Groupoidified Linear Algebra

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(Joint work with John Baez and James Dolan)

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November 22, 2008
There is a systematic process (first described by James Dolan) called Degroupoidification which turns:

Groupoids $\Rightarrow$ Vector Spaces

"Spans" of Groupoids $\Rightarrow$ Linear Operators

Groupoidification is an attempt to reverse this process. As with any categorification process, this "reverse" direction is not systematic.
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Why groupoids and “spans” of groupoids?
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- We use groupoids because they give us a way of categorifying the positive real numbers.
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- We use groupoids because they give us a way of categorifying the positive real numbers.
- We use “spans” because they give us a way to describe the matrix of a linear operator.
A little bit of category theory.
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Groupoidified Linear Algebra
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There are actually two categories we are working in:

- **Grpd** - the category of groupoids as objects and functors as morphisms
- **Span** - the category of groupoids as objects and ‘spans’ of groupoids as morphisms
What is a span?
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**Definition**

Given groupoids $X$ and $Y$, a **span** from $X$ to $Y$ is defined as

```
\begin{tikzpicture}
  \node (S) at (0,0) {$S$};
  \node (X) at (1,0) {$X$};
  \node (Y) at (-1,0) {$Y$};
  \draw[->] (S) to node {$q$} (Y);
  \draw[->] (S) to node {$p$} (X);
\end{tikzpicture}
```

where $S$ is another groupoid and $p : S \to X$ and $q : S \to Y$ are functors.

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We can compose two spans using the “weak pullback” over the matching legs of the spans.
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Definition

Given two functors $f: S \rightarrow Y$ and $g: T \rightarrow Y$, we define the weak pullback as:

$$
\begin{array}{c}
\pi_T \\
\downarrow & & \downarrow \\
T & & \Downarrow g \\
\downarrow & & \downarrow \\
TS & & S \\
\downarrow & & \Downarrow f \\
\pi_T \\
\end{array}
$$

where $TS$ is the groupoid whose objects are triples $(s, t, \alpha)$ where $\alpha: f(s) \rightarrow g(t)$. 

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Groupoidified Linear Algebra
Where do the positive real numbers come from?

Definition
Given a groupoid $X$, we define the cardinality of $X$ to be:

$$|X| = \sum [x] \cdot |\text{Aut}_x|$$

where $x$ ranges over all isomorphism classes in $X$.

When this sum converges, we call the groupoid tame, and we can obtain any positive real number. This gives us access to a field to define a vector space over.

Example - Let $E$ be the groupoid of finite sets and bijections. Then:

$$|E| = \sum n \cdot |S_n| = \sum n \cdot n! = e$$
Where do the positive real numbers come from?

**Definition**

Given a groupoid $X$, we define the **cardinality** of $X$ to be:

$$|X| = \sum_{[x]} \frac{1}{|\text{Aut } x|}$$

where $x$ ranges over all isomorphism classes in $X$. 

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**Example** - Let $E$ be the groupoid of finite sets and bijections. Then:

$$|E| = \sum_{n} \frac{1}{|S_n|}$$
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**Definition**

Given a groupoid $X$, we define the **cardinality** of $X$ to be:

$$|X| = \sum_{[x]} 1/|\text{Aut } x|$$

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**Groupoidified Linear Algebra**
Definition

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In order to produce a single vector (function) in $\mathbb{R}^X$, we consider a groupoid over $X$, $p : \Psi \rightarrow X$. We say $\Psi$ is **tame** if $p^{-1}(x)$ is tame for all $x$, where $p^{-1}(x)$ is the essential preimage of $x$. We then define the function:

$$\Psi([x]) = |p^{-1}(x)|$$

Example - Again, consider $E$. Since $E \sim = \mathbb{N}$, $\mathbb{R}^E \sim = \mathbb{R}^{[\mathbb{N}]}$. Also $E$ is a tame groupoid over itself with the identity functor.

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**Example** - Again, consider $E$. Since $E \cong \mathbb{N}$, $\mathbb{R}^E \cong \mathbb{R}[[x]]$. Also $E$ is a tame groupoid over itself with the identity functor.

$E([n]) = \frac{1}{n!}$  
$E = e^x$.  

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Groupoidified Linear Algebra
We now produce a linear operator out of a span of groupoids.

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Groupoidified Linear Algebra
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**Definition**

Given a tame span of groupoids

\[
\begin{array}{c}
S \\
q \\
Y \quad X
\end{array}
\]

the linear operator \( S : \mathbb{R}^X \to \mathbb{R}^Y \) is given by \( \widetilde{S}\Psi = \widetilde{S}\Psi \), where \( \Psi \) is a groupoid over \( X \), \( \nu : \Psi \to X \), and \( \widetilde{S}\Psi \) is the weak pullback:

\[
\begin{array}{c}
\Psi \\
\pi_S \\
\downarrow \quad \downarrow \\
S \\
\downarrow \quad \downarrow \\
Y \quad X
\end{array}
\]
Considering the definition of weak pullback, we get a nice formula for the matrix entries of $\mathcal{S}$:
Considering the definition of weak pullback, we get a nice formula for the matrix entries of $\tilde{S}$:

$$S_{[x][y]} = \sum_{[s] \in p^{-1}(x) \cap q^{-1}(y)} \frac{|\text{Aut } x|}{|\text{Aut } s|}$$
Addition We can add vectors and operators as follows:
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**Proposition**

*Give two groupoids Φ and Ψ over X, the disjoint union Φ + Ψ forms a groupoid over X, and*

\[ \Phi + \Psi = \Phi + \Psi \]
**Addition** We can add vectors and operators as follows:

**Proposition**

*Give two groupoids $\Phi$ and $\Psi$ over $X$, the disjoint union $\Phi + \Psi$ forms a groupoid over $X$, and*

\[
\Phi + \Psi = \Phi + \Psi
\]

**Proposition**

*Give two spans:*

\[
\begin{array}{ccc}
S & \xrightarrow{q_S} & Y \\
\quad & p_S & \quad \\
\downarrow & & \downarrow \\
X & \xrightarrow{q_T} & Y \\
\quad & p_T & \quad \\
\downarrow & & \downarrow \\
Y & \xrightarrow{p_T} & X
\end{array}
\]

*the disjoint union $S + T$ forms a span from $X$ to $Y$, and*

\[
S + T = \tilde{S} + \tilde{T}
\]

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Groupoidified Linear Algebra
Scalar Multiplication We can multiply both vectors and linear operators by scalars as follows:
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**Proposition**

Given a groupoid $\Lambda$ and a groupoid $\Phi$ over $X$, the groupoid $\Lambda \times \Phi$ over $X$ satisfies

$$\Lambda \times \Phi = |\Lambda| \Phi.$$
Scalar Multiplication  We can multiply both vectors and linear operators by scalars as follows:

**Proposition**

*Given a groupoid \( \Lambda \) and a groupoid \( \Phi \) over \( X \), the groupoid \( \Lambda \times \Phi \) over \( X \) satisfies*

\[
\Lambda \times \Phi = |\Lambda|\Phi.
\]

**Proposition**

*Given a groupoid \( \Lambda \) and a span*

\[
\begin{array}{ccc}
S & \rightarrow & Y \\
\downarrow & & \downarrow \\
& \Lambda \times S & \rightarrow \ X
\end{array}
\]

*then*

\[
\Lambda \times S = |\Lambda|S.
\]
We also have a concept of inner product, which turns our vector space into a Hilbert space.
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**Definition**

*Given groupoids $\Phi$ and $\Psi$ over $X$, we define the **inner product** $\langle \Phi, \Psi \rangle$ to be this weak pullback:*

\[
\begin{array}{c}
\Phi \\
\downarrow \\
\langle \Phi, \Psi \rangle \\
\downarrow \\
X
\end{array} \quad \begin{array}{c}
\Psi \\
\downarrow \\
\langle \Phi, \Psi \rangle \\
\downarrow \\
X
\end{array}
\]

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Groupoidified Linear Algebra
This definition holds to the standard properties of inner products.
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**Proposition**

*Given a groupoid \( \Lambda \) and groupoids \( \Phi, \Psi, \) and \( \Psi' \) over \( X \), the following properties hold:*

- \( \langle \Phi, \Psi \rangle \simeq \langle \Psi, \Phi \rangle. \)
- \( \langle \Phi, \Psi + \Psi' \rangle \simeq \langle \Phi, \Psi \rangle + \langle \Phi, \Psi' \rangle. \)
- \( \langle \Phi, \Lambda \times \Psi \rangle \simeq \Lambda \times \langle \Phi, \Psi \rangle. \)
Definition

A groupoid $\Phi$ over $X$ is called **square-integrable** if $\langle \Phi, \Phi \rangle$ is tame. We define $L^2(X)$ to be the subspace of $\mathbb{R}^X$ consisting of finite real linear combinations of vectors $\Phi$ where $\Phi$ is square-integrable.
Definition

A groupoid $\Phi$ over $X$ is called **square-integrable** if $\langle \Phi, \Phi \rangle$ is tame. We define $L^2(X)$ to be the subspace of $\mathbb{R}^X$ consisting of finite real linear combinations of vectors $\Phi$ where $\Phi$ is square-integrable.

We then produce our Hilbert space.

Proposition

$L^2(X)$ forms a Hilbert space with the inner product $\langle \psi, \Phi \rangle = |\langle \psi, \Phi \rangle|$. 
Examples
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- Hecke Algebras and Hecke Operators (Alex Hoffnung)
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- Quantum Harmonic Oscillators (Jeffrey Morton)
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- Jordan-Schwinger Representations
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- Quiver Representations and Hall Algebras