Lectures on Higher Gauge Theory – II

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February 6, 2009
Čech Cohomology for Bundles

If $G$ is a topological group and $M$ is a topological space, we can describe a principal $G$-bundle $P \to M$ using a Čech cocycle. This consists of an open cover $\mathcal{U} = \{U_i\}$ of $M$ together with transition functions

$$g_{ij} : U_i \cap U_j \to G$$

making these triangles commute for all $x \in U_i \cap U_j \cap U_k$:

\[
\begin{array}{c}
\bullet \\
\downarrow \quad \downarrow \\
\bullet \\
\downarrow \quad \downarrow \\
\bullet \\
\end{array}
\]

\[
g_{ij}(x) \quad \quad g_{jk}(x) \quad \quad g_{ik}(x)
\]
Two Čech cocycles define isomorphic bundles iff they are \textit{cohomologous}, meaning there are functions 

\[ f_i : U_i \rightarrow G \]

making these squares commute for all \( x \in U_i \cap U_j \):

\[
\begin{array}{c}
\bullet \\
g_{ij}(x) \\
\downarrow \quad f_i(x) \quad \downarrow \quad g_{ij}'(x) \\
\bullet \\
\bullet \\
f_j(x) \\
\end{array}
\]
The set of cohomology classes of Čech cocycles is called $\check{H}(\mathcal{U}, G)$. Taking the limit as we refine the open cover, we obtain the (first) Čech cohomology of $M$ with coefficients in $G$:

$$\check{H}(M, G) = \lim_{\mathcal{U}} \check{H}(\mathcal{U}, G)$$

There is a bijection between $\check{H}(M, G)$ and the set of isomorphism classes of principal $G$-bundles over $M$. 
A Famous Old Theorem

Here is the result we’d like to categorify:

**Thm.** Let $G$ be a well-pointed topological group. Let $BG$, the **classifying space** of $G$, be the geometric realization of the nerve of $G$. Then for any paracompact Hausdorff space $M$, there is a bijection

$$[M, BG] \cong \check{H}(M, G)$$

(A topological group $G$ is **well-pointed** if $1 \in G$ has a neighborhood of which it is a deformation retract.)
Topological 2-Groupoids

So far we’ve generalized this famous old theorem only to ‘strict’ topological 2-groups, where the group laws hold as equations. We can think of these as strict topological 2-groupoids with one object.
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**Defn.** A **strict 2-groupoid** is a strict 2-category where all morphisms and 2-morphisms are strictly invertible.

**Defn.** A **strict topological 2-groupoid** $\mathcal{G}$ is a strict 2-groupoid with:

- a topological space of objects,
- a topological space of morphisms,
- a topological space of 2-morphisms,

such that all the 2-groupoid operations are continuous.
**Defn.** A strict topological 2-group is a strict topological 2-groupoid with one object.

From this viewpoint, it has one object: \( \bullet \)

together with a space of morphisms: \( \bullet \xrightarrow{g} \bullet \)

and a space of 2-morphisms: \( \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \)

\( \xrightarrow{g'} \)

We'll use this viewpoint henceforth.
The Čech 2-Groupoid

Let $\mathcal{U} = \{U_i\}$ be an open cover of a topological space $M$.

**Defn.** The Čech 2-groupoid $\hat{\mathcal{U}}$ is the strict topological 2-groupoid where:

- objects are pairs $(x, i)$ with $x \in U_i$,
- there is a single morphism from $(x, i)$ to $(x, j)$ when $x \in U_i \cap U_j$, and none otherwise,
- there are only identity 2-morphisms.
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- there are only identity 2-morphisms.

(This is just a topological groupoid promoted to a 2-groupoid by throwing in identity 2-morphisms.)
Čech Cohomology for 2-Bundles

Defn. A Čech cocycle with coefficients in the strict topological 2-group $\mathcal{G}$ is continuous pseudofunctor $g : \hat{\mathcal{U}} \to \mathcal{G}$. 
Čech Cohomology for 2-Bundles

**Defn.** A Čech cocycle with coefficients in the strict topological 2-group $G$ is continuous pseudofunctor $g : \hat{U} \rightarrow G$.

**Defn.** Two Čech cocycles $g, g'$ are **cohomologous** if there is a continuous pseudonatural isomorphism $f : g \Rightarrow g'$.
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Defn. Two Čech cocycles \( g, g' \) are cohomologous if there is a continuous pseudonatural isomorphism \( f : g \Rightarrow g' \).

Defn. Let \( \check{H}(U, G) \) be the set of cohomology classes of Čech cocycles. Let the Čech cohomology of \( M \) with coefficients in \( G \) be the limit as we refine the cover:

\[
\check{H}(M, G) = \lim_{U} \check{H}(U, G)
\]
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It sends every object of $\hat{U}$ to the one object $\bullet \in \mathcal{G}$.

It sends each morphism $(x, i) \to (x, j)$ to a morphism $g_{ij}(x) : \bullet \to \bullet$ depending continuously on $x$. 
A Čech cocycle \( g : \hat{U} \to \mathcal{G} \) is a recipe for building a **principal \( \mathcal{G} \)-2-bundle** over \( M \) using transition functions.

It sends every object of \( \hat{U} \) to the one object \( \bullet \in \mathcal{G} \).

It sends each morphism \( (x, i) \to (x, j) \) to a morphism \( g_{ij}(x) : \bullet \to \bullet \) depending continuously on \( x \).

Composition of morphisms is weakly preserved:

\[
\begin{array}{c}
\bullet \\
\uparrow g_{ij}(x) \\
\downarrow g_{jk}(x) \\
\bullet \\
\downarrow g_{ik}(x) \\
\bullet \\
\end{array}
\]

for some 2-morphism \( h_{ijk}(x) \) depending continuously on \( x \in U_i \cap U_j \cap U_k \).
Finally, the $h_{ijk}$ must make these tetrahedra commute for each $x \in U_i \cap U_j \cap U_k \cap U_l$:
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Bartels has shown we can assume without loss of generality that $g_{ii}(x) = 1$ and that $h_{ijk}(x) = 1$ whenever two or more of the indices $i, j$ and $k$ agree. Then we have a **normalized** cocycle.
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$f$ sends each object $(x, i)$ of $\hat{U}$ to a morphism $f_i(x): \bullet \to \bullet$ depending continuously on $x$. 
Given Čech cocycles \( g, g' : \hat{U} \rightarrow \mathcal{G} \), a continuous pseudonatural isomorphism \( f : g \Rightarrow g' \) gives an isomorphism between the corresponding 2-bundles.

\( f \) sends each object \((x, i)\) of \( \hat{U} \) to a morphism \( f_i(x) : \bullet \rightarrow \bullet \) depending continuously on \( x \).

It sends each morphism \((x, i) \rightarrow (x, j)\) of \( \hat{U} \) to a 2-morphism \( k_{ij}(x) \) depending continuously on \( x \) and filling in this naturality square:
Finally, the $k_{ij}$ must make these prisms commute:
Categorifying That Famous Old Theorem

**Thm.** Suppose $\mathcal{G}$ is a well-pointed strict topological 2-group and $M$ is a paracompact Hausdorff space admitting good covers. Then there is a bijection

$$\check{H}(M, \mathcal{G}) \cong [M, BJ\mathcal{N}\mathcal{G}]$$

where the topological group $|\mathcal{N}\mathcal{G}|$ is the geometric realization of the nerve of $\mathcal{G}$. So, we call $BJ\mathcal{N}\mathcal{G}$ the **classifying space** of $\mathcal{G}$.

(A topological 2-group $\mathcal{G}$ is **well-pointed** if both the topological groups in its corresponding crossed module are well-pointed. An open cover is **good** if each nonempty finite intersection of sets in the cover is contractible.)
How to Build the Classifying Space

First we think of $\mathcal{G}$ as a group in $\text{TopGpd}$ and apply the nerve construction:

$$N: \text{TopGpd} \to \text{Top}^{\Delta^{\text{op}}}$$

to get a group in simplicial spaces, $N\mathcal{G}$.

Then we use geometric realization:

$$|\cdot|: \text{Top}^{\Delta^{\text{op}}} \to \text{Top}$$

to get a topological group $|N\mathcal{G}|$.

Then we think of $|N\mathcal{G}|$ as a 1-object topological groupoid, and take the nerve and the geometric realization of this to get our space $B|N\mathcal{G}|$. 
A Corollary: Bundles vs. 2-Bundles

Cor. There is a 1-1 correspondence between:

- equivalence classes of principal \( G \)-2-bundles over \( M \)
- elements of the Čech cohomology \( \check{H}(M, G) \)
- homotopy classes of maps \( f : M \to B|NG| \)
- elements of the Čech cohomology \( \check{H}(M, |NG|) \)
- isomorphism classes of principal \( |NG| \)-bundles over \( M \).
Example: Abelian Gerbes

The abelian topological group $\mathbb{U}(1)$ gives a topological 2-group $G$ with:

- one morphism,
- $\mathbb{U}(1)$ as the topological group of 2-morphisms.
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The abelian topological group $\mathbb{U}(1)$ gives a topological 2-group $\mathcal{G}$ with:

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In this case a principal $\mathcal{G}$-2-bundle is the same as an abelian $\mathbb{U}(1)$-gerbe. We have

$$|\mathcal{N}\mathcal{G}| = B\mathbb{U}(1) = K(\mathbb{Z}, 2)$$

so the classifying space of $\mathcal{G}$ is

$$B|\mathcal{N}\mathcal{G}| = B(B\mathbb{U}(1)) = K(\mathbb{Z}, 3)$$

and abelian $\mathbb{U}(1)$-gerbes over $M$ are classified by

$$[M, K(\mathbb{Z}, 2)] \cong H^3(M, \mathbb{Z}).$$
Example: Nonabelian Gerbes

Any topological group $G$ gives a topological 2-group $\mathcal{G}$ with:

- $\text{Aut}(G)$ as morphisms,
- elements $g \in G$ as 2-morphisms $g : h \Rightarrow gh(\cdot)g^{-1}$. 
Example: Nonabelian Gerbes

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- elements $g \in G$ as 2-morphisms $g : h \Rightarrow gh(\cdot)g^{-1}$.

In this case a principal $\mathcal{G}$-2-bundle is the same as an nonabelian $G$-gerbe. So, such gerbes are classified by

$$[M, B|NG]$$

which might be called the nonabelian cohomology $H^2(M, G)$. 
Example: String 2-Bundles

Let $G$ be a simply-connected compact simple Lie group. Then
$\pi_3(G) = \mathbb{Z}$. 
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There is a topological group $\hat{G}$ called the $3$-connected cover of $G$, with $\pi_3\hat{G} = 0$ and a continuous homomorphism

$$p: \hat{G} \rightarrow G$$

that induces an isomorphism on $\pi_n$ except for $n = 3$. 
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that induces an isomorphism on $\pi_n$ except for $n = 3$.

When $G = \text{Spin}(n)$, $\hat{G}$ is called the string group.

In fact, we can build $\hat{G}$ from a topological 2-group!
Last time we saw $G$ gives a topological 2-group $\text{String}_k G$ for each $k \in \mathbb{Z}$. 
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Let

$$\text{String} G = \text{String}_1 G$$

**Thm.** $\hat{G}$ is the geometric realization of the nerve of $\text{String} G$:

$$\hat{G} \simeq |N\text{String} G|$$
The continuous homomorphism

\[ p: |N\text{String}G| \to G \]

gives an algebra homomorphism:

\[ H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|N\text{String}G|, \mathbb{R}) \]

**Thm.** The homomorphism \( p^* \) is onto, with kernel generated by the ‘2nd Chern class’ \( c_2 \in H^4(BG, \mathbb{R}) \).

So, the real characteristic classes of String\( G \)-2-bundles are just like those of \( G \)-bundles, but with \( c_2 \) set to zero!