Lectures on Higher Gauge Theory – II

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Čech Cohomology for Bundles

If G is a topological group and M is a topological space, we can describe a principal G-bundle $P \to M$ using a **Čech cocycle**. This consists of an open cover $\mathcal{U} = \{U_i\}$ of M together with **transition functions**

$$g_{ij} \colon U_i \cap U_j \to G$$

making these triangles commute for all $x \in U_i \cap U_j \cap U_k$:

$$\begin{tikzcd}
& \bullet \arrow[dr, "g_{jk}(x)"] & \\
\bullet \arrow[ur, "g_{ij}(x)"] \arrow[rr, "g_{ik}(x)"'] & & \bullet
\end{tikzcd}$$

Two Čech cocycles define isomorphic bundles iff they are **cohomologous**, meaning there are functions

$$f_i \colon U_i \to G$$

making these squares commute for all $x \in U_i \cap U_j$:

$$\begin{tikzcd}
\bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "f_i(x)"'] & \bullet \arrow[d, "f_j(x)"] \\
\bullet \arrow[r, "g'_{ij}(x)"'] & \bullet
\end{tikzcd}$$

The set of cohomology classes of Čech cocycles is called $\check{H}(\mathcal{U}, G)$. Taking the limit as we refine the open cover, we obtain the (first) **Čech cohomology** of M with coefficients in G:

$$\check{H}(M, \mathcal{G}) = \varinjlim_{\mathcal{U}} \check{H}(\mathcal{U}, G)$$

There is a bijection between $\check{H}(M, G)$ and the set of isomorphism classes of principal G-bundles over M.

A Famous Old Theorem

Here is the result we'd like to categorify:

Thm. Let G be a well-pointed topological group. Let BG, the **classifying space** of G, be the geometric realization of the nerve of G. Then for any paracompact Hausdorff space M, there is a bijection

$$[M, BG] \cong \check{H}(M, G)$$

(A topological group G is **well-pointed** if $1 \in G$ has a neighborhood of which it is a deformation retract.)

Topological 2-Groupoids

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Defn. A **strict 2-groupoid** is a strict 2-category where all morphisms and 2-morphisms are strictly invertible.

Defn. A **strict topological 2-groupoid** $\mathcal{G}$ is a strict 2-groupoid with:

- a topological space of objects,
- a topological space of morphisms,
- a topological space of 2-morphisms,

such that all the 2-groupoid operations are continuous.

Defn. A **strict topological 2-group** is a strict topological 2-groupoid with one object.

From this viewpoint, it has one object: $\bullet$

together with a space of morphisms: $\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$

and a space of 2-morphisms: $\begin{tikzcd} \bullet \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "g'"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "h"] \end{tikzcd}$

We'll use this viewpoint henceforth.

The Čech 2-Groupoid

Let $\mathcal{U} = \{U_i\}$ be an open cover of a topological space M.

Defn. The **Čech 2-groupoid** $\widehat{\mathcal{U}}$ is the strict topological 2-groupoid where:

- objects are pairs (x, i) with $x \in U_i$,
- there is a single morphism from (x, i) to (x, j) when $x \in U_i \cap U_j$, and none otherwise,
- there are only identity 2-morphisms.

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- there are only identity 2-morphisms.

(This is just a topological groupoid promoted to a 2-groupoid by throwing in identity 2-morphisms.)

Čech Cohomology for 2-Bundles

Defn. A **Čech cocycle** with coefficients in the strict topological 2-group $\mathcal{G}$ is continuous pseudofunctor $g\colon \widehat{\mathcal{U}} \to \mathcal{G}$.

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Defn. Let $\check{H}(\mathcal{U}, \mathcal{G})$ be the set of cohomology classes of Čech cocycles. Let the **Čech cohomology** of M with coefficients in $\mathcal{G}$ be the limit as we refine the cover:

$$\check{H}(M, \mathcal{G}) = \varinjlim_{\mathcal{U}} \check{H}(\mathcal{U}, \mathcal{G})$$

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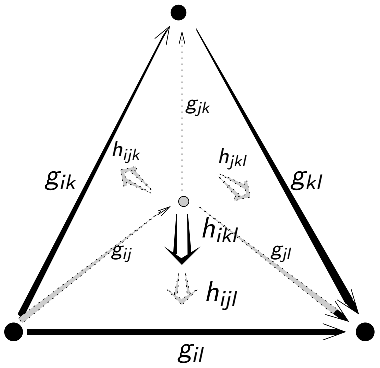
It sends each morphism $(x,i) \to (x,j)$ to a morphism $g_{ij}(x)\colon \bullet \to \bullet$ depending continuously on x.

Composition of morphisms is weakly preserved:

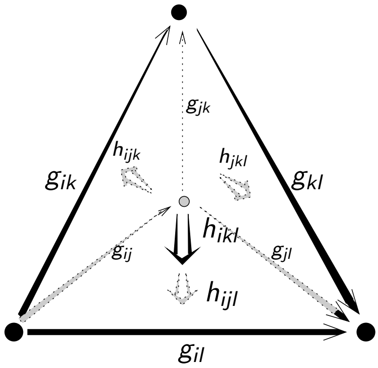
$$
\begin{tikzcd}
& \bullet \arrow[dr, "g_{jk}(x)"] \arrow[d, phantom, "\Downarrow h_{ijk}(x)"] & \\
\bullet \arrow[ur, "g_{ij}(x)"] \arrow[rr, "g_{ik}(x)"'] & & \bullet
\end{tikzcd}
$$

for some 2-morphism $h_{ijk}(x)$ depending continuously on $x \in U_i \cap U_j \cap U_k$.

Finally, the h_{ijk} must make these tetrahedra commute for each $x \in U_i \cap U_j \cap U_k \cap U_l$:



Finally, the h_{ijk} must make these tetrahedra commute for each $x \in U_i \cap U_j \cap U_k \cap U_l$:



Bartels has shown we can assume without loss of generality that $g_{ii}(x) = 1$ and that $h_{ijk}(x) = 1$ whenever two or more of the indices i, j and k agree. Then we have a **normalized** cocycle.

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It sends each morphism $(x, i) \to (x, j)$ of $\hat{\mathcal{U}}$ to a 2-morphism $k_{ij}(x)$ depending continuously on x and filling in this naturality square:

$$
\begin{tikzcd}
\bullet \arrow[r, "g_{ij}(x)"] \arrow[d, "f_i(x)"'] \arrow[dr, phantom, "\swarrow k_{ij}(x)"] & \bullet \arrow[d, "f_j(x)"] \\
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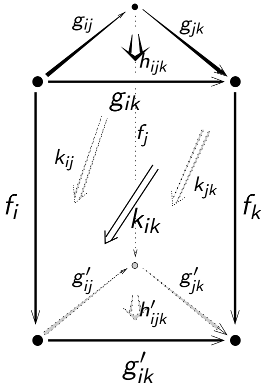
\end{tikzcd}
$$

Finally, the k_{ij} must make these prisms commute:



Categorifying That Famous Old Theorem

Thm. Suppose $\mathcal{G}$ is a well-pointed strict topological 2-group and M is a paracompact Hausdorff space admitting good covers. Then there is a bijection

$$\check{H}(M, \mathcal{G}) \cong [M, B|N\mathcal{G}|]$$

where the topological group $|N\mathcal{G}|$ is the geometric realization of the nerve of $\mathcal{G}$. So, we call $B|N\mathcal{G}|$ the **classifying space** of $\mathcal{G}$.

(A topological 2-group G is **well-pointed** if both the topological groups in its corresponding crossed module are well-pointed. An open cover is **good** if each nonempty finite intersection of sets in the cover is contractible.)

How to Build the Classifying Space

First we think of $\mathcal{G}$ as a group in TopGpd and apply the nerve construction:

$$N\colon \mathrm{TopGpd} \to \mathrm{Top}^{\Delta^{\mathrm{op}}}$$

to get a group in simplicial spaces, $N\mathcal{G}$.

Then we use geometric realization:

$$|\cdot|\colon \mathrm{Top}^{\Delta^{\mathrm{op}}} \to \mathrm{Top}$$

to get a topological group $|N\mathcal{G}|$.

Then we think of $|N\mathcal{G}|$ as a 1-object topological groupoid, and take the nerve and the geometric realization *of this* to get our space $B|N\mathcal{G}|$.

A Corollary: Bundles vs. 2-Bundles

Cor. There is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over M
- elements of the Čech cohomology $\check{H}(M, \mathcal{G})$
- homotopy classes of maps $f \colon M \to B|N\mathcal{G}|$
- elements of the Čech cohomology $\check{H}(M, |N\mathcal{G}|)$
- isomorphism classes of principal $|N\mathcal{G}|$-bundles over M.

Example: Abelian Gerbes

The abelian topological group $\mathrm{U}(1)$ gives a topological 2-group $\mathcal{G}$ with:

- one morphism,
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In this case a principal $\mathcal{G}$-2-bundle is the same as an **abelian $\mathbf{U(1)}$-gerbe**. We have

$$|N\mathcal{G}| = B\mathrm{U}(1) = K(\mathbb{Z}, 2)$$

so the classifying space of $\mathcal{G}$ is

$$B|N\mathcal{G}| = B(B\mathrm{U}(1)) = K(\mathbb{Z}, 3)$$

and abelian $\mathrm{U}(1)$-gerbes over M are classified by

$$[M, K(\mathbb{Z}, 2)] \cong H^3(M, \mathbb{Z}).$$

Example: Nonabelian Gerbes

Any topological group G gives a topological 2-group $\mathcal{G}$ with:

- $\mathrm{Aut}(G)$ as morphisms,
- elements $g \in G$ as 2-morphisms $g\colon h \Rightarrow gh(\cdot)g^{-1}$.

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In this case a principal $\mathcal{G}$-2-bundle is the same as an **nonabelian G-gerbe**. So, such gerbes are classified by

$$[M, B|N\mathcal{G}|]$$

which might be called the **nonabelian cohomology** $H^2(M, G)$.

Example: String 2-Bundles

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There is a topological group $\widehat{G}$ called the **3-connected cover** of G, with $\pi_3 \widehat{G} = 0$ and a continuous homomorphism

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When $G = \mathrm{Spin}(n)$, $\widehat{G}$ is called the **string group**.

In fact, we can build $\widehat{G}$ from a topological 2-group!

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Thm. $\widehat{G}$ is the geometric realization of the nerve of $\mathrm{String} G$:

$$\widehat{G} \simeq |N \mathrm{String} G|$$

The continuous homomorphism

$$p \colon |N\mathrm{String}G| \to G$$

gives an algebra homomorphism:

$$H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|N\mathrm{String}G|, \mathbb{R})$$

Thm. The homomorphism p^* is onto, with kernel generated by the '2nd Chern class' $c_2 \in H^4(BG, \mathbb{R})$.

So, the real characteristic classes of $\mathrm{String}G$-2-bundles are just like those of G-bundles, but with c_2 set to zero!