Lectures on Higher Gauge Theory – I

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Gauge Theory

Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a Lie group element to each path:

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while reversing the direction of a path corresponds to taking inverses:

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "g^{-1}"'] \end{tikzcd}$$

The associative law makes the holonomy along a triple composite unambiguous:

$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \arrow[r, bend left, "g''"] & \bullet
\end{tikzcd}
$$

So: *the topology dictates the algebra!*

Higher Gauge Theory

Higher gauge theory describes the parallel transport not only of point particles, but also 1-dimensional strings. For this we must categorify the notion of a group! A '2-group' has objects:

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and also morphisms:

$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "g'"'{name=D}] & \bullet
\arrow[from=U, to=D, Rightarrow, "f"]
\end{tikzcd}
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$$\begin{tikzcd} \bullet \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_1'"'] \arrow[r, phantom, "f_1 \Downarrow"] & \bullet \arrow[r, bend left, "g_2"] \arrow[r, bend right, "g_2'"'] \arrow[r, phantom, "f_2 \Downarrow"] & \bullet \end{tikzcd}$$

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\end{tikzcd}
$$

multiply morphisms:

$$
\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=40, "g_1"{name=A}] \arrow[r, bend right=40, "g_1'"'{name=B}] \arrow[from=A, to=B, Rightarrow, "f_1"'] & \bullet \arrow[r, bend left=40, "g_2"{name=C}] \arrow[r, bend right=40, "g_2'"'{name=D}] \arrow[from=C, to=D, Rightarrow, "f_2"'] & \bullet
\end{tikzcd}
$$

and also compose morphisms:

$$
\begin{tikzcd}[column sep=huge]
\bullet \arrow[r, bend left=70, "g"{name=A}] \arrow[r, "g'"{name=B, description}] \arrow[r, bend right=70, "g''"'{name=C}] \arrow[from=A, to=B, Rightarrow, "f"] \arrow[from=B, to=C, Rightarrow, "f'"] & \bullet
\end{tikzcd}
$$

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Let's make this precise!

1. In this lecture we'll categorify the theory of Lie groups and Lie algebras.
2. Then we'll categorify principal bundles and their classifying spaces.
3. Finally we'll categorify connections and parallel transport.

The resulting mathematics has fascinating relations to string theory.

2-Groups

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A **2-group** is a monoidal category where every object g has a 'weak inverse':

$$g \otimes \bar{g} \cong 1, \qquad \bar{g} \otimes g \cong 1$$

and every morphism $f: g \to g'$ has an inverse:

$$ff^{-1} = 1, \qquad f^{-1}f = 1.$$

A **homomorphism** between 2-groups is a monoidal functor.

A **2-homomorphism** is a monoidal natural transformation.

So, the 2-groups $\mathcal{G}$ and $\mathcal{G}'$ are **equivalent** if there are homomorphisms

$$F\colon \mathcal{G} \to \mathcal{G}' \qquad \bar{F}\colon \mathcal{G}' \to \mathcal{G}$$

that are inverses up to 2-isomorphism:

$$F\bar{F} \cong 1, \qquad \bar{F}F \cong 1.$$

Theorem. 2-groups are classified up to equivalence by quadruples consisting of:

- a group G,
- an abelian group H,
- an action α of G as automorphisms of H,
- an element $[a] \in H^3(G, H)$.

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- G is the group of isomorphism classes of objects of $\mathcal{G}$.
- H is the group of automorphisms of $1 \in \mathcal{G}$.
- The action of G on H is defined like this:

$$\begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[r, bend left, "1", ""{name=U, below}] \arrow[r, bend right, "1"', ""{name=D, above}] & \bullet \arrow[r, "g^{-1}"] & \bullet \arrow[Rightarrow, from=U, to=D, "h"'] \end{tikzcd}$$

- $a \colon G^3 \to H$ comes from the associator, and the pentagon identity says it's a cocycle!

Lie 2-Algebras

To categorify the concept of 'Lie algebra' we must first treat the concept of 'vector space':

A **2-vector space** L is a category for which the set of objects and the set of morphisms are vector spaces, and all the category operations are linear.

We can also define **linear functors** between 2-vector spaces, and **linear natural transformations** between these, in the obvious way.

Theorem. The 2-category of 2-vector spaces, linear functors and linear natural transformations is equivalent to the 2-category of:

- 2-term chain complexes $C_1 \xrightarrow{d} C_0$,
- chain maps between these,
- chain homotopies between these.

The objects of the 2-vector space form the space C_0. The morphisms $f\colon 0 \to x$ form the space C_1, and $df = x$.

A **Lie** 2-**algebra** consists of:

- a 2-vector space L

equipped with:

- a functor called the **bracket**:

$$[\cdot,\cdot]\colon L \times L \to L,$$

  bilinear and skew-symmetric as a function of objects and morphisms,

- a natural isomorphism called the **Jacobiator**:

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$

  trilinear and antisymmetric as a function of the objects x, y, z.

We also impose the **Jacobiator identity**:

$$
\begin{tikzcd}[column sep=small]
& {[[[w,x],y],z]} \arrow[dl, "{[J_{w,x,y},z]}"'] \arrow[dr, "{J_{[w,x],y,z}}"] & \\
{[[[w,y],x],z]+[[w,[x,y]],z]} \arrow[d, "{J_{[w,y],x,z}+J_{w,[x,y],z}}"'] & & {[[[w,x],z],y]+[[w,x],[y,z]]} \arrow[d, "{[J_{w,x,z},y]+1}"] \\
\begin{array}{c}[[[w,y],z],x]+[[w,y],[x,z]]\\+[w,[[x,y],z]]+[[w,z],[x,y]]\end{array} \arrow[d, "{[J_{w,y,z},x]+1}"'] & & \begin{array}{c}[[w,[x,z]],y]\\+[[w,x],[y,z]]+[[[w,z],x],y]\end{array} \arrow[d, "{J_{w,[x,z],y}+J_{[w,z],x,y}+J_{w,x,[y,z]}}"] \\
\begin{array}{c}[[[w,z],y],x]+[[w,[y,z]],x]\\+[[w,y],[x,z]]+[w,[[x,y],z]]\\+[[w,z],[x,y]]\end{array} \arrow[rr, "{[w,J_{x,y,z}]+1}"'] & & \begin{array}{c}[[[w,z],y],x]+[[w,z],[x,y]]+[[w,y],[x,z]]\\+[w,[[x,z],y]]+[[w,[y,z]],x]\\+[w,[x,[y,z]]]\end{array}
\end{tikzcd}
$$

must commute.

Just as 2-vector spaces are the same as 2-term chain complexes, Lie 2-algebras are the same as 2-term L_∞-algebras.

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We can define homomorphisms between Lie 2-algebras, and 2-homomorphisms between these. The Lie 2-algebras L and L' are **equivalent** if there are homomorphisms

$$F\colon L \to L' \qquad \bar{F}\colon L' \to L$$

that are inverses up to 2-isomorphism.

Theorem. Lie 2-algebras are classified up to equivalence by quadruples consisting of:

- a Lie algebra $\mathfrak{g}$,
- a vector space $\mathfrak{h}$,
- a representation ρ of $\mathfrak{g}$ on $\mathfrak{h}$,
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This is just like the classification of 2-groups, but with Lie algebra cohomology replacing group cohomology!

The 3-cocycle $j \colon \mathfrak{g}^{\otimes 3} \to \mathfrak{h}$ comes from the Jacobiator.

The Lie 2-Algebra $\mathfrak{g}_k$

Suppose $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{R}$. To get a Lie 2-algebra with $\mathfrak{g}$ as objects we need:

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Suppose ρ is irreducible. To get Lie 2-algebras with nontrivial Jacobiator, we need $H^3(\mathfrak{g}, \mathfrak{h}) \neq 0$. This only happens when $\mathfrak{h} = \mathbb{R}$ is the trivial representation. Then we have $H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$, with a nontrivial 3-cocycle given by:

$$j(x, y, z) = \langle x, [y, z] \rangle$$

Using $k \in \mathbb{R}$ times this to define the Jacobiator, we get a Lie 2-algebra we call $\mathfrak{g}_k$.

In short: *every simple Lie algebra gives a one-parameter family of Lie 2-algebras!*

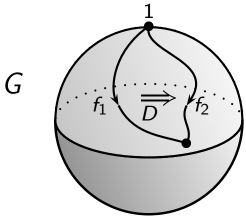
Does $\mathfrak{g}_k$ Come From a Lie 2-Group?

There is a 2-group that 'wants' to have $\mathfrak{g}_k$ as its Lie 2-algebra. It has G as its set of objects and $\mathrm{U}(1)$ as the endomorphisms of any object. However, unless $k = 0$ we cannot make its associator smooth globally — only locally. Henriques has formalized this quite nicely.

On the other hand, when k is an integer, $\mathfrak{g}_k$ is *equivalent* to a Lie 2-algebra that *does* come from a Lie 2-group:

Theorem. For any $k \in \mathbb{Z}$, there is an infinite-dimensional Lie 2-group $\mathrm{String}_k G$ whose Lie 2-algebra is equivalent to $\mathfrak{g}_k$.

An object of $\mathrm{String}_k G$ is a smooth path in G starting at the identity. A morphism from f_1 to f_2 is an equivalence class of pairs (D, α) consisting of a smooth homotopy D from f_1 to f_2 together with $\alpha \in \mathrm{U}(1)$:



Any two such pairs (D_1, α_1) and (D_2, α_2) have a 3-ball B whose boundary is $D_1 \cup D_2$. The pairs are equivalent when

$$\exp\left(2\pi i k \int_B \nu\right) = \alpha_2/\alpha_1$$

where ν is the left-invariant closed 3-form on G with

$$\nu(x, y, z) = \langle [x, y], z \rangle$$

and $\langle \cdot, \cdot \rangle$ is the smallest invariant inner product on $\mathfrak{g}$ such that ν gives an integral cohomology class.

There's an easy way to compose morphisms in $\mathrm{String}_k G$, and the resulting category inherits a Lie 2-group structure from the Lie group structure of G.

Relation to Loop Groups

We can also describe $\mathrm{String}_k G$ using central extensions of the loop group of G:

Theorem. An object of $\mathrm{String}_k G$ is a smooth path in G starting at the identity. Given objects $f_1, f_2 \in \mathrm{String}_k G$, a morphism

$$\widehat{\ell} \colon f_1 \to f_2$$

is an element $\widehat{\ell} \in \widehat{\Omega_k G}$ with

$$p(\widehat{\ell}) = f_2/f_1 \in \Omega G$$

where $\widehat{\Omega_k G}$ is the level-k central extension of the loop group ΩG:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \xrightarrow{p} \Omega G \longrightarrow 1$$