2-Vector Spaces and Groupoids

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Groupoidfest '08
November 23, 2008

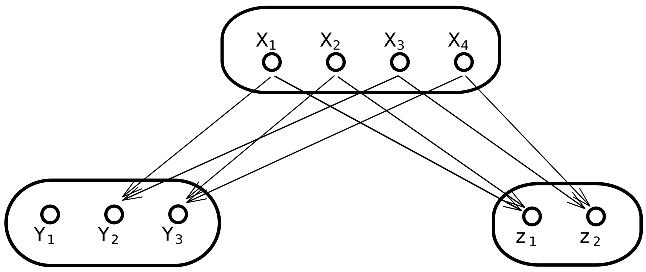
Goal

We'll describe a (weak) 2-functor

$$\Lambda : \mathrm{Span}(\mathbf{Gpd}) \to \mathbf{2Vect}$$

where $\mathrm{Span}(\mathbf{Gpd})$ is a 2-category of *spans of groupoids* and **2Vect** is the 2-category of *Kapranov-Voevodsky* 2-vector spaces:

This is analogous to the operation of *degroupoidification*, which in turn generalizes the obvious way to get vector spaces and linear maps from spans of sets:



by "pulling and pushing" complex functions through the span. (This construction is ubiquitous!)

Span(Gpd)

This bicategory has:

- **Objects** - Groupoids
- **Morphisms** - Spans:

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B \end{tikzcd}$$

with composition defined by weak pullback:

$$\begin{tikzcd} & & X' \circ X \arrow[dl, "S"'] \arrow[dr, "T"] \arrow[dll, bend right, "s \circ S"'] \arrow[drr, bend left, "t' \circ T"] & & \\ & X \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dl, "s"'] \arrow[dr, "t"] & & X' \arrow[dl, "s'"'] \arrow[dr, "t'"] & \\ A_1 & & A_2 & & A_3 \end{tikzcd} \tag{1}$$

- **2-Morphisms**: Isomorphism classes of *spans of* span maps. A *span map* f between two spans consists of a compatible map of the central objects:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] \arrow[drr, "t" near end] & X' \arrow[dl, "s'"' near end] \arrow[dr, "t'"] & \\ A & & B \end{tikzcd}$$

This bicategory has monoidal structure, and duals for morphisms and 2-morphisms.

2-Vector Spaces

The 2-category **2Vect** has:

- *objects*: 2-vector spaces
- *morphisms*: 2-linear maps
- *2-morphisms*: natural transformations

Definition: A **Kapranov–Voevodsky 2-vector space** is a $\mathbb{C}$-linear finitely semisimple additive category (one generated by simple objects x, where $\hom(x, x) \cong \mathbb{C}$). A **2-linear map** between 2-vector spaces is a $\mathbb{C}$-linear additive functor.

Theorem:Any 2-vector spaces is equivalent to $\mathbf{Vect^k}$ (objects k-tuples of vector spaces, morphisms k-tuples of linear maps) for some k.

Any 2-linear map $T : \mathbf{Vect}^n \to \mathbf{Vect}^m$ is naturally isomorphic to a map of the form

$$\begin{pmatrix} V_{1,1} & \dots & V_{1,k} \\ \vdots & & \vdots \\ V_{l,1} & \dots & V_{l,k} \end{pmatrix} \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} = \begin{pmatrix} \bigoplus_{i=1}^k V_{1,i} \otimes W_i \\ \vdots \\ \bigoplus_{i=1}^k V_{l,i} \otimes W_i \end{pmatrix}$$

Any natural transformation can be written as a matrix of linear maps between the components.

Example

Given a finite group G, the category **Rep(G)** has:

- **Objects**: Representations of G
- **Morphisms**: Intertwining operators between reps

Theorem:For any finite group G, **Rep(G)** is a 2-vector space

Any representation is a direct sum of irreducible reps - these form a *basis* for the 2-vector space.

By *Schur's Lemma*, if V_j is irreducible,

$$\hom(V_j, V_j) \cong \mathbb{C} \cdot 1$$

so these are indeed simple objects

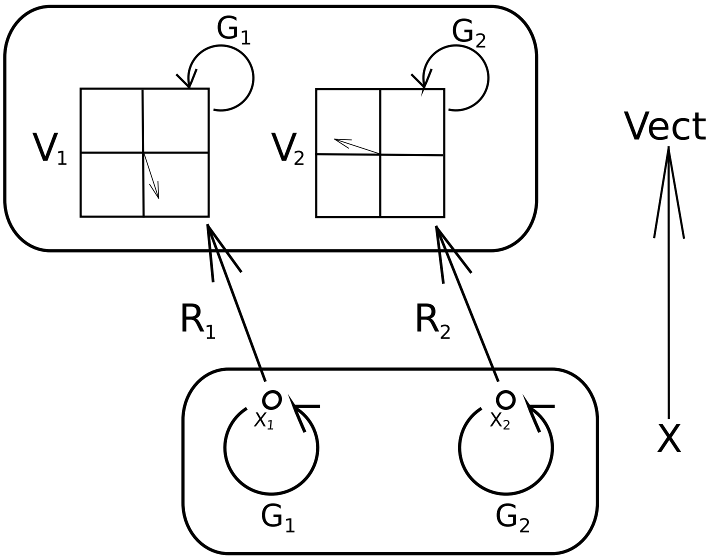
We can make a similar construction for groupoids as for groups. Taking a group G as a one-object groupoid:

$$\mathbf{Rep(G)} \cong [G, \mathbf{Vect}]$$

where we use the notation $[\mathbf{X}, \mathbf{Vect}] = \hom(\mathbf{X}, \mathbf{Vect})$.

Lemma 1 *If* $\mathbf{X}$ *is an essentially finite groupoid, the functor category* $\Lambda(\mathbf{X}) = [\mathbf{X}, \mathbf{Vect}]$ *is a KV 2-vector space.*

Here is an illustration of a **Vect**-valued functor on $\mathbf{X}$:



Note: If the automorphism groups of (isomorphism classes of) objects of $\mathbf{X}$ are $G_1, \ldots, G_n$, then we have

$$[X, \mathbf{Vect}] \cong \prod_j \mathbf{Rep(G_j)}$$

So the "basis elements" (simple objects) in $[X, \mathbf{Vect}]$ are labeled by $([x], V)$, where $[x]$ is an isomorphism class of objects of $\mathbf{X}$ and V an irreducible rep of $Aut(x)$.

Theorem:If $\mathbf{X}$ and $\mathbf{Y}$ are essentially finite groupoids, a functor $f : \mathbf{X} \to \mathbf{Y}$ gives two 2-linear maps between the 2-vector spaces of **Vect**-presheaves:

$$f^* : \Lambda(\mathbf{Y}) \to \Lambda(\mathbf{X})$$

namely pullback along f, with $f^* F = F \circ f$ and

$$f_* : \Lambda(\mathbf{X}) \to \Lambda(\mathbf{Y})$$

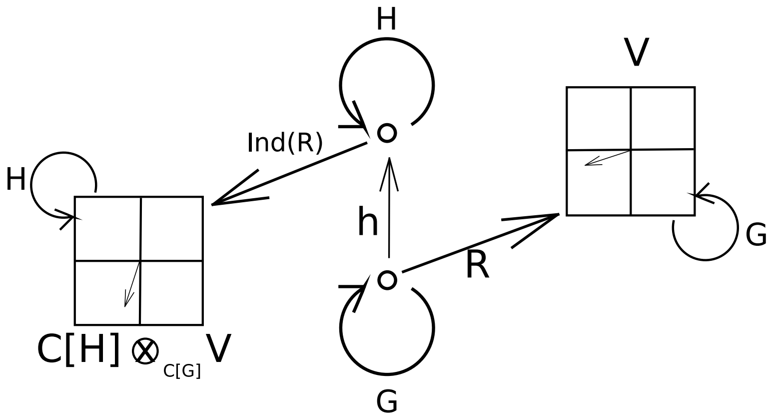
called "pushforward along f". Furthermore, f_* is the two-sided (and *2-linear*) adjoint to f^*.

The adjoint map

$$f_* : [X, \mathbf{Vect}] \to [Y, \mathbf{Vect}]$$

gives the *induced representation*.

Given a group homomorphism $h : G \to H$, and a representation $R : G \to GL(V)$, there is an induced representation of H, namely $\mathbb{C}[H] \otimes_{\mathbb{C}[G]} V$:



For a map of groupoids $t : X \to B$, we can push forward a **Vect**-presheaf in the same way. If more than one object is sent to the same $b \in B$, we get a direct sum of all their contributions. Assuming all groupoids are skeletal, this is:

$$t_*(F)(b) = \bigoplus_{t(x)=b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

This is a direct sum over the (essential) preimage of $b \in B$.

V: 2-Linear Maps for Spans

Given a span of groupoids:

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ A & & B \end{tikzcd}$$

we can apply the functor $[-, \mathbf{Vect}]$ to the whole diagram. This functor is contravariant, so we get a cospan:

$$\begin{tikzcd} & {[X, \mathbf{Vect}]} & \\ \Lambda(A) \arrow[ur, "s^*"] & & \Lambda(B) \arrow[ul, "t^*"'] \end{tikzcd}$$

Then we use these to define a 2-linear map corresponding to a span:

Definition: For a span of groupoids $X : A \to B$ in $\mathrm{Span}(\mathbf{Gpd})$ define the 2-linear map:

$$\Lambda(X) = t_* \circ s^* : \Lambda(A) \longrightarrow \Lambda(B)$$

So then:

$$\Lambda(X)(F)(b) = \bigoplus_{t(x)=b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} (F \circ s)(x)$$

Picking basis elements $([a], V) \in \Lambda(A)$, and $([b], W) \in \Lambda(B)$, and using that

$$\mathbb{C}[G] \cong \bigoplus_i V_i \otimes V_i^*$$

where the sum is over irreps of G, Schur's lemma means that

$$\Lambda(X)_{([a],V),([b],W)} = \hom_{Rep(Aut(b))}[t_* \circ s^*(V), W]$$
$$\simeq \bigoplus_{[x] \in \underline{(s,t)^{-1}([a],[b])}} \hom_{Rep(\mathrm{Aut}(x))}[s^*(V), t^*(W)]$$

by adjointness (a.k.a. Frobenius reciprocity).

V: 2-Morphisms

Given a span between spans, $Y : X_1 \to X_2$ for $X_1, X_2 : A \to B$:

$$\begin{tikzcd}
 & X_1 \arrow[dl, "s_1"'] \arrow[dr, "t_1"] & \\
A & Y \arrow[u, "s"] \arrow[d, "t"'] & B \\
 & X_2 \arrow[ul, "s_2"] \arrow[ur, "t_2"'] &
\end{tikzcd}$$

we want a natural transformation

$$\Lambda(Y) : \Lambda(X_1) \to \Lambda(X_2)$$

We saw that $\Lambda(Y)$ is given as a matrix of linear operators between corresponding entries of $\Lambda(X_1)$ and $\Lambda(X_2)$:

$$\Lambda(Y)_{[([a],V),([b],W)]} : \Lambda(X_1)_{[([a],V),([b],W)]} \to \Lambda(X_2)_{[([a],V),([b],W)]}$$

This amounts to:

$$\Lambda(Y)_{[([a],V),([b],W)]} : \bigoplus_{[x_1]} \hom_{Rep(\mathrm{Aut}(x))}[s_1^*(V), t_2^*(W)]$$
$$\to \bigoplus_{[x_2]} \hom_{Rep(\mathrm{Aut}(x_2))}[s_2^*(V), t_2^*(W)]$$

such that for each block $([x_1],[x_2])$, the corresponding linear operator behaves as follows: for any intertwiner $f \in \hom[s_1^*(V), t_1^*(W)]$ we get:

$$\Lambda(Y)_{[([a],V),([b],W)]}|_{([x_1],[x_2])} = \frac{|\widehat{(x_1, x_2)}|}{|\,\mathrm{Aut}(x_2)|} \sum_{g \in \mathrm{Aut}(x_2)} g f g_1$$

where we are using the Baez-Dolan groupoid cardinality of $\widehat{(x_1, x_2)}$, the essential preimage of (x_1, x_2) under (s,t).

Given all this, the main fact is:

Theorem:The process $\Lambda : \mathrm{Span}(\mathbf{Gpd}) \to \mathbf{2Vect}$ is a weak 2-functor.

Extension to Smooth Groupoids

What is different for smooth groupoids? First, the 2-vector space is generally infinite dimensional. There are different possible approaches to defining infinite-dimensional 2-vector spaces. One would involve:

Definition:If (X, μ) is a measurable space $\mathbf{Meas(X)}$ is the category with:

- Objects: *measurable fields of Hilbert spaces* on $(X, \mathcal{M})$: i.e. X-indexed families of Hilbert spaces $\mathcal{H}_x$ with a Hilbert space of *measurable sections*
- Morphisms: *measurable fields of bounded linear maps* between Hilbert spaces, $f_x : \mathcal{H}_x \rightarrow \mathcal{K}_x$ so that $||f||$ (the operator norm of f) is measurable.

Then **Meas** is the 2-category of all categories $\mathbf{Meas(X)}$, and we could look for a 2-functor

$$\Lambda : \mathrm{Span}(\mathbf{Gpd}) \rightarrow \mathbf{Meas}$$

(Another variant would involve defining a 2-category of *2-Hilbert spaces* - which are built on measure spaces, rather than measurable spaces - using a measure μ to define an inner product as with ordinary $L^2(X)$. *Work in progress.*)

What changes are needed in the infinite dimensional case?

The category of measurable functors $\mathcal{H} : \mathcal{G} \to \mathbf{Meas}$ amounts to a category of *equivariant* measurable fields of Hilbert spaces on the objects (so each $\mathcal{H}_x$ carries a representation of Aut(x).

• For a span $\mathcal{S} : \mathcal{G} \to \mathcal{G}'$, there will be a 2-linear map given by a *direct integral*:

$$(\Lambda(S)(\mathcal{H}))_y = \int^{\oplus} \mathcal{H}_x S_{([x],V),([y],W)} d\mu$$

Where the $S_{([x],V),([y],W)}$ are defined similarly to the above.

Generally:

• Finite product of $Rep(Aut(x_i)) \mapsto \mathbf{Meas}_{\mathcal{G}}(\mathcal{G}^{(0)})$
• Direct sum $\mapsto$ direct integral
• Counting measure $\mapsto$ measure on object space
• Groupoid cardinality $\mapsto$ volume of groupoid (requires measure and Haar system)

The algebraic part of the construction is the same... (We leave the rest as an exercise to the speaker.)