triangulation of the spacetime and sum only over spin foams that live in the dual 2-skeleton of this triangulation. This approach has been very successful for models that give topological quantum field theories, such as:

- The Fukuma-Hosono-Kawai version of the G/G gauged WZW model in 2 dimensions, where G is a compact Lie group [16].
- The Turaev-Viro model of 3-dimensional SU(2) BF theory with cosmological term, and its subsequent generalization to other compact Lie groups [7, 13, 32, 33].
- The Crane-Yetter model for 4-dimensional SU(2) BF theory with cosmological term, and its subsequent generalization to other compact Lie groups [9, 10].
- The Dijkgraaf-Witten model, and its generalization to spacetimes of arbitrary dimension [12, 14, 23, 35]. Here the gauge group is a finite group.

The reason is that in these cases, the sum over spin foams is finite and the result is independent of the choice of triangulation. This allows us to have our cake and eat it too: we obtain the benefits of finiteness that come from working with a triangulation, while still being able to regard the triangulation as a mere computational device, rather than something 'physical'.

However, in the existing spin foam models of 4-dimensional quantum gravity that involve fixing a triangulation of spacetime [5, 6, 18, 25], the results appear to depend on the triangulation. There are various attitudes one can take to this. One can accept the triangulation as an inevitable fixed background structure. However, this goes against the desire for a background-free theory of quantum gravity. Alternatively, one can try to eliminate the triangulation-dependence somehow: for example, either by summing over triangulations, or by taking a limit as the triangulation becomes ever finer. Both these options lead to questions of convergence.

Since spin foam models corresponding to topological quantum field theories are well-understood, while others are not, it seems natural to develop a perturbation theory for spin foam models. This might allow us to study nontopological theories as perturbations of topological ones. This idea is already present in the work of Freidel and Krasnov [15], who consider a large class of spin foam models as perturbations of BF theory. It is also related to Smolin's work on strings as perturbations of evolving spin networks [30].

In this paper we start with any spin foam model that involves a fixed triangulation of a manifold representing spacetime, and consider an arbtrary perturbation of the vertex amplitudes of this model. (Perturbing the edge and face amplitudes works similarly, so to reduce the complexity of our notation we do not consider such perturbations.) We expand the evolution operators of the perturbed model as convergent power series in the coupling constant governing the perturbation.

The terms in this power series can be efficiently computed when the unperturbed model is a topological quantum field theory, such as those listed above. In fact, in this case the power series can be explicitly summed in the limit where the number of top-dimensional simplices in the triangulation goes to infinity, as long as we also renormalize the coupling constant. This limit corresponds to a 'dilute gas of interactions'. Taking this limit, we obtain models that are triangulation-independent but not topological quantum field theories. Examples include 2d Yang-Mills theory and a host of other theories in 2 dimensions. However, we show that the dilute gas limit does not give interesting spin foam models in higher dimensions. We discuss the implications of this fact in the Conclusions.

In what follows, we assume some familiarity with the basic definitions concerning spin networks and spin foams. These can be found in the references [2, 3, 4, 29].

2 Partition Functions

In this section we give a power series for the partition function of a spin foam model obtained by perturbing the vertex amplitudes of a fixed 'unperturbed' model. For simplicity we restrict attention to the case when spacetime is a compact manifold without boundary. In the next section we generalize this result to the case when spacetime is compact manifold with boundary (i.e., a cobordism).

Let M be a compact connected oriented piecewise-linear n-manifold equipped with a triangulation Δ. Let $\mathcal{V}$ (resp. $\mathcal{E}$, $\mathcal{F}$) be the set of 0-cells (resp. 1-cells, 2-cells) of the dual 2-skeleton of this triangulation. We call these 'vertices', 'edges', and 'faces', respectively. We assume that each face and edge is equipped with an orientation. By Poincaré duality, we may also think of $\mathcal{V}$ (resp. $\mathcal{E}$, $\mathcal{F}$) as the set of simplices in Δ with dimension n (resp. $n-1$, $n-2$). We find it useful to freely switch between these points of view.

We now fix a spin foam model to perturb about. We shall not need much detailed information about this model. First we fix a suitable category — typically a category of representations of some group or quantum group. We assume that this category is equipped with a tensor product, and that every representation in the category is a direct sum of representations chosen from some finite set R of inequivalent irreducible representations. Finally, we assume that the hom-sets of our category are finite-dimensional Hilbert spaces. The topological quantum field theories listed in the Introduction satisfy all these assumptions.

Given this, a 'spin foam' is a way of labelling each face in $\mathcal{F}$ with a representation in R and labelling each edge in $\mathcal{E}$ with an intertwiner chosen from a fixed orthonormal basis of $\mathrm{hom}(\rho_1 \otimes \cdots \otimes \rho_n, \rho'_1 \otimes \cdot \otimes \rho'_m)$, where ρ_i are the representations labelling the faces incoming to this edge, and ρ'_j are the representations labelling faces outgoing to this edge.

We define the partition function of our unperturbed spin foam model by:

$$Z_0(M) = \sum_F \prod_{f\in\mathcal{F}} A_0(F,f) \prod_{e\in\mathcal{E}} A_0(F,e) \prod_{v\in\mathcal{V}} A_0(F,v).$$

Here the sum is taken over all spin foams F. The 'face amplitude' $A_0(F,f)$ is a complex number depending on the representation that F assigns to the face f. The 'edge amplitude' $A_0(F,e)$ is a complex number depending on the intertwiner that F assigns to the edge e. Finally, the 'vertex amplitude' $A_0(F,v)$ is a complex number depending on the representations and intertwiners labelling faces and edges incident to v.

Next we perturb the vertex amplitudes of this spin foam model:

$$\begin{aligned} A(F,f) &= A_0(F,f) \\ A(F,e) &= A_0(F,e) \\ A(F,v) &= A_0(F,v) + \lambda A_1(F,v). \end{aligned} \tag{1}$$

The partition function of the perturbed model is given by:

$$Z(M) = \sum_F \prod_{f\in\mathcal{F}} A(F,f) \prod_{e\in\mathcal{E}} A(F,e) \prod_{v\in\mathcal{V}} A(F,v).$$

We may expand this partition function as a power series in the coupling constant λ:

$$Z(M) = \sum_{j=0}^{\infty} \lambda^j Z_j(M)$$

where

$$Z_j(M) = \sum_{S\subseteq\mathcal{V},\, |C|=j} \sum_F \prod_{f\in\mathcal{F}} A(F,f) \prod_{e\in\mathcal{E}} A(F,e) \prod_{v\notin C} A_0(F,v) \prod_{v\in C} A_1(F,v)$$

In other words, the jth term in the perturbation expansion for the partition function is a sum over spin foams where nontrivial interactions occur at exactly j vertices: the vertices in C. We call elements of C 'interaction vertices'. The zeroth-order term, $Z_0(M)$, is just the partition function of the original unperturbed spin foam model. Note that this perturbation expansion converges for a very simple reason: it is a finite sum! The reason is that $Z_j(M) = 0$ when j exceeds the total number of vertices in the dual 2-skeleton of the triangulation of M. We denote this number by $|\mathcal{V}|$.

To write the formula for $Z_j(M)$ more tersely, let us define a 'configuration' to be a triangulation Δ of M together with a given subset $C \subseteq \mathcal{V}$. (We may also think of C as a subset of the n-simplices in the triangulation.) For any configuration (Δ, C), define

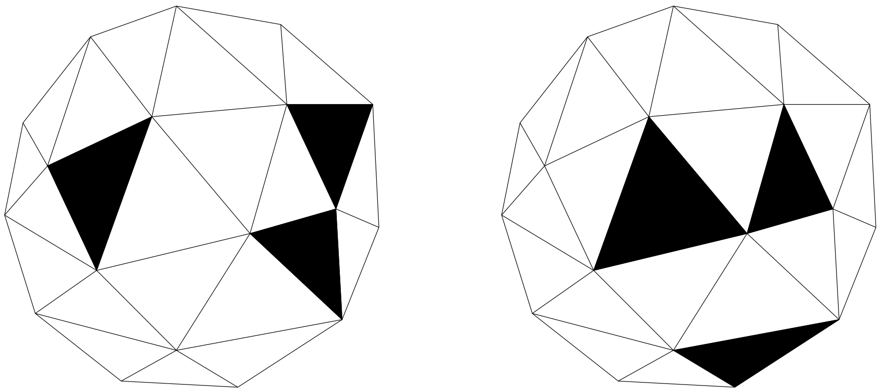
$$Z(\Delta, C) = \sum_F \prod_{f \in \mathcal{F}} A(F, f) \prod_{e \in \mathcal{E}} A(F, e) \prod_{v \notin C} A_0(F, v) \prod_{v \in C} A_1(F, v)$$

where we sum over spin foams F living in the dual 2-skeleton of the triangulation. Then we have

$$Z_j(M) = \sum_{C \subseteq \mathcal{V},\, |C| = j} Z(\Delta, C). \tag{2}$$

To make further progress, we need to know a bit more about our original unperturbed spin foam model. Let us assume that in the unperturbed model, the partition function is independent of the triangulation of M, including the choice of orientations for faces and edges. More importantly, let us assume that this can shown using purely *local* calculations, by checking invariance under the Pachner moves [22]. All the topological quantum field theories listed in the Introduction meet this assumption.

Define two configurations to be 'equivalent' if they become combinatorially equivalent after repeatedly applying Pachner moves to n-simplices that do not lie in the given subset. Figure 2 shows two equivalent configurations with the same underlying triangulation. In this example the manifold M is the 2-sphere, and the 2-simplices in C are colored black.



1. Two equivalent configurations with the same underlying triangulation

Define a 'j-element configuration' to be one for which $|C| = j$. By our assumptions on the unperturbed spin foam model, $Z(\Delta, C)$ is the same for any two configurations in the same equivalence class. This allows us to rewrite the sum in equation (2) as a sum over equivalence classes of j-element configurations (Δ, C), weighted by how many configurations are in each equivalence class.

If we fix a triangulation Δ, the number of j-element configurations (Δ, C) is $\binom{|\mathcal{V}|}{j}$. This is always less than $|\mathcal{V}|^j/j!$, so keeping only the low-order terms in the perturbation expansion for the partition function of the perturbed spin foam model should give a good approximation when

$$|\lambda| \ll \frac{1}{|\mathcal{V}|}.$$

When j is small, there should not be many equivalence classes of j-element configurations. We thus have a practical recipe for approximately computing the partition function in this case.

3 Evolution Operators

Now we turn to the evolution operators associated to cobordisms. Our unperturbed spin foam model assigns a Hilbert space $Z(S)$ of 'kinematical states' to any compact oriented $(n-1)$-manifold S equipped with a triangulation, and to any oriented cobordism $M\colon S \to S'$ equipped with a triangulation Δ compatible with those of S and S', it assigns an 'evolution operator' $Z_0(M)\colon Z(S) \to Z(S')$. We begin by recalling how these are defined, and then give formula for the evolution operators of the perturbed spin foam model. All the formulas are very much like those in the previous section. In fact, partition functions are just the special case of evolution operators where S and S' are empty.

The Hilbert space $Z(S)$ is defined to have an orthonormal basis given by spin networks in the dual 2-skeleton of the triangulation of S. To describe the evolution operator $Z_0(M)\colon Z(S) \to Z(S')$, it thus suffices to give a formula for the transition amplitude between spin network states $\psi \in Z(S)$, $\psi' \in Z(S')$. The formula generalizes that for the partition function of a closed manifold:

$$\langle \psi', Z_0(M)\psi \rangle =$$
$$\sum_F \prod_{f\in\mathcal{F}} A_0(F,f) \prod_{f\in\mathcal{F}'} A_0(F,f)^{\frac{1}{2}} \prod_{e\in\mathcal{E}} A_0(F,e) \prod_{e\in\mathcal{E}'} A_0(F,e)^{\frac{1}{2}} \prod_{v\in\mathcal{V}} A_0(F,v).$$

Here $\mathcal{V}$ (resp. $\mathcal{E}, \mathcal{F}$) denotes the set of vertices (resp. edges, faces) of the triangulation of M that do not intersect the boundary of M, while $\mathcal{E}'$ (resp. $\mathcal{F}'$) denotes the set of edges (resp. faces) that intersect, but do not lie in, the boundary of M. It is important in this formula to make a consistent choice of the square roots involved. It then follows that

$$Z_0(M)Z_0(M') = Z_0(MM')$$

whenever M and M' are composable.

Next we perturb the vertex amplitudes of our spin foam model as in equation (1), and define perturbed evolution operators $Z(M)\colon Z(S) \to Z(S')$ by

$$\langle \psi', Z(M)\psi \rangle =$$
$$\sum_F \prod_{f\in\mathcal{F}} A(F,f) \prod_{f\in\mathcal{F}'} A(F,f)^{\frac{1}{2}} \prod_{e\in\mathcal{E}} A(F,e) \prod_{e\in\mathcal{E}'} A(F,e)^{\frac{1}{2}} \prod_{v\in\mathcal{V}} A(F,v).$$

As before, this has a perturbation expansion

$$Z(M) = \sum_{j=0}^{\infty} \lambda^j Z_j(M) \tag{3}$$

with only finitely many nonzero terms. Moreover, we have

$$Z(M)Z(M') = Z(MM')$$

whenever M and M' are composable.

To give a formula for $Z_j(M)$, we define a 'configuration' as in the previous section, and for any configuration (Δ, C) we let the operator $Z(\Delta, C)\colon Z(S) \to Z(S')$ be given by

$$\langle \psi', Z(\Delta, C)\psi \rangle =$$
$$\sum_F \prod_{f\in\mathcal{F}} A(F,f) \prod_{f\in\mathcal{F}'} A(F,f)^{\frac{1}{2}} \prod_{e\in\mathcal{E}} A(F,e) \prod_{e\in\mathcal{E}'} A(F,e)^{\frac{1}{2}} \prod_{v\notin C} A_0(F,v) \prod_{v\in C} A_1(F,v).$$

Then we have

$$Z_j(M) = \sum_{C\subseteq\mathcal{V},\ |C|=j} Z(\Delta, C) \tag{4}$$

As before, if our unperturbed spin foam model is invariant under the Pachner moves, we can rewrite this as a sum over equivalence classes of j-element configurations. And as before, the first few terms of equation (3) will be easy to compute in this case, and should give a good approximation to $Z(M)$ when

$$|\lambda| \ll \frac{1}{|\mathcal{V}|}.$$

4 The 'Dilute Gas' Limit

As a simple application of the ideas above, we now calculate the evolution operators for a perturbed spin foam model in the $|\mathcal{V}| \to \infty$ limit, assuming that the original unperturbed spin foam model is invariant under the Pachner moves. As we take the limit, we rescale the coupling constant as follows:

$$\lambda = \frac{g}{|\mathcal{V}|} \tag{5}$$

for some fixed value of g. This is a simple form of 'coupling constant renormalization', with g playing the role of the renormalized coupling constant. Rescaling λ this way ensures that the sum over spin foams is dominated by those where the interaction vertices are separated from each other. Physically speaking, this amounts to considering the limit in which the interaction vertices form a 'dilute gas'.

In what follows, we fix a connected cobordism $M\colon S \to S'$ and let the triangulation Δ of M vary. We fix constants $k > 0$ and $0 < \epsilon < 1$, and consider only triangulations for which fewer than k simplices of any dimension share a given 0-simplex, and for which fewer than $\epsilon|\mathcal{V}|^{1/2}$ of the n-simplices intersect the boundary of M. We claim that with these assumptions, as $|\mathcal{V}| \to \infty$ the sum (4) becomes dominated by configurations in a single equivalence class.

Our argument relies on two plausible conjectures that we have not proved. Define a configuration (Δ, C) to be 'separated' if no two n-simplices in C share a 0-simplex, and no n-simplex in C intersects the boundary of M. We conjecture that:

(A) All separated j-element configurations are equivalent.

(B) Letting Δ vary as above and letting $C \subset \mathcal{V}$ be arbitrary, there are only finitely many equivalence classes of j-element configurations (Δ, C).

Note that conjecture A depends crucially on the fact that M is connected, and also the fact that no n-simplex in a separated configuration touches the boundary of M.

Given conjecture A, the value of $Z(\Delta, C)$ is the same for every separated j-element configuration (Δ, C). Call this value $\alpha_j(M)$. Then we claim that as $|\mathcal{V}| \to \infty$, we have the asymptotic formula

$$Z_j(M) \sim \frac{|\mathcal{V}|^j}{j!}\alpha_j(M). \tag{6}$$

In other words, separated configurations dominate the sum over spin foams.

To prove equation (6), note that in equation (4) we are computing $Z_j(M)$ as a sum over all j-element configurations with the same underlying triangulation. There are exactly $\binom{|\mathcal{V}|}{j}$ such configurations. In the limit as $|\mathcal{V}| \to \infty$, this number is asymptotic to $|\mathcal{V}|^j/j!$. Of these configurations, fewer than

$$(k(j-1) + \epsilon|\mathcal{V}|^{1/2})\binom{|\mathcal{V}|}{j-1}$$

are nonseparated, since to form a nonseparated configuration we can choose the first $j-1$ of the n-simplices arbitrarily, but the last one must either share at least one 0-simplex with the rest, or intersect the boundary. As $|\mathcal{V}| \to \infty$, this number is on the order of $|\mathcal{V}|^{j-1/2}$, so in the $|\mathcal{V}| \to \infty$ limit almost all the configurations being summed over are separated. Each such configuration contributes $\alpha_j(M)$ to the sum for $Z_j(M)$, so the total contribution of the separated configurations is asymptotic to $\alpha_j(M)|\mathcal{V}|^j/j!$. Thus to finish proving equation (6) it suffices to check that the nonseparated configurations contribute an amount of order at most $|\mathcal{V}|^{j-1/2}$ to the sum for $Z_j(M)$. Because there are on the order of $|\mathcal{V}|^{j-1/2}$, and two configurations in the same equivalence contribute the same amount, this follows from conjecture B.

Equations (5) and (6) imply that

$$\lim_{|\mathcal{V}|\to\infty} \lambda^j Z_j(M) = \frac{g^j}{j!}\alpha_j(M). \tag{7}$$

This means that if we renormalize the coupling constant appropriately while taking the $|\mathcal{V}| \to \infty$ limit, each term in the power series expansion for the evolution operator $Z(M)$ converges to a quantity that can be explicitly computed using any triangulation of M that is large enough to contain a separated j-element configuration.

Naively one might wish to conclude from equation (7) that

$$\lim_{|\mathcal{V}|\to\infty} Z(M) = \sum_{j=0}^{\infty} \frac{g^j}{j!}\alpha_j(M),$$

but of course there are two problems. First, the power series on the right-hand side of this equation might not converge. Second, the fact that one power series converges to another term-by-term does not imply that the sum of the first converges to the sum of the second. In what follows we shall treat the first problem but not the second. In other words, we show that for any choice of the interaction amplitude $A_1(F,v)$, the power series

$$Z_g(M) = \sum_{j=0}^{\infty} \frac{g^j}{j!}\alpha_j(M). \tag{8}$$

converges for all values of g.

When M, M' are connected cobordisms and MM' is their composite, we have

$$\alpha_j(M)\alpha_k(M') = \alpha_{j+k}(MM').$$

This implies, at the level of formal power series, that

$$Z_g(M)Z_{g'}(M') = Z_{g+g'}(MM'). \tag{9}$$

A particularly important special case of this equation occurs when $M\colon S \to S$ is the cylinder $S\times I$, where I is the closed unit interval. In this case we have $(S\times I)^2 = S\times I$, so

$$Z_g(M)Z_{g'}(S\times I) = Z_{g+g'}(S\times I).$$

This implies that

$$\begin{aligned} Z_g(S\times I) &= \exp(-gH_S)Z_0(S\times I) \\ &= Z_0(S\times I)\exp(-gH_S) \end{aligned} \tag{10}$$

for some operator H_S, which is given explicitly by

$$H_S = -\alpha_1(S \times I).$$

We are now in a position to see that the power series for $Z_g(M)$ in equation (8) converges for all values of g. When M is the cylinder $S \times I$, the power series for $Z_g(M)$ converges by equation (10), because the power series for $\exp(-gH_S)$ converges. In general, we can write any connected cobordism M as a composite $M_1M_2M_3$ where M_2 is a cylinder, and use equation (9) to write $Z_g(M)$ as $Z_0(M_1)Z_g(M_1)Z_0(M_2)$, thus reducing to the cylinder case. For the sake of completeness, it is also useful to define $Z_g(M)$ when M is not connected. In this case, we define $Z_g(M)$ to be the (tensor) product of evolution operators for its connected components. This again allows us to write $Z_g(M)$ as a convergent power series in g.

It is clear that the perturbative spin foam models under consideration are not topological quantum field theories. In a topological theory, the evolution operator associated to a cylinder is just the projection onto the subspace of physical states. Here, however, we have 1-parameter semigroup of evolution operators depending on g. At $g = 0$, this reduces to the projection operator for the topological quantum field theory being perturbed about.

It may seem puzzling that the parameter g, originally introduced as a renormalized coupling constant, is now playing a role a bit like 'time'. In fact, we should not think of g as 'time', but as an extensive parameter measuring the mean number of interaction vertices. The reason is that in formula (8) for the evolution operator, the term corresponding to a spin foam with j interaction vertices is weighted by a factor of $g^j/j!$. Up to a normalization factor, this is just the Poisson distribution with mean g. (To normalize this Poisson distribution, we could divide $Z_g(M)$ by $\exp(g)$.) We can thus think of $Z_g(M)$ as the evolution operator for a spacetime manifold M containing a large spin foam for which the interaction vertices are separated and their number is distributed according to the Poisson distribution with mean g.

This interpretation of g as an extensive parameter is clearest in the case of 2-dimensional Yang-Mills theory. Witten [34] showed that 2d Yang-Mills theory can be described as a perturbation of 2d BF theory, and his work fits nicely into the language of spin foam models [15]. In this case, the renormalized coupling constant g turns out to have the physical significance of *area*. To illustrate the results above, we now turn to this example.

Let R be a complete set of inequivalent irreducible representations of a compact Lie group G. Of course, R is finite only when G is, so our results strictly apply only to that case, but apart from some convergence issues that we mention below, they generalize straightforwardly to the compact case. To simplify our description of the face, edge, and vertex amplitudes of 2d BF theory, we can use the orientation of M to consistently orient all the faces. The amplitude for a face labelled by the representation ρ is just $\dim(\rho)$. Each edge has two faces incident to it, which must be labelled with the same representation ρ for there to be an intertwiner labelling the edge; the amplitude for the edge is then $\dim(\rho)^{-1}$. Finally, each vertex has a collection of faces incident to it, which must all be labelled by the same representation ρ, and the amplitude for the vertex is then $\dim(\rho)$.

It follows that when M is a connected surface without boundary, its partition function is

$$Z_0(M) = \sum_{\rho \in R} \dim(\rho)^{\chi(M)}.$$

(This may not converge when G is not finite.) We can consider an arbitrary perturbation of the vertex amplitudes, which is necessarily of the form

$$A_1(F, v) = f(\rho)$$

where $f\colon R \to \mathbb{C}$ and ρ is the representation labelling all the faces incident to v. If we take S to be the circle triangulated using a single 1-simplex, then $Z(S)$ has an orthonormal basis ψ_ρ with one element for each irreducible representation $\rho \in R$, and the operator H_S given in equation (10) has the form

$$H_S \psi_\rho = -f(\rho)\psi_\rho.$$

It follows that

$$Z_g(M) = \sum_{\rho \in R} \dim(\rho)^{\chi(M)}\, e^{g f(\rho)}.$$

This may converge even when $Z_0(M)$ does not. In particular, if the Lie algebra of G is equipped with an invariant inner product, and we let $f(\rho)$ be the minus the Casimir of the representation ρ, the sum for $Z_g(M)$ converges whenever $g > 0$. In this case H_S is the Hamiltonian for 2d Yang-Mills theory, and $Z_g(M)$ is the partition function for 2d Euclidean Yang-Mills theory on a surface with area g. Taking g imaginary gives the Lorentzian theory.

Besides Yang-Mills theory, this setup gives many other triangulation-independent spin foam models in 2 dimensions. If we take G to be finite, any function $f\colon R \to \mathbb{C}$ will give such a model. The most interesting models are the 'unitary' ones, where f is real-valued. If G is compact, we obtain a unitary model with convergent Euclidean partition functions for surfaces with high genus from any function $f\colon R \to \mathbb{R}$ that grows sufficiently fast. We can also work with a quantum group at root of unity instead of a group. This gives theories that are perturbations of the G/G gauged WZW model instead of BF theory.

Since the dilute gas limit gives a host of triangulation-independent spin foam models in 2 dimensions, we might hope for similar results in higher dimensions. However, these hopes are in vain. To see this, let $Z_0(S)$ be the Hilbert space of physical states associated to the manifold S, i.e., the range of the projection $Z_0(S \times I)$. Since $H_S = H_S Z_0(S \times I)$, H_S is completely determined by its value on physical states. In what follows, we show that on physical states, H_S is just a multiple of the identity in theories for which $Z_0(S^{n-1})$ is 1-dimensional. This includes all the theories of Turaev-Viro type in 3 dimensions and of Crane-Yetter type in 4 dimensions. Thus for such theories, any perturbation of the vertex amplitudes has a trivial effect in the dilute gas limit: for any cobordism M, we have

$$Z_g(M) = \exp(-cg) Z_0(M)$$

for some constant c.

To see that H_S is a multiple of the identity on physical states when $Z_0(S^{n-1})$ is 1-dimensional, we use some facts about the structure of topological quantum field theories [28]. First, the obvious cobordism from $S^{n-1} \cup S^{n-1}$ to S^{n-1} makes $A = Z_0(S^{n-1})$ into a commutative algebra. Second, for every compact connected oriented $(n-1)$-manifold S, the obvious cobordism

$$M \colon S^{n-1} \cup S \to S$$

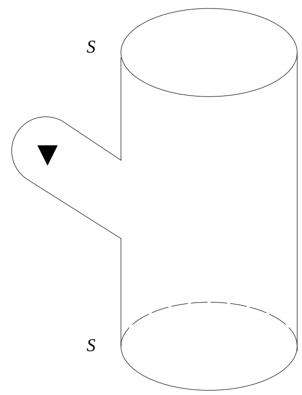
makes $Z_0(S)$ into an A-module. The cobordism $D^n \colon \emptyset \to S^{n-1}$ gives an operator

$$\alpha_1(D^n) \colon \mathbb{C} \to A,$$

and the image of $1 \in \mathbb{C}$ under this operator gives a special element of A which we call $-H$. This has the property that for any physical state $\psi \in Z_0(S)$,

$$H_S \psi = H \psi,$$

where the right-hand side is defined using the action of $H \in A$ on $Z_0(S)$. In Figure 2 we show this action of H on $Z_0(S)$; here we have omitted all the n-simplices in the triangulation except the one corresponding to the single interaction vertex, shown in black. If $Z_0(S)$ is 1-dimensional, H is a multiple of the identity in A, so H_S is a fixed multiple of the identity, independent of S.



2. The action of H on physical states

5 Conclusions

While the spin foam perturbation theory developed here is applicable to a wide variety of situations, we have seen that one simple-minded way of using it to construct triangulation-independent spin foam models gives trivial results except when spacetime has dimension 2. It is worth reflecting on exactly why this happens. Technically, the reason is that in higher dimensions the TQFTs being perturbed about assign a 1-dimensional physical Hilbert space to the sphere. The reason for this, in turn, is that these theories are obtained by quantizing theories of flat connections, and the moduli space of flat connections on S^{n-1} is just a point when $n \neq 2$.

A perhaps more illuminating explanation is that in the dilute gas limit, nontrivial interactions occur at a discrete set of points. Removing a point from a manifold changes its moduli space of flat connections only in the 2-dimensional case, because only then is there a noncontractible loop going around the removed point. Thus, only in this case can the operator shown in Figure 2 have an interesting effect.

This suggests a number of ideas. Naively, one might hope that perturbing edge or face amplitudes could give more interesting results in higher dimensions. However, it does not. Everything we have done for perturbations of vertex amplitudes works very similarly for perturbations of edge and face amplitudes. A more interesting idea is to consider more general state sum models in which all the cells of the dual skeleton are labelled, not just those of dimensions 0 and 1. Such models correspond to field theories based not on connections, but on categorified analogues of connections, such as connective structures on gerbs [8]. Some topological quantum field theories of this type have been studied by Yetter [35], Porter [24], and Mackaay [19, 20]. In the dilute gas limit, perturbations of these should give triangulation-independent models that are not topological quantum field theories. These should be more interesting in dimensions above 2 than the examples considered here.

Ultimately, to make contact with real physics, one must study theories with propagating degrees of freedom. Whether perturbation about a topological quantum field theory can really be useful here is not clear. However, for it to have any chance of being useful, we must go beyond the dilute gas limit, and explore the effect of nonseparated configurations in the perturbation expansion.

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