The Algebra of Grand Unified Theories

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May 7, 2009

Abstract

The Standard Model is the best tested and most widely accepted theory of elementary particles we have today. It may seem complicated and arbitrary, but it has hidden patterns that are revealed by the relationship between three 'grand unified theories': theories that unify forces and particles by extending the Standard Model symmetry group $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ to a larger group. These three are Georgi and Glashow's $\mathrm{SU}(5)$ theory, Georgi's theory based on the group $\mathrm{Spin}(10)$, and the Pati–Salam model based on the group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$. In this expository account for mathematicians, we explain only the portion of these theories that involves finite-dimensional group representations. This allows us to reduce the prerequisites to a bare minimum while still giving a taste of the profound puzzles that physicists are struggling to solve.

1 Introduction

The Standard Model of particle physics is one of the greatest triumphs of physics. This theory is our best attempt to describe all the particles and all the forces of nature... *except* gravity. It does a great job of fitting experiments we can do in the lab. But physicists are dissatisfied with it. There are three main reasons. First, it leaves out gravity: that force is described by Einstein's theory of general relativity, which has not yet been reconciled with the Standard Model. Second, astronomical observations suggest that there may be forms of matter not covered by the Standard Model—most notably, 'dark matter'. And third, the Standard Model is complicated and seemingly arbitrary. This goes against the cherished notion that the laws of nature, when deeply understood, are simple and beautiful.

For the modern theoretical physicist, looking beyond the Standard Model has been an endeavor both exciting and frustrating. Most modern attempts are based on string theory. There are also other interesting approaches, such as loop quantum gravity and theories based on noncommutative geometry. But back in the mid 1970's, before any of the currently popular approaches rose to prominence, physicists pursued a program called 'grand unification'. This sought to unify the forces and particles of the Standard Model using the mathematics of Lie groups, Lie algebras, and their representations. Ideas from this era remain influential, because grand unification is still one of the most fascinating attempts to find order and beauty lurking within the Standard Model.

This paper is a gentle introduction to the group representations that describe particles in the Standard Model and the most famous grand unified theories. To make the material more approachable for mathematicians, we limit our scope by not discussing particle interactions or 'symmetry breaking'—the way a theory with a large symmetry group can mimic one with a smaller group at low energies. These topics lie at the very heart of particle physics. But by omitting them, we can focus on ideas from algebra that many mathematicians will find familiar, while introducing the unfamiliar ways that physicists use these ideas.

In fact, the essential simplicity of the representation theory involved in the Standard Model and grand unified theories is quite striking. The usual textbook approach to particle physics proceeds through quantum field theory and gauge theory. While these subjects are very important to modern mathematics, learning them is a major undertaking. We have chosen to focus on the algebra of grand unified theories because many mathematicians have the prerequisites to understand it with only a little work.

A full-fledged treatment of particle physics requires quantum field theory, which uses representations of a noncompact Lie group called the Poincaré group on *infinite-dimensional* Hilbert spaces. This brings in a lot of analytical subtleties, which make it hard to formulate theories of particle physics in a mathematically rigorous way. In fact, no one has yet succeeded in doing this for the Standard Model. But by neglecting the all-important topic of particle interactions, we can restrict attention to *finite-dimensional* Hilbert spaces: that is, finite-dimensional complex inner product spaces. This makes our discussion purely algebraic in flavor.

Every theory we consider has an 'internal symmetry group' or 'gauge group'. This is a compact Lie group, say G. Particles then live in representations of G on a finite-dimensional Hilbert space V. More precisely: V can always be decomposed as a direct sum of irreducible representations, or **irreps**—and for our limited purposes, *particles are basis vectors of irreps*. This provides a way to organize particles, which physicists have been exploiting since the 1960s.

The idea of 'unification' has a clear meaning in these terms. Suppose V is a representation, not only of G, but also some larger group H having G as a subgroup. Then we expect V to decompose into fewer irreps as a representation of H than as a representation of G, because elements of H can mix different irreps of G. So: by introducing a larger symmetry group, particles can be unified into larger irreps.

'Grand unification' occurs when the compact Lie group G is simple, and thus not a product of other groups. A gauge theory based on G requires an invariant inner product on its Lie algebra. When G is simple, this form is unique up to a scale factor, which physicists call a 'coupling constant': this measures the strength of the force corresponding to G. When G is the product of simple factors, there is one coupling constant for each factor of G. So, by using a simple Lie group as gauge group, we minimize the number of coupling constants.

In this paper, we give an account of the algebra behind the Standard Model and three attempts at unification: Georgi and Glashow's $\mathrm{SU}(5)$ theory, Georgi's theory based on the group $\mathrm{Spin}(10)$ (physicists call this the $\mathrm{SO}(10)$ theory), and the Pati–Salam model. All three date to around 1974. The first two are known as grand unified theories, or GUTs, because they are based on simple Lie groups. The Pati–Salam model is different: while it is called a GUT by some authors, and does indeed involve unification, it is based on the Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$, which is merely semisimple.

It is important to note that these theories have their problems. The $\mathrm{SU}(5)$ theory predicts that protons will decay more quickly than they do, and it requires certain trends to hold among the relative strengths of forces at high energies—trends which the data do not support. The SO(10) theory may still be viable, especially if at low enough energies it reduces to the Pati–Salam model. However, the issues involved are complex. For details, see the paper by Bertolini *et al* and the many references therein [4].

Nonetheless, it is still very much worthwhile for mathematicians to study the algebra of grand unified theories. First, even apart from their physical significance, these theories are intrinsically beautiful mathematical structures. Second, they provide a nice way for mathematicians to get some sense of the jigsaw puzzle that physicists are struggling to solve. It is certainly hopeless trying to understand what physicists are trying to accomplish with string theory without taking a look at grand unified theories. Finally, grand unified theories can be generalized by adding 'supersymmetry'—and the resulting generalizations are considered serious contenders for describing the real world. For some recent overviews of their prospects, see Pati [26, 27] and Peskin [29].

This is how we shall proceed. In Section 2 we start by describing the Standard Model. After a brief nod to the electron and photon, we explain some nuclear physics in Section 2.1. We start with Heisenberg's old attempt to think of the proton and neutron as two states of a single particle, the 'nucleon', described by a 2-dimensional representation of SU(2). The idea of unification through representation theory traces its origins back to this notion.

After this warmup we tour the Standard Model in its current form. In Section 2.2 we describe the particles called 'fundamental fermions', which constitute matter. In Section 2.3 we describe the particles called 'gauge bosons', which carry forces. Apart from the elusive Higgs boson, all particles in the Standard Model are of these two kinds. In Section 2.4 we give a more mathematical treatment of these ideas: the gauge bosons are determined by the Standard Model gauge group

$$G_{\mathrm{SM}} = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3),$$

while the fundamental fermions and their antiparticles are basis vectors of a highly reducible representation of this group, which we call $F \oplus F^*$. Here F describes the fermions, while F^* describes their antiparticles.

Amazingly, using the ideas of gauge theory and quantum field theory, plus the 'Higgs mechanism' for symmetry breaking, we can recover the dynamical laws obeyed by these particles from the representation of G_{SM} on $F \oplus F^*$. This information is enough to decode the physics of these particles and make predictions about what is seen in the gigantic accelerators that experimental physicists use to probe the natural world at high energies. Unfortunately, to explain all this would go far beyond the modest goals of this paper. For a guide to further study, see Section 1.1.

Having acquainted the reader with the Standard Model of particle physics in Section 2, we then go on to talk about grand unified theories in Section 3. These theories go beyond the Standard Model by 'extending' the gauge group. That is, we pick a way to include G_{SM} in some larger group G, and choose a representation V of G which reduces to the Standard Model representation $F \oplus F^*$ when we restrict it to G_{SM}. We describe how this works for the SU(5) theory (Section 3.1), the SO(10) theory (Section 3.2), and the Pati–Salam model (Section 3.3). Of course, since we do not discuss the dynamics, a lot will go unsaid about these GUTs.

As we proceed, we explain how the SU(5) theory and the Pati–Salam model are based on two distinct visions about how to extend the Standard Model. However, we will see that the SO(10) theory is an extension of both the SU(5) theory (Section 3.2) and the Pati–Salam model (Section 3.4). Moreover, these two routes to the SO(10) theory are compatible in a precise sense: we get a commuting square of groups, and a commuting square of representations, which fit together to form a commuting cube (Section 3.5).

In Section 4, we conclude by discussing what this means for physics: namely, how the Standard Model reconciles the two visions of physics lying behind the SU(5) theory and the Pati–Salam model. In a sense, it is the intersection of the SU(5) theory and the Pati–Salam model within their common unification, SO(10).

Throughout the course of the paper, we occasionally summarize our progress in theorems, most phrased in terms of commutative diagrams:

- Section 3.1, Theorem 1: the SU(5) theory extends the Standard Model.
- Section 3.2, Theorem 2: the Spin(10) theory extends the SU(5) theory.
- Section 3.3, Theorem 3: the Pati–Salam model extends the Standard Model.
- Section 3.4, Theorem 4: the Pati–Salam model is isomorphic to a theory involving $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$.
- Section 3.4, Theorem 5: the $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ theory extends the Standard Model.
- Section 3.4, Theorem 6: the Spin(10) theory extends the $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ theory.
- Section 3.5, Theorem 7: the Standard Model, the SU(5) theory, the $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ theory and the Spin(10) theory fit together in a commutative cube.
- Section 4, Theorem 8: The true gauge group of the Standard Model is the intersection of SU(5) and $\mathrm{SO}(4) \times \mathrm{SO}(6)$ in SO(10).
- Section 4, Theorem 9: The true gauge group of the Standard Model is the intersection of SU(5) and $(\mathrm{Spin}(4) \times \mathrm{Spin}(6))/\mathbb{Z}_2$ in Spin(10).

1.1 Guide to Further Reading

We have tried to limit our prerequisites to the bare minimum. This includes basic facts about Lie groups, Lie algebras, and their representations—especially finite-dimensional unitary representations of compact Lie groups. We will not need the structure theory for simple Lie groups. We do, however, assume a little familiarity with the classical Lie groups $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$, as well as their Lie algebras.

There are countless books on Lie groups, Lie algebras and their representations, but the text by Hall [14] has everything we need, and more. Sternberg's introduction to group theory and physics [34] includes an excellent account of applications to particle physics. To see the subject more through the eyes of a physicist, try the books by Lipkin [20] or Tinkham [38]. Georgi's text [11] shows how the subject looks to one of the inventors of grand unified theories.

Starting in Section 3.1 we assume familiarity with exterior algebras, and in Section 3.2 we also use Clifford algebras. For what we need, Chevalley's book [6] is more than sufficient.

For the interested reader, there are many introductions to particle physics where one can learn the dynamics after getting a taste of the algebra here. It might be good to start by reading Griffiths' introductory book [13] together with Sudbery's text specially designed for mathematicians [36]. The book by Huang [17] delves as deep as one can go into the Standard Model without a heavy dose of quantum field theory, and the book by Lee [19] is full of practical wisdom. For more information on grand unified theories, see the textbooks by Ross [31] and Mohapatra [21].

Particle physics relies heavily on quantum field theory. There are many books on this subject, none of which make it easy. Prerequisites include a good understanding of classical mechanics, classical field theory and quantum mechanics. Many physicists consider the books by Brown [5] and Ryder [32] to be the most approachable. The text by Peskin and Schroeder [30] offers a lot of physical insight, and we have also found Zee's book [40] very useful in this respect. Srednicki's text [35] is clear about many details that other books gloss over—and it costs nothing! Of course, these books are geared toward physicists: mathematicians may find the lack of rigor frustrating. Ticciati [37] provides a nice introduction for mathematicians, but anyone serious about this subject should quickly accept the fact that quantum field theory has not been made rigorous: this is a project for the century to come.

Particle physics also relies heavily on geometry, especially gauge theory. This subject is easier to develop in a rigorous way, so there are plenty of texts that describe the applications to physics, but which a mathematician can easily understand. Naber's books are a great place to start [22, 23], and one of us has also written an elementary introduction [3]. Isham's text is elegant and concise [18], and many people swear by Nakahara [24]. The quantum field theory texts mentioned above also discuss gauge theory, but in language less familiar to mathematicians.

Finally, few things are more enjoyable than the history of nuclear and particle physics—a romantic tale full of heroic figures and tightly linked to the dark drama of World War II, the Manhattan Project, and the ensuing Cold War. Crease and Mann [8] give a very readable introduction. To dig deeper, try the book by Segrè [33], or the still more detailed treatments by Pais [25] and Hoddeson *et al* [16].

2 The Standard Model

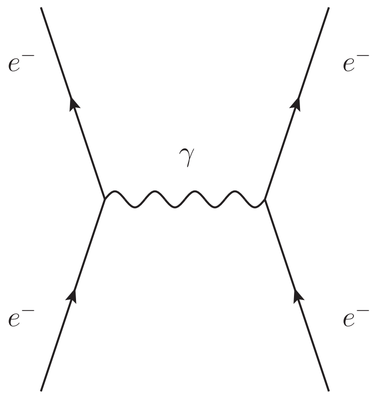
Today, most educated people know that the world is made of atoms, and that atoms, in turn, are made of electrons, protons, and neutrons. The electrons orbit a dense nucleus made of protons and neutrons, and as the outermost layer of any atom's structure, they are responsible for all chemistry. They are held close to the nucleus by electromagnetic forces: The electrons carry a negative electric charge, and protons carry a positive charge. Opposite charges attract, and this keeps the electrons and the nucleus together.

At one point in time, electrons, protons, and neutrons were all believed to be fundamental and without any constituent parts, just as atoms themselves were once believed to be, before the discovery of the electron. Electrons are the only one of these subatomic particles still considered fundamental, and it is with this venerable particle that we begin a table of the basic constituents of matter, called 'fundamental fermions'. We will see more soon.

Fundamental Fermions (first try)		
Name	Symbol	Charge
Electron	e^-	-1

Since the electron is charged, it participates in electromagnetic interactions.

From the modern perspective of quantum field theory, electromagnetic interactions are mediated by the exchange of virtual photons, particles of light that we never see in the lab, but whose effects we witness whenever like charges are repelled or opposite charges are attracted. We depict this process with a diagram:



Here, time runs up the page. Two electrons come in, exchange a photon, and leave, slightly repelled from each other by the process.

The photon is our next example of a fundamental particle, though it is of a different character than the electron and quarks. As a mediator of forces, the photon is known as a **gauge boson** in modern parlance. It is massless, and interacts only with charged particles, though it carries no charge itself. So, we begin our list of gauge bosons as follows:

Gauge Bosons (first try)		
Force	Gauge Boson	Symbol
Electromagnetism	Photon	γ

2.1 Isospin and SU(2)

Because like charges repel, it is remarkable that the atomic nucleus stays together. After all, the protons are all positively charged and are repelled from each other electrically. To hold these particles so closely together, physicists hypothesized a new force, the **strong force**, strong enough to overcome the electric repulsion of the protons. It must be strongest only at short distances (about 10^{-15} m), and then it must fall off rapidly, for protons are repelled electrically unless their separation is that small. Neutrons must also experience it, because they are bound to the nucleus as well.

Physicists spent several decades trying to understand the strong force; it was one of the principal problems in physics in the mid-twentieth century. About 1932, Werner Heisenberg, pioneer in quantum mechanics, discovered one of the first clues to its nature. He proposed, in [15], that the proton and neutron might really be two states of the same particle, now called the **nucleon**. In modern terms, he attempted to unify the proton and neutron.

To understand how, we need to know a little quantum mechanics. In quantum mechanics, the state of any physical system is given by a unit vector in a complex Hilbert space, and it is possible to take complex linear combinations of the system in different states. For example, the state for a quantum system, like a particle on a line, is a complex-valued function

$$\psi \in L^2(\mathbb{R}).$$

Or if the particle is confined to a 1-dimensional box, so that its position lies in the unit interval $[0,1]$, then its state lives in the Hilbert space $L^2([0,1])$.

We have special rules for combining quantum systems. If, say, we have two particles in a box, particle 1 and particle 2, then the state is a function of both particle 1's position and particle 2's:

$$\psi \in L^2([0,1] \times [0,1]),$$

but this is isomorphic to the tensor product of particle 1's Hilbert space with particle 2's:

$$L^2([0,1] \times [0,1]) \cong L^2([0,1]) \otimes L^2([0,1]).$$

This is how we combine systems in general. If a system consists of one part with Hilbert space V *and* another part with Hilbert space W, their tensor product $V \otimes W$ is the Hilbert space of the combined system. Heuristically,

$$\mathit{and} = \otimes.$$

We just discussed the Hilbert space for two particles in a single box. We now consider the Hilbert space for a single particle in two boxes, by which we mean a particle that is in one box, say $[0,1]$, *or* in another box, say $[2,3]$. The Hilbert space here is

$$L^2([0,1] \cup [2,3]) \cong L^2([0,1]) \oplus L^2([2,3]).$$

In general, if a system's state can lie in a Hilbert space V *or* in a Hilbert space W, the total Hilbert space is then

$$V \oplus W.$$

Heuristically,

$$\mathit{or} = \oplus.$$

Back to nucleons. According to Heisenberg's theory, a nucleon is a proton *or* a neutron. If we use the simplest nontrivial Hilbert space for both the proton and neutron, namely $\mathbb{C}$, then the Hilbert space for the nucleon should be

$$\mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}.$$

The proton and neutron then correspond to basis vectors of this Hilbert space:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2$$

and

$$n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2.$$

But, we can also have a nucleon in a linear combination of these states. More precisely, the state of the nucleon can be represented by any unit vector in $\mathbb{C}^2$.

The inner product in $\mathbb{C}^2$ then allows us to compute probabilities, using the following rule coming from quantum mechanics: the probability that a system in state $\psi \in H$, a given Hilbert space, will be observed in state $\phi \in H$ is

$$|\langle \psi, \phi \rangle|^2 .$$

Since p and n are orthogonal, there is no chance of seeing a proton as a neutron or vice versa, but for a nucleon in the state

$$\alpha p + \beta n \in \mathbb{C}^2,$$

there is probability $|\alpha|^2$ that measurement will result in finding a proton, and $|\beta|^2$ that measurement will result in finding a neutron. The condition that our state be a unit vector ensures that these probabilities add to 1.

In order for this to be interesting, however, there must be processes that can turn protons and neutrons into different states of the nucleon. Otherwise, there would be no point in having the full $\mathbb{C}^2$ space of states. Conversely, if there are processes which can change protons into neutrons and back, it turns out we need all of $\mathbb{C}^2$ to describe them.

Heisenberg believed in such processes, because of an analogy between nuclear physics and atomic physics. The analogy turned out to be poor, based on the faulty notion that the neutron was composed of a proton and an electron, but the idea of the nucleon with states in $\mathbb{C}^2$ proved to be a breakthrough.

The reason is that in 1936 a paper by Cassen and Condon [7] appeared suggesting that the nucleon's Hilbert space $\mathbb{C}^2$ is acted on by the symmetry group SU(2). They emphasized the analogy between this and the spin of the electron, which is also described by vectors in $\mathbb{C}^2$, acted on by the double cover of the 3d rotation group, which is also SU(2). In keeping with this analogy, they invented a concept called 'isospin'. The proton was declared the **isospin up** state or $I_3 = \frac{1}{2}$ state, and the neutron was declared the **isospin down** or $I_3 = -\frac{1}{2}$ state. Cassen and Condon's paper put isospin on its way to becoming a useful tool in nuclear physics.

Isospin proved useful because it formalized the following idea, which emerged from empirical data around the time of Cassen and Condon's paper. Namely: the strong force, unlike the electromagnetic force, is the same whether the particles involved are protons or neutrons. Protons and neutrons are interchangeable, as long as we neglect the small difference in their mass, and most importantly, as long as we neglect electromagnetic effects. One can phrase this idea in terms of group representation theory as follows: the strong force is *invariant* under the action of SU(2).

Though this idea was later seen to be an oversimplification, it foreshadowed modern ideas about unification. The proton, living in the representation $\mathbb{C}$ of the trivial group, and the neutron, living in a different representation $\mathbb{C}$ of the trivial group, were unified into the nucleon, with representation $\mathbb{C}^2$ of SU(2). These symmetries hold for the strong force, but not for electromagnetism: we say this force 'breaks' SU(2) symmetry.

But what does it mean, exactly, to say that a force is invariant under the action of some group? It means that when we are studying particles interacting via this force, the Hilbert space of each particle should be equipped with a unitary representation of this group. Moreover, any physical process caused by this force should be described by an 'intertwining operator': that is, a linear operator that respects the action of this group. A bit more precisely, suppose V and W are finite-dimensional Hilbert spaces on which some group G acts as unitary operators. Then a linear operator $F\colon V \to W$ is an **intertwining operator** if

$$F(g\psi) = gF(\psi)$$

for every $\psi \in V$ and $g \in G$.

Quite generally, symmetries give rise to conserved quantities. In quantum mechanics this works as follows. Suppose that G is a Lie group with a unitary representation on the finite-dimensional Hilbert space V and W. Then V and W automatically become representations of $\mathfrak{g}$, the Lie algebra of G, and any intertwining operator $F\colon V \to W$ respects the action of $\mathfrak{g}$. In other words,

$$F(T\psi) = TF(\psi)$$

for every $\psi \in V$ and $T \in \mathfrak{g}$. Next suppose that $\psi \in V$ is an eigenvector of T:

$$T\psi = i\lambda\psi$$

for some real number λ. Then it is easy to check $F(\psi)$ is again an eigenvector of T with the same eigenvalue:

$$TF(\psi) = i\lambda F(\psi).$$

So, the number λ is 'conserved' by the operator F.

The element $T \in \mathfrak{g}$ will act as a skew-adjoint operator on any unitary representation of G. Physicists prefer to work with self-adjoint operators since these have real eigenvalues. In quantum mechanics, self-adjoint operators are called 'observables'. We can get an observable by dividing T by i.

In Casson and Condon's isospin theory of the strong interaction, the symmetry group G is $\mathrm{SU}(2)$. The Lie algebra $\mathfrak{su}(2)$ has a basis consisting of three elements, and the quantity I_3 arises as above: it is just the eigenvalue of one of these elements, divided by i to get a real number. Because any physical process caused by the strong force is described by an intertwining operator, I_3 is conserved. In other words, the total I_3 of any system remains unchanged after a process that involves only strong interactions.

Nevertheless, for the states in $\mathbb{C}^2$ which mix protons and neutrons to have any meaning, there must be a mechanism which can convert protons into neutrons and vice versa. Mathematically, we have a way to do this: the action of $\mathrm{SU}(2)$. What does this correspond to, physically?

The answer originates in the work of Hideki Yukawa. The early 1930s, he predicted the existence of a particle that mediates the strong force, much as the photon mediates the electromagnetic force. From known properties of the strong force, he was able to predict that this particle should be about 200 times as massive as the electron, or about a tenth the mass of a proton. He predicted that experimentalists would find a particle with a mass in this range, and that it would interact strongly when it collided with nuclei.

Partially because of the intervention of World War II, it took over ten years for Yukawa's prediction to be vindicated. After a famous false alarm (see Section 2.5), it became clear by 1947 that a particle with the expected properties had been found. It was called the **pion** and it came in three varieties: one with positive charge, the π^+, one neutral, the π^0, and one with negative charge, the π^-.

The pion proved to be the mechanism that can transform nucleons. To wit, we observe processes like those in Figure 1, where we have drawn the Feynman diagrams which depict the nucleons absorbing pions, transforming where they are allowed to by charge conservation.

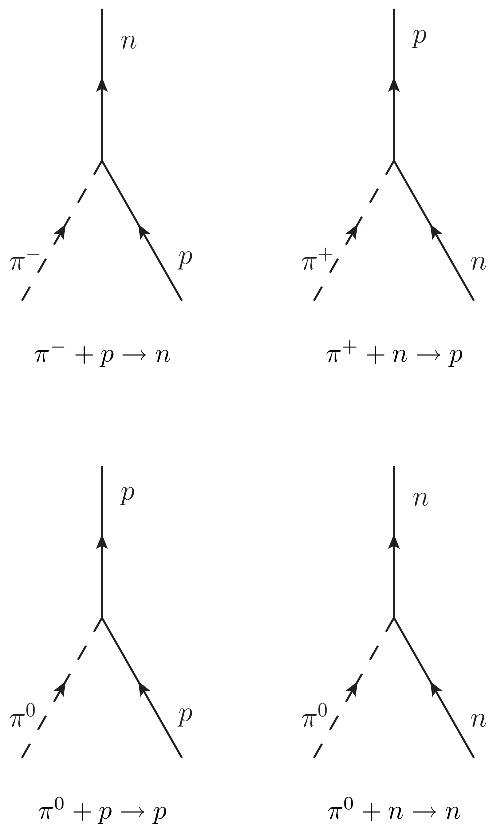


Figure 1: The nucleons absorbing pions.

Because of isospin conservation, we can measure the I_3 of a pion by looking at these interactions with the nucleons. It turns out that the I_3 of a pion is the same as its charge:

Pion	I_3
π^+	$+1$
π^0	0
π^-	-1

Here we pause, because we can see the clearest example of a pattern that lies at the heart of the Standard Model. It is the relationship between isospin I_3 and charge Q. For the pion, they are equal:

$$Q(\pi) = I_3(\pi).$$

But they are also related for the nucleon, though in a subtler way:

Nucleon	I_3	Charge
p	$\frac{1}{2}$	1
n	$-\frac{1}{2}$	0

The relationship for nucleons is

$$Q(N) = I_3(N) + \frac{1}{2}$$

This is nearly the most general relationship. It turns out that, for any given family of particles that differ only by I_3, we have the **Gell-Mann–Nishijima formula**:

$$Q = I_3 + Y/2$$

where Q and I_3 depend on the particle, but a new quantity, the **hypercharge** Y, depends only on the family. For example, pions all have hypercharge $Y = 0$, while nucleons both have hypercharge $Y = 1$.

Mathematically, Y being constant on 'families' just means it is constant on representations of the isospin symmetry group, SU(2). The three pions, like the proton and neutron, are nearly identical in terms of mass and their strong interactions. In Heisenberg's theory, the different pions are just different isospin states of the same particle. Since there are three, they have to span a three-dimensional representation of SU(2).

Up to isomorphism, there is only one three-dimensional complex irrep of SU(2), which is $\mathrm{Sym}^2\mathbb{C}^2$, the symmetric tensors of rank 2. In general, the unique $(n+1)$-dimensional irrep of SU(2) is given by $\mathrm{Sym}^n\mathbb{C}^2$. Physicists call this the **spin-$\boldsymbol{n/2}$ representation** of SU(2), or in the present context, the 'isospin-$n/2$ representation'. This representation has a basis of vectors where I_3 ranges from $-n/2$ to $n/2$ in integer steps. Nucleons lie in the isospin-$\frac{1}{2}$ representation, while pions lie in the isospin-1 representation.

This sets up an interesting puzzle. We know two ways to transform nucleons: the mathematical action of SU(2), and their physical interactions with pions. How are these related?

The answer lies in the representation theory. Just as the two nucleons span the two-dimensional irrep of $\mathbb{C}^2$ of SU(2), the pions span the three-dimensional irrep $\mathrm{Sym}^2\mathbb{C}^2$ of SU(2). But there is another way to write this representation which sheds light on the pions and the way they interact with nucleons: because SU(2) is itself a three-dimensional *real* manifold, its Lie algebra $\mathfrak{su}(2)$ is a three-dimensional *real* vector space. SU(2) acts on itself by conjugation, which fixes the identity and thus induces linear transformations of $\mathfrak{su}(2)$, giving a representation of SU(2) on $\mathfrak{su}(2)$ called the adjoint representation.

For simple Lie groups like SU(2), the adjoint representation is irreducible. Thus $\mathfrak{su}(2)$ is a three-dimensional *real* irrep of SU(2). This is different from the three-dimensional *complex* irrep $\mathrm{Sym}^2\mathbb{C}^2$, but very related. Indeed, $\mathrm{Sym}^2\mathbb{C}^2$ is just the complexification of $\mathfrak{su}(2)$:

$$\mathrm{Sym}^2\mathbb{C}^2 \cong \mathbb{C} \otimes \mathfrak{su}(2) \cong \mathfrak{sl}(2, \mathbb{C}).$$

The pions thus live in $\mathfrak{sl}(2, \mathbb{C})$, a complex Lie algebra, and this acts on $\mathbb{C}^2$ because SU(2) does. To be precise, Lie group representations induce Lie algebra representations, so the real Lie algebra $\mathfrak{su}(2)$ has a representation on $\mathbb{C}^2$. This then extends to a representation of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. And this representation is even familiar—it is the fundamental representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$.

Quite generally, whenever $\mathfrak{g}$ is the Lie algebra of a Lie group G, and $\rho\colon G\times V \to V$ is a representation of G on some finite-dimensional vector space V, we get a a representation of the Lie algebra $\mathfrak{g}$ on V, which we can think of as a linear map

$$d\rho\colon \mathfrak{g}\otimes V \to V.$$

And this map is actually an intertwining operator, meaning that it commutes with the action of G: since $\mathfrak{g}$ and V are both representations of G this is a sensible thing to say, and it is easy to check.

Pions act on nucleons via precisely such an intertwining operator:

$$\mathfrak{sl}(2,\mathbb{C})\otimes\mathbb{C}^2 \to \mathbb{C}^2.$$

So, the interaction between pions and nucleons arises naturally from the action of SU(2) on $\mathbb{C}^2$ after we complexify the Lie algebra of this group!

Physicists have invented a nice way to depict such intertwining operators—**Feynman diagrams**:

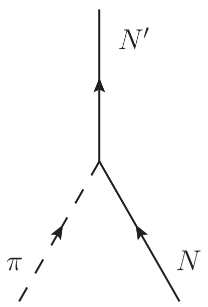
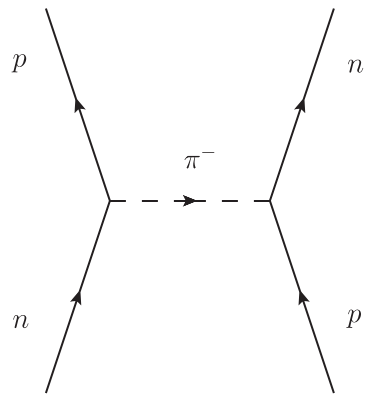


Figure 2: A nucleon absorbs a pion.

Here we see a nucleon coming in, absorbing a pion, and leaving. That is, this diagram depicts a basic *interaction* between pions and nucleons.

Feynman diagrams are calculational tools in physics, though to actually use them as such, we need quantum field theory. Then, instead of just standing for intertwining operators between representations of a compact groups like SU(2), they depict intertwining operators between representations of the product of this group and the Poincaré group, which describes the symmetries of spacetime. Unfortunately, the details are beyond the scope of this paper. By ignoring the Poincaré group, we are, in the language of physics, restricting our attention to 'internal degrees of freedom', and their 'internal' (i.e., gauge) symmetries.

Nonetheless, we can put basic interactions like the one in Figure 2 together to form more complicated ones, like this:



Here, two nucleons interact by exchanging pions. This is the mechanism for the strong force proposed by Yukawa, still considered approximately right today. Better, though, it depicts all the representation-theoretic ingredients of a modern gauge theory in physics. That is, it shows two nucleons, which live in a representation $\mathbb{C}^2$ of the gauge group $\mathrm{SU}(2)$, interacting by the exchange of a pion, which lives in the complexified adjoint rep, $\mathbb{C} \otimes \mathfrak{su}(2)$. In the coming sections we will see how these ideas underlie the Standard Model.

2.2 The Fundamental Fermions

2.2.1 Quarks

In the last section, we learned how Heisenberg unified the proton and neutron into the nucleon, and that Yukawa proposed nucleons interact by exchanging pions. This viewpoint turned out to be at least approximately true, but it was based on the idea that the proton, neutron and pions were all fundamental particles without internal structure, which was not ultimately supported by the evidence.

Protons and neutrons are not fundamental. They are made of particles called **quarks**. There are a number of different types of quarks, called **flavors**. However, it takes only two flavors to make protons and neutrons: the **up quark**, u, and the **down quark**, d. The proton consists of two up quarks and one down:

$$p = uud$$

while the neutron consists of one up quark and two down:

$$n = udd$$

Protons have an electric charge of $+1$, exactly opposite the electron, while neutrons are neutral, with 0 charge. These two conditions are enough to determine the charge of their constituents, which are fundamental fermions much like the electron:

Fundamental Fermions (second try)		
Name	Symbol	Charge
Electron	e^-	-1
Up quark	u	$+\frac{2}{3}$
Down quark	d	$-\frac{1}{3}$

There are more quarks than these, but these are the lightest ones, comprising the **first generation**. They are all we need to make protons and neutrons, and so, with the electron in tow, the above list contains all the particles we need to make atoms.

Yet quarks, fundamental as they are, are never seen in isolation. They are always bunched up into particles like the proton and neutron. This phenomenon is called **confinement**. It makes the long, convoluted history of how we came to understand quarks, despite the fact that they are never seen, all the more fascinating. Unfortunately, we do not have space for this history here, but it can be found in the books by Crease and Mann [8], Segrè [33], and Pais [25].

It is especially impressive how physicists were able to discover that each flavor of quark comes in three different states, called **colors**: **red** r, **green** g, and **blue** b. These 'colors' have nothing to do with actual colors; they are just cute names—though as we shall see, the names are quite well chosen. Mathematically, all that matters is that the Hilbert space for a single quark is $\mathbb{C}^3$; we call the standard basis vectors r, g and b. The **color symmetry group** SU(3) acts on this Hilbert space in the obvious way, via its fundamental representation.

Since both up and down quarks come in three color states, there are really six kinds of quarks in the matter we see around us. Three up quarks, spanning a copy of $\mathbb{C}^3$:

$$u^r, u^g, u^b \in \mathbb{C}^3.$$

and three down quarks, spanning another copy of $\mathbb{C}^3$:

$$d^r, d^g, d^b \in \mathbb{C}^3.$$

The group SU(3) acts on each space. All six quarks taken together span this vector space:

$$\mathbb{C}^3 \oplus \mathbb{C}^3 \cong \mathbb{C}^2 \otimes \mathbb{C}^3$$

where $\mathbb{C}^2$ is spanned by the flavors u and d. Put another way, a first-generation quark comes in one of six flavor-color states.

How could physicists discover the concept of color, given that quarks are confined? In fact confinement was the key to this discovery! Confinement amounts to the following decree: all observed states must be **white**, i.e., invariant under the action of SU(3). It turns out that this has many consequences.

For starters, this decree implies that we cannot see an individual quark, because they all transform nontrivially under SU(3). Nor do we ever see a particle built from two quarks, since no unit vectors in $\mathbb{C}^3 \otimes \mathbb{C}^3$ are fixed by SU(3). But we *do* see particles made of three quarks: namely, nucleons! This is because there *are* unit vectors in

$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$$

fixed by SU(3). Indeed, as a representation of SU(3), $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ contains precisely one copy of the trivial representation: the antisymmetric rank three tensors, $\Lambda^3\mathbb{C}^3 \subseteq \mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$. This one dimensional vector space is spanned by the wedge product of all three basis vectors:

$$r \wedge b \wedge g \in \Lambda^3\mathbb{C}^3.$$

So, up to normalization, this *must* be the color state of a nucleon. And now we see why the 'color' terminology is well-chosen: an equal mixture of red, green and blue light is white. This is just a coincidence, but it is too cute to resist.

So: color is deeply related to confinement. Flavor, on the other hand, is deeply related to isospin. Indeed, the flavor $\mathbb{C}^2$ is suspiciously like the isospin $\mathbb{C}^2$ of the nucleon. We even call the quark flavors 'up' and 'down'. This is no accident. The proton and neutron, which are the two isospin states of the nucleon, differ only by their flavors, and only the flavor of one quark at that. If one could interchange u and d, one could interchange protons and neutrons.

Indeed, we can use quarks to explain the isospin symmetry of Section 2.1. Protons and neutrons are so similar, with nearly the same mass and strong interactions, because u and d quarks are so similar, with nearly the same mass and truly identical colors.

So as in Section 2.1, let SU(2) act on the flavor states $\mathbb{C}^2$. By analogy with that section, we call this SU(2) the isospin symmetries of the quark model. Unlike the color symmetries SU(3), these symmetries are not exact, because u and d quarks have different mass and charge. Nevertheless, they are useful.

The isospin of the proton and neutron then arises from the isospin of its quarks. Define $I_3(u) = \frac{1}{2}$ and $I_3(d) = -\frac{1}{2}$, making u and d the isospin up and down states at which their names hint. To find the I_3 of a composite, like a proton or neutron, add the I_3 for its constituents. This gives the proton and neutron the right I_3:

$$\begin{aligned} I_3(p) &= \tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2} &=& \quad \tfrac{1}{2} \\ I_3(n) &= \tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2} &=& -\tfrac{1}{2}. \end{aligned}$$

Of course, having the right I_3 is not the whole story for isospin. The states p and n must still span a copy of the fundamental rep $\mathbb{C}^2$ of SU(2). Whether or not this happens depends on how the constituent quark flavors transform under SU(2).

The states $u\otimes u\otimes d$ and $u\otimes d\otimes d$ do *not* span a copy of the fundamental rep of SU(2) inside $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$. So, as with color, the equations

$$p = uud, \quad n = udd$$

fail to give us the whole story. For the proton, we actually need some linear combination of the $I_3 = \frac{1}{2}$ flavor states, which are made of two u's and one d:

$$u\otimes u\otimes d, \quad u\otimes d\otimes u, \quad d\otimes u\otimes u \quad \in \quad \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2.$$

And for the neutron, we need some linear combination of the $I_3 = -\frac{1}{2}$ flavor states, which are made of one u and two d's:

$$u\otimes d\otimes d, \quad d\otimes u\otimes d, \quad d\otimes d\otimes u \quad \in \quad \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2.$$

Writing $p = uud$ and $n = udd$ is just a sort of shorthand for saying that p and n are made from basis vectors with those quarks in them.

In physics, the linear combination required to make p and n work also involves the 'spin' of the quarks, which lies outside of our scope. We will content ourselves with showing that it *can* be done. That is, we will show that $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ really does contain a copy of the fundamental rep $\mathbb{C}^2$ of $\mathrm{SU}(2)$. To do this, we use the fact that any rank 2 tensor can be decomposed into symmetric and antisymmetric parts; for example,

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathrm{Sym}^2\mathbb{C}^2 \oplus \Lambda^2\mathbb{C}^2$$

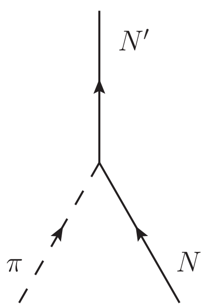
and this is actually how $\mathbb{C}^2 \otimes \mathbb{C}^2$ decomposes into irreps. $\mathrm{Sym}^2\mathbb{C}^2$, as we noted in Section 2.1, is the unique 3-dimensional irrep of $\mathrm{SU}(2)$; its orthogonal complement $\Lambda^2\mathbb{C}^2$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$ is thus also a subrepresentation, but this space is 1-dimensional, and must therefore be the trivial irrep, $\Lambda^2\mathbb{C}^2 \cong \mathbb{C}$. In fact, for any $\mathrm{SU}(n)$, the top exterior power of its fundamental rep, $\Lambda^n\mathbb{C}^n$, is trivial.

As a representation of $\mathrm{SU}(2)$, we thus have

$$\begin{aligned}\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 &\cong \mathbb{C}^2 \otimes (\mathrm{Sym}^2\mathbb{C}^2 \oplus \mathbb{C}) \\ &\cong \mathbb{C}^2 \otimes \mathrm{Sym}^2\mathbb{C}^2 \quad \oplus \quad \mathbb{C}^2.\end{aligned}$$

So indeed, $\mathbb{C}^2$ is a subrepresentation of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

As in the last section, there is no reason to have the full $\mathbb{C}^2$ of isospin states for nucleons unless there is a way to change protons into neutrons. There, we discussed how the pions provide this mechanism. The pions live in $\mathfrak{sl}(2,\mathbb{C})$, the complexification of the adjoint representation of $\mathrm{SU}(2)$, and this acts on $\mathbb{C}^2$:



This Feynman diagram is a picture of the intertwining operator $\mathfrak{sl}(2,\mathbb{C}) \otimes \mathbb{C}^2 \to \mathbb{C}^2$ given by the representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$. Now we know that nucleons are made of quarks and that isospin symmetry comes from their flavor symmetry. What about pions?

Pions also fit into this model, but they require more explanation, because they are made of quarks and 'antiquarks'. To every kind of particle, there is a corresponding antiparticle, which is just like the original particle but with opposite charge and isospin. The antiparticle of a quark is called an **antiquark**.

In terms of group representations, passing from a particle to its antiparticle corresponds to taking the dual representation. Since the quarks live in $\mathbb{C}^2 \otimes \mathbb{C}^3$, a representation of $\mathrm{SU}(2) \times \mathrm{SU}(3)$, the antiquarks live in the dual representation $\mathbb{C}^{2*} \otimes \mathbb{C}^{3*}$. Since $\mathbb{C}^2$ has basis vectors called **up** and **down**:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$$

the space $\mathbb{C}^{2*}$ has a dual basis

$$\overline{u} = (1,0) \in \mathbb{C}^{2*} \qquad \overline{d} = (0,1) \in \mathbb{C}^{2*}$$

called **antiup** and **antidown**. Similarly, since the standard basis vectors for $\mathbb{C}^3$ are called red green and blue, the dual basis vectors for $\mathbb{C}^{3*}$ are known as **anticolors**: namely **antired** $\overline{r}$, **antigreen** $\overline{g}$, and **antiblue** $\overline{b}$. When it comes to actual colors of light, antired is called 'cyan': this is the color of light which blended with red gives white. Similarly, antigreen is magenta, and antiblue is yellow. But few physicists dare speak of 'magenta antiquarks'—apparently this would be taking the joke too far.

All pions are made from one quark and one antiquark. The flavor state of the pions must therefore live in

$$\mathbb{C}^2 \otimes \mathbb{C}^{2*}.$$

We can use the fact that pions live in $\mathfrak{sl}(2,\mathbb{C})$ to find out how they decompose into quarks and antiquarks, since

$$\mathfrak{sl}(2,\mathbb{C}) \subseteq \mathrm{End}(\mathbb{C}^2).$$

First, express the pions as matrices:

$$\pi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \pi^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \pi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We know they have to be these matrices, up to normalization, because these act the right way on nucleons in $\mathbb{C}^2$:

$$\begin{aligned} \pi^- + p &\to n \\ \pi^+ + n &\to p \\ \pi^0 + p &\to p \\ \pi^0 + n &\to n. \end{aligned}$$

Now, apply the standard isomorphism $\mathrm{End}(\mathbb{C}^2) \cong \mathbb{C}^2 \otimes \mathbb{C}^{2*}$ to write these matrices as linear combinations of quarks and antiquarks:

$$\pi^+ = u \otimes \overline{d}, \quad \pi^0 = u \otimes \overline{u} - d \otimes \overline{d}, \quad \pi^- = d \otimes \overline{u}.$$

Note these all have the right I_3, because isospins reverse for antiparticles. For example, $I_3(\overline{d}) = +\frac{1}{2}$, so $I_3(\pi^+) = 1$.

In writing these pions as quarks and antiquarks, we have once again neglected to write the color, because this works the same way for all pions. As far as color goes, pions live in

$$\mathbb{C}^3 \otimes \mathbb{C}^{3*}.$$

Confinement says that pions need to be white, just like nucleons, and there is only a one-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^{3*}$ on which $\mathrm{SU}(3)$ acts trivially, spanned by

$$r \otimes \overline{r} + g \otimes \overline{g} + b \otimes \overline{b} \in \mathbb{C}^3 \otimes \mathbb{C}^{3*}.$$

So, this must be the color state of all pions.

Finally, the Gell-Mann–Nishijima formula also still works for quarks, provided we define the hypercharge for both quarks to be $Y = \frac{1}{3}$:

$$\begin{aligned} Q(u) &= I_3(u) + Y/2 &= \tfrac{1}{2} + \tfrac{1}{6} &= \tfrac{2}{3} \\ Q(d) &= I_3(d) + Y/2 &= -\tfrac{1}{2} + \tfrac{1}{6} &= -\tfrac{1}{3}. \end{aligned}$$

Since nucleons are made of three quarks, their total hypercharge is $Y = 1$, just as before.

2.2.2 Leptons

With the quarks and electron, we have met all the fundamental fermions required to make atoms, and almost all of the particles we need to discuss the Standard Model. Only one player remains to be introduced—the **neutrino**, ν. This particle completes the **first generation** of fundamental fermions:

The First Generation of Fermions — Charge

Name	Symbol	Charge
Neutrino	ν	0
Electron	e^-	-1
Up quark	u	$+\frac{2}{3}$
Down quark	d	$-\frac{1}{3}$

Neutrinos are particles which show up in certain interactions, like the decay of a neutron into a proton, an electron, and an antineutrino

$$n \to p + e^- + \overline{\nu}.$$

Indeed, neutrinos ν have antiparticles $\overline{\nu}$, just like quarks and all other particles. The electron's antiparticle, denoted e^+, was the first discovered, so it wound up subject to an inconsistent naming convention: the 'antielectron' is called a **positron**.

Neutrinos carry no charge and no color. They interact very weakly with other particles, so weakly that they were not observed until the 1950s, over 20 years after they were hypothesized by Pauli. Collectively, neutrinos and electrons, the fundamental fermions that do not feel the strong force, are called **leptons**.

In fact, the neutrino only interacts via the **weak force**. Like the electromagnetic force and the strong force, the weak force is a fundamental force, hypothesized to explain the decay of the neutron, and eventually required to explain other phenomena.

The weak force cares about the 'handedness' of particles. It seems that every particle that we have discussed comes in left- and right-handed varieties, which (quite roughly speaking) spin in opposite ways. There are are left-handed leptons, which we denote as

$$\nu_L \quad e^-_L$$

and left-handed quarks, which we denote as

$$u_L \quad d_L$$

and similarly for right-handed fermions, which we will denote with a subscript R. As the terminology suggests, looking in a mirror interchanges left and right—in a mirror, the left-handed electron e^-_L looks like a right-handed electron, e^-_R, and vice versa. More precisely, applying any of the reflections in the Poincaré group to the (infinite-dimensional) representation we use to describe these fermions interchanges left and right.

Remarkably, the weak force interacts only with left-handed particles and right-handed antiparticles. For example, when the neutron decays, we always have

$$n_L \to p_L + e_L^- + \overline{\nu}_R$$

and never

$$n_R \to p_R + e_R^- + \overline{\nu}_L.$$

This fact about the weak force, first noticed in the 1950s, left a deep impression on physicists. No other physical law is asymmetric in left and right. That is, no other physics, classical or quantum, looks different when viewed in a mirror. Why the weak force, and only the weak force, exhibits this behavior is a mystery.

Since neutrinos only feel the weak force, and the weak force only involves left-handed particles, the right-handed neutrino ν_R has never been observed directly. For a long time, physicists believed this particle did not even exist, but recent observations of neutrino oscillations suggest otherwise. In this paper, we will assume there are right-handed neutrinos, but the reader should be aware that this is still open to some debate. In particular, even if they do exist, we know very little about them.

Note that isospin is not conserved in weak interactions. After all, we saw in the last section that I_3 is all about counting the number of u quarks over the number of d quarks. In a weak process like neutron decay

$$udd \to uud + e^- + \overline{\nu},$$

the right-hand side has $I_3 = -\frac{1}{2}$, while the left has $I_3 = \frac{1}{2}$.

Yet maybe we are not being sophisticated enough. Perhaps isospin can be extended beyond quarks, and leptons can also carry I_3. Indeed, if we define $I_3(\nu_L) = \frac{1}{2}$ and $I_3(e^-) = -\frac{1}{2}$, we get

$$\begin{array}{ccccccccc} & n_L & \to & p_L & + & e_L^- & + & \overline{\nu}_R \\ I_3: & -\frac{1}{2} & = & \frac{1}{2} & - & \frac{1}{2} & - & \frac{1}{2} \end{array}$$

where we have used the rule that I_3 reverses sign for antiparticles.

This extension of isospin is called **weak isospin** since it extends the concept to weak interactions. Indeed, it turns out to be fundamental to the theory of weak interactions. Unlike regular isospin symmetry, which is only approximate, weak isospin symmetry turns out to be exact.

So from now on we shall discuss only weak isospin, and call it simply **isospin**. For this I_3 equals zero for right-handed particles, and $\pm\frac{1}{2}$ for left-handed particles:

The First Generation of Fermions — Charge and Isospin			
Name	Symbol	Q	I_3
Left-handed neutrino	ν_L	0	$\frac{1}{2}$
Left-handed electron	e_L^-	-1	$-\frac{1}{2}$
Left-handed up quark	u_L	$+\frac{2}{3}$	$\frac{1}{2}$
Left-handed down quark	d_L	$-\frac{1}{3}$	$-\frac{1}{2}$
Right-handed neutrino	ν_R	0	0
Right-handed electron	e_R^-	-1	0
Right-handed up quark	u_R	$+\frac{2}{3}$	0
Right-handed down quark	d_R	$-\frac{1}{3}$	0

The antiparticle of a left-handed particle is right-handed, and the antiparticle of a right-handed particle is left-handed. The isospins also change sign. For example, $I_3(e_R^+) = +\frac{1}{2}$, while $I_3(e_L^+) = 0$.

In Section 2.3.2, we will see that the Gell-Mann–Nishijima formula, when applied to weak isospin, defines a fundamental quantity, the 'weak hypercharge', that is vital to the Standard Model. But first, in Section 2.3.1, we discuss how to generalize the SU(2) symmetries from isospin to weak isospin.

2.3 The Fundamental Forces

2.3.1 Isospin and SU(2), Redux

The tale we told of isospin in Section 2.1 only concerned the strong force, which binds nucleons together into nuclei. We learned about an approximation in which nucleons live in the fundamental rep $\mathbb{C}^2$ of the isospin symmetry group SU(2), and that they interact by exchanging pions, which live in the complexified adjoint rep of this group, namely $\mathfrak{sl}(2, \mathbb{C})$.

But this tale is mere prelude to the modern story, where *weak* isospin, defined in Section 2.2.2, is the star of the show. This story is not about the strong force, but rather the weak force. This story parallels the old one, but it involves left-handed fermions instead of nucleons. The left-handed fermions, with $I_3 = \pm\frac{1}{2}$, are paired up into fundamental representations of SU(2), the **weak isospin symmetry group**. There is one spanned by left-handed leptons:

$$\nu_L, e_L^- \in \mathbb{C}^2,$$

and one spanned by each color of left-handed quarks:

$$u_L^r, d_L^r \in \mathbb{C}^2, \quad u_L^g, d_L^g \in \mathbb{C}^2, \quad u_L^b, d_L^b \in \mathbb{C}^2.$$

The antiparticles of the left-handed fermions, the right-handed antifermions, span the dual representation $\mathbb{C}^{2*}$.

Because these particles are paired up in the same SU(2) representation, physicists often write them as **doublets**:

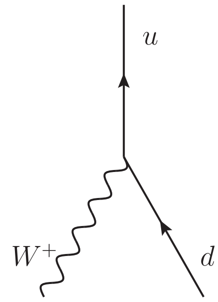
$$\begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix} \qquad \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

with the particle of higher I_3 written on top. Note that we have suppressed color on the quarks. This is conventional, and is done because SU(2) acts the same way on all colors.

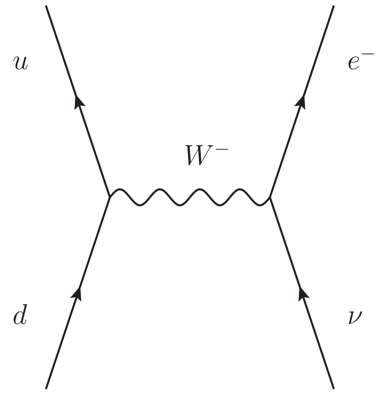
The particles in these doublets then interact via the exchange of W bosons, which are the weak isospin analogues of the pions. Like the pions, there are three W bosons:

$$W^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad W^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad W^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

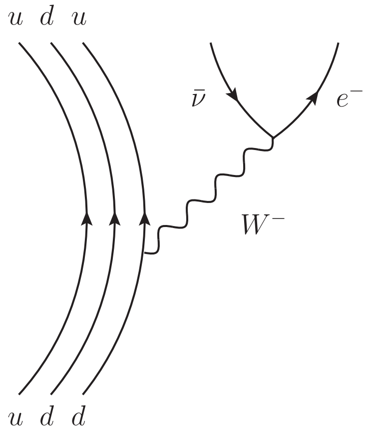
They span the complexified adjoint rep of SU(2), $\mathfrak{sl}(2, \mathbb{C})$, and they act on each of the doublets like the pions act on the nucleons, via the action of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathbb{C}^2$. For example,



Again, Feynman diagrams are the physicists' way of drawing intertwining operators. Since all the $\mathbb{C}^2$'s are acted on by the same SU(2), they can interact with each other via W boson exchange. For example, quarks and leptons can interact via W's:



This is in sharp contrast to the old isospin theory. In the new theory, it is processes like these that are responsible for the decay of the neutron:



The fact that only left-handed particles are combined into doublets reflects the fact that only they take part in weak interactions. Every right-handed fermion, on the other hand, is trivial under SU(2). Each one spans the trivial rep, $\mathbb{C}$. An example is the right-handed electron

$$e_R^- \in \mathbb{C}.$$

Physicists call these particles **singlets**, meaning they are trivial under SU(2). This is just the representation theoretic way of saying the right-handed electron, e_R^-, does not participate in weak interactions.

In summary, left-handed fermions are grouped into doublets (nontrivial representations of SU(2) on $\mathbb{C}^2$), while right-handed fermions are singlets (trivial representations on $\mathbb{C}$). So, the left-handed ones interact via the exchange of W bosons, while the right-handed ones do not.

The First Generation of Fermions — SU(2) Representations

Name	Symbol	I_3	SU(2) rep
Left-handed leptons	$\begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$	$\pm\frac{1}{2}$	$\mathbb{C}^2$
Left-handed quarks	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\pm\frac{1}{2}$	$\mathbb{C}^2$
Right-handed neutrino	ν_R	0	$\mathbb{C}$
Right-handed electron	e_R^-	0	$\mathbb{C}$
Right-handed up quark	u_R	0	$\mathbb{C}$
Right-handed down quark	d_R	0	$\mathbb{C}$

2.3.2 Hypercharge and U(1)

In Section 2.2.2, we saw how to extend the notion of isospin to weak isospin, which proved to be more fundamental, since we saw in Section 2.3.1 how this gives rise to interactions among left-handed fermions mediated via W bosons.

We grouped all the fermions into SU(2) representations. When we did this in Section 2.1, we saw that the SU(2) representations of particles were labeled by a quantity, the hypercharge Y, which relates the isospin I_3 to the charge Q via the Gell-Mann–Nishijima formula

$$Q = I_3 + Y/2.$$

We can use this formula to extend the notion of hypercharge to **weak hypercharge**, a quantity which labels the weak isospin representations. For left-handed quarks, this notion, like weak isospin, coincides with the old isospin and hypercharge. We have weak hypercharge $Y = \frac{1}{3}$ for these particles:

$$\begin{array}{rcrcrcr} Q(u_L) & = & I_3(u_L) + Y/2 & = & \frac{1}{2} + \frac{1}{6} & = & \frac{2}{3} \\ Q(d_L) & = & I_3(d_L) + Y/2 & = & -\frac{1}{2} + \frac{1}{6} & = & -\frac{1}{3}. \end{array}$$

But just as weak isospin extended isospin to leptons, weak hypercharge extends hypercharge to leptons. For left-handed leptons the Gell-Mann–Nishijima formula holds if we set $Y = -1$:

$$\begin{array}{rcrcrcr} Q(\nu_L) & = & I_3(\nu_L) + Y/2 & = & \frac{1}{2} - \frac{1}{2} & = & 0 \\ Q(e^-_L) & = & I_3(e^-_L) + Y/2 & = & -\frac{1}{2} - \frac{1}{2} & = & -1. \end{array}$$

Note that the weak hypercharge of quarks comes in units one-third the size of the weak hypercharge for leptons, a reflection of the fact that quark charges come in units one-third the size of lepton charges. Indeed, thanks to the Gell-Mann–Nishijima formula, these facts are equivalent.

For right-handed fermions, weak hypercharge is even simpler. Since $I_3 = 0$ for these particles, the Gell-Mann–Nishijima formula reduces to

$$Q = Y/2.$$

So, the hypercharge of a right-handed fermion is twice its charge. In summary, the fermions have these hypercharges:

The First Generation of Fermions — Hypercharge

Name	Symbol	Y
Left-handed leptons	$\begin{pmatrix} \nu_L \\ e^-_L \end{pmatrix}$	-1
Left-handed quarks	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\frac{1}{3}$
Right-handed neutrino	ν_R	0
Right-handed electron	e^-_R	-2
Right-handed up quark	u_R	$\frac{4}{3}$
Right-handed down quark	d_R	$-\frac{2}{3}$

But what is the meaning of hypercharge? We can start by reviewing our answer for the quantity I_3. This quantity, as we have seen, is related to how particles interact via W bosons, because particles with $I_3 = \pm\frac{1}{2}$ span the fundamental representation of SU(2), while the W bosons span the complexified adjoint representation, which acts on any other representation. Yet there is a deeper connection.

In quantum mechanics, observables like I_3 correspond to self-adjoint operators. We will denote the operator corresponding to an observable with a caret, for example $\hat{I}_3$ is the operator corresponding to I_3. A state of specific I_3, like ν_L which has $I_3 = \frac{1}{2}$, is an eigenvector,

$$\hat{I}_3 \nu_L = \frac{1}{2}\nu_L$$

with an eigenvalue that is the I_3 of the state. This makes it easy to write $\hat{I}_3$ as a matrix when we let it act on the $\mathbb{C}^2$ with basis ν_L and e_L^-, or any other doublet. We get

$$\hat{I}_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Note that this is an element of $\mathfrak{su}(2)$ divided by i. So, it lies in $\mathfrak{sl}(2,\mathbb{C})$, the complexified adjoint representation of SU(2). In fact it equals $\frac{1}{2}W^0$, one of the gauge bosons. So, up to a constant of proportionality, the observable $\hat{I}_3$ *is* one of the gauge bosons!

Similarly, corresponding to hypercharge Y is an observable $\hat{Y}$. This is also, up to proportionality, a gauge boson, though this gauge boson lives in the complexified adjoint rep of U(1).

Here are the details. Particles with hypercharge Y span irreps $\mathbb{C}_Y$ of U(1). Since U(1) is abelian, all of its irreps are one-dimensional. By $\mathbb{C}_Y$ we denote the one-dimensional vector space $\mathbb{C}$ with action of U(1) given by

$$\alpha \cdot z = \alpha^{3Y} z.$$

The factor of 3 takes care of the fact that Y might not be an integer, but is only guaranteed to be an integral multiple of $\frac{1}{3}$. For example, the left-handed leptons ν_L and e_L^- both have hypercharge $Y = -1$, so each one spans a copy of $\mathbb{C}_{-1}$:

$$\nu_L \in \mathbb{C}_{-1}, \quad e_L^- \in \mathbb{C}_{-1}$$

or, more compactly,

$$\nu_L, e_L^- \in \mathbb{C}_{-1} \otimes \mathbb{C}^2$$

where $\mathbb{C}^2$ is trivial under U(1).

In summary, the fermions we have met thus far lie in these U(1) representations:

The First Generation of Fermions — U(1) Representations		
Name	Symbol	U(1) rep
Left-handed leptons	$\begin{pmatrix} \nu_L \\ e^-_L \end{pmatrix}$	$\mathbb{C}_{-1}$
Left-handed quarks	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\mathbb{C}_{\frac{1}{3}}$
Right-handed neutrino	ν_R	$\mathbb{C}_0$
Right-handed electron	e^-_R	$\mathbb{C}_{-2}$
Right-handed up quark	u_R	$\mathbb{C}_{\frac{4}{3}}$
Right-handed down quark	d_R	$\mathbb{C}_{-\frac{2}{3}}$

Now, the adjoint representation $\mathfrak{u}(1)$ of U(1) is just the tangent space to the unit circle in $\mathbb{C}$ at 1. It is thus parallel to the imaginary axis, and can be identified as $i\mathbb{R}$. Is is generated by i. i also generates the complexification, $\mathbb{C} \otimes \mathfrak{u}(1) \cong \mathbb{C}$, though this also has other convenient generators, like 1. Given a particle $\psi \in \mathbb{C}_Y$ of hypercharge Y, we can differentiate the action of U(1) on ψ

$$e^{i\theta} \cdot \psi = e^{3iY\theta}\psi$$

and set $\theta = 0$ to find out how $\mathfrak{u}(1)$ acts:

$$i \cdot \psi = 3iY\psi.$$

Dividing by i we obtain

$$1 \cdot \psi = 3Y\psi.$$

In other words, we have

$$\hat{Y} = \frac{1}{3} \in \mathbb{C}$$

as an element of the complexified adjoint rep of U(1).

Particles with hypercharge interact by exchange of a boson, called the B boson, which spans the complexified adjoint rep of U(1). Of course, since $\mathbb{C}$ is one-dimensional, any nonzero element spans it. Up to a constant of proportionality, the B boson is just $\hat{Y}$, and we might as well take it to be equal to $\hat{Y}$, but calling it B is standard in physics.

The B boson is a lot like another, more familiar U(1) gauge boson—the photon! The hypercharge force which the B boson mediates is a lot like electromagnetism, which is mediated by photons, but its strength is proportional to hypercharge rather than charge.

2.3.3 Electroweak Symmetry Breaking

In the Standard Model, electromagnetism and the weak force are unified into the electroweak force. This is a U(1) × SU(2) gauge theory, and without saying so, we just told you all about it in sections 2.3.1 and 2.3.2. The fermions live in representations of hypercharge U(1) and weak isospin SU(2), exactly as we described in those sections, and we tensor these together to get representations of $\mathrm{U}(1)\times\mathrm{SU}(2)$:

The First Generation of Fermions — $\mathrm{U}(1) \times \mathrm{SU}(2)$ Representations				
Name	Symbol	Y	I_3	$\mathrm{U}(1) \times \mathrm{SU}(2)$ rep
Left-handed leptons	$\begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$	-1	$\pm\frac{1}{2}$	$\mathbb{C}_{-1} \otimes \mathbb{C}^2$
Left-handed quarks	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\frac{1}{3}$	$\pm\frac{1}{2}$	$\mathbb{C}_{\frac{1}{3}} \otimes \mathbb{C}^2$
Right-handed neutrino	ν_R	0	0	$\mathbb{C}_0 \otimes \mathbb{C}$
Right-handed electron	e_R^-	-2	0	$\mathbb{C}_{-2} \otimes \mathbb{C}$
Right-handed up quark	u_R	$\frac{4}{3}$	0	$\mathbb{C}_{\frac{4}{3}} \otimes \mathbb{C}$
Right-handed down quark	d_R	$-\frac{2}{3}$	0	$\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C}$

These fermions interact by exchanging B and W bosons, which span $\mathbb{C}\oplus\mathfrak{sl}(2,\mathbb{C})$, the complexified adjoint representation of $\mathrm{U}(1) \times \mathrm{SU}(2)$.

Yet despite the electroweak unification, electromagnetism and the weak force are very different at low energies, including most interactions in the everyday world. Electromagnetism is a force of infinite range that we can describe by a U(1) gauge theory, with the photon as gauge boson. The photon lives in $\mathbb{C}\oplus\mathfrak{sl}(2,\mathbb{C})$, alongside the B and W bosons. It is given by a linear combination

$$\gamma = W^0 + B/2$$

that parallels the Gell-Mann–Nishijima formula, $Q = I_3 + Y/2$. The weak force is of very short range and mediated by the W and Z bosons. The Z boson lives in $\mathbb{C}\oplus\mathfrak{sl}(2,\mathbb{C})$, and is given by the linear combination

$$Z = W^0 - B/2$$

which is in some sense 'perpendicular' to the photon. So, we can expand our chart of gauge bosons to include a basis for all of $\mathbb{C}\oplus\mathfrak{sl}(2,\mathbb{C})$ as follows:

Gauge Bosons (second try)		
Force	Gauge boson	Symbol
Electromagnetism	Photon	γ
Weak force	W and Z bosons	W^+, W^- and Z

What makes the photon (and electromagnetism) so different from the W and Z bosons (and the weak force)? It is symmetry breaking. Symmetry breaking allows the full electroweak $\mathrm{U}(1)\times\mathrm{SU}(2)$ symmetry group to be hidden away at high energy, replaced with the electromagnetic subgroup U(1) at lower energies. This electromagnetic U(1) is not the obvious factor of U(1) given by $\mathrm{U}(1) \times 1$. It is another copy, one which wraps around inside $\mathrm{U}(1) \times \mathrm{SU}(2)$ in a manner given by the Gell-Mann–Nishijima formula.

The dynamics behind symmetry breaking are beyond the scope of this paper. We will just mention that, in the Standard Model, electroweak symmetry breaking is believed to be due to the 'Higgs mechanism'. In this mechanism, all particles in the Standard Model, including the photon and the W and Z bosons, interact with a particle called the 'Higgs boson', and it is their differing interactions with this particle that makes them appear so different at low energies.

The Higgs boson has yet to be observed, and remains one of the most mysterious parts of the Standard Model. As of this writing, the Large Hadron Collider at CERN is beginning operations; searching for the Higgs boson is one of its primary aims.

For the details on symmetry breaking and the Higgs mechanism, which is essential to understanding the Standard Model, see Huang [17]. For a quick overview, see Zee [40].

2.3.4 Color and SU(3)

There is one more fundamental force in the Standard Model: the **strong force**. We have already met this force, as the force that keeps the nucleus together, but we discussed it before we knew that protons and neutrons are made of quarks. Now we need a force to keep quarks together inside the nucleons, and quark confinement tells us it must be a very strong force indeed. It is this force that, in modern parlance, is called the strong force and considered fundamental. The force between nucleons is a side effect of these more fundamental interactions among quarks.

Like all three forces in the Standard Model, the strong force is explained by a gauge theory, this time with gauge group SU(3), the color symmetry group of the quarks. The picture is simpler than that of electromagnetism and the weak force, however, because this symmetry is 'unbroken'.

By now you can guess how this goes. Every kind of quark spans the fundamental representation $\mathbb{C}^3$ of SU(3). For example, the left-handed up quark, with its three colors, lives in

$$u_L^r, u_L^g, u_L^b \in \mathbb{C}^3$$

and the left-handed down quark, with its three colors, spans another copy of $\mathbb{C}^3$,

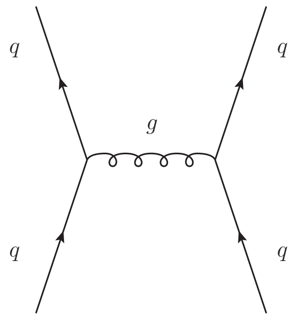
$$d_L^r, d_L^g, d_L^b \in \mathbb{C}^3$$

Together, these span the SU(3) representation

$$\mathbb{C}^2 \otimes \mathbb{C}^3$$

where $\mathbb{C}^2$ is trivial under SU(3).

The quarks interact by the exchange of **gluons**, the gauge bosons of the strong force. These gauge bosons live in $\mathbb{C} \otimes \mathfrak{su}(3) \cong \mathfrak{sl}(3, \mathbb{C})$, the complexified adjoint representation of SU(3). The interactions are drawn as Feynman diagrams, which now depict intertwining operators between representations of SU(3):



The gluons are fundamental particles, gauge bosons of the strong force, and they complete our table of gauge bosons:

Gauge Bosons		
Force	Gauge Boson	Symbol
Electromagnetism	Photon	γ
Weak force	W and Z bosons	W^+, W^- and Z
Strong force	Gluons	g

On the other hand, the leptons are 'white': they transform trivially under SU(3). So, they do not exchange gluons. In other words, they are not affected by the strong force. We can capture all of this information in a table, where we give the SU(3) representations in which all our fermions live.

The First Generation of Fermions — SU(3) Representations			
Name	Symbol	Colors	SU(3) rep
Left-handed neutrino	ν_L	white	$\mathbb{C}$
Left-handed electron	e^-_L	white	$\mathbb{C}$
Left-handed up quarks	u^r_L, u^g_L, u^b_L	r, g, b	$\mathbb{C}^3$
Left-handed down quarks	d^r_L, d^g_L, d^b_L	r, g, b	$\mathbb{C}^3$
Right-handed electron	e^-_R	white	$\mathbb{C}$
Right-handed neutrino	ν_R	white	$\mathbb{C}$
Right-handed up quarks	u^r_R, u^g_R, u^b_R	r, g, b	$\mathbb{C}^3$
Right-handed down quarks	d^r_R, d^g_R, d^b_R	r, g, b	$\mathbb{C}^3$

2.4 The Standard Model Representation

We are now in a position to put the entire Standard Model together in a single picture, much as we combined the isospin SU(2) and hypercharge U(1) into the electroweak gauge group, $\mathrm{U}(1) \times \mathrm{SU}(2)$, in Section 2.3.3. We then tensored the hypercharge U(1) representations with the isospin SU(2) representations to get the electroweak representations.

Now let us take this process one step further, by bringing in a factor of SU(3), for the color symmetry, and tensoring the representations of $\mathrm{U}(1) \times \mathrm{SU}(2)$ with the representations of SU(3). Doing this, we get the Standard Model. The Standard Model has this gauge group:

$$G_{\mathrm{SM}} = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3).$$

The fundamental fermions described by the Standard Model combine to form representations of this group. We know what these are, and describe all of them in Table 1.

The Standard Model Representation		
Name	Symbol	$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ rep
Left-handed leptons	$\begin{pmatrix} \nu_L \\ e^-_L \end{pmatrix}$	$\mathbb{C}_{-1} \otimes \mathbb{C}^2 \otimes \mathbb{C}$
Left-handed quarks	$\begin{pmatrix} u^r_L, u^g_L, u^b_L \\ d^r_L, d^g_L, d^b_L \end{pmatrix}$	$\mathbb{C}_{\frac{1}{3}} \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$
Right-handed neutrino	ν_R	$\mathbb{C}_0 \otimes \mathbb{C} \otimes \mathbb{C}$
Right-handed electron	e^-_R	$\mathbb{C}_{-2} \otimes \mathbb{C} \otimes \mathbb{C}$
Right-handed up quarks	(u^r_R, u^g_R, u^b_R)	$\mathbb{C}_{\frac{4}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3$
Right-handed down quarks	(d^r_R, d^g_R, d^b_R)	$\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3$

Table 1: Fundamental fermions as representations of $G_{\mathrm{SM}} = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$

All of the representations of G_{SM} in the left-hand column are irreducible, since they are made by tensoring irreps of this group's three factors, U(1), SU(2) and SU(3). This is a general fact: if V is an irrep of G, and W is an irrep of H, then $V \otimes W$ is an irrep of $G \times H$. Moreover, all irreps of $G \times H$ arise in this way.

On the other hand, if we take the direct sum of all these irreps,

$$F = (\mathbb{C}_{-1} \otimes \mathbb{C}^2 \otimes \mathbb{C}) \quad \oplus \quad \cdots \quad \oplus \quad (\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3),$$

we get a reducible representation containing all the first-generation fermions in the Standard Model. We call F the **fermion representation**. If we take the dual of F, we get a representation describing all the antifermions in the first generation. And taking the direct sum of these spaces:

$$F \oplus F^*$$

we get a representation of G_{SM} that we will call the **Standard Model representation**. It contains all the first-generation elementary particles in the Standard Model. It does not contain the gauge bosons or the mysterious Higgs.

The fermions living in the Standard Model representation interact by exchanging gauge bosons that live in the complexified adjoint representation of G_{SM}. We have already met all of these, and we collect them in Table 2.

Gauge Bosons		
Force	Gauge Boson	Symbol
Electromagnetism	Photon	γ
Weak force	W and Z bosons	W^+, W^- and Z
Strong force	Gluons	g

Table 2: Gauge bosons

Of all the particles and antiparticles in $F \oplus F^*$, exactly two of them are fixed by the action of G_{SM}. These are the right-handed neutrino

$$\nu_R \in \mathbb{C}_0 \otimes \mathbb{C} \otimes \mathbb{C}$$

and its antiparticle

$$\overline{\nu}_L \in (\mathbb{C}_0 \otimes \mathbb{C} \otimes \mathbb{C})^*,$$

both of which are trivial representations of G_{SM}; they thus do not participate in any forces mediated by the gauge bosons of the Standard Model. They might interact with the Higgs boson, but very little about right-handed neutrinos is known with certainty at this time.

2.5 Generations

Our description of the Standard Model is almost at an end. We have told you about its gauge group, G_{SM}, its representation $F \oplus F^*$ on the the first-generation of fermions and antifermions, and a bit about how these fermions interact by exchanging gauge bosons, which live in the complexified adjoint rep of G_{SM}. For the grand unified theories we are about to discuss, that is all we need. The stage is set.

Yet we would be derelict in our duty if we did not mention the *second* and *third* generation of fermions. The first evidence for these came in the 1930s, when a charged particle 207 times as heavy as the electron was found. At first researchers thought it was the particle predicted by Yukawa—the one that mediates the strong force between nucleons. But then it turned out the newly discovered particle was *not* affected by the strong force. This came as a complete surprise. As the physicist Rabi quipped at the time: "Who ordered that?"

Dubbed the **muon** and denoted μ^-, this new particle turned out to act like an overweight electron. Like the electron, it feels only the electromagnetic and weak force—and like the electron, it has its own neutrino! So, the neutrino we have been discussing so far is now called the **electron neutrino**, ν_e, to distinguish it from the **muon neutrino**, ν_μ. Together, the muon and the muon neutrino comprise the second generation of leptons. The muon decays via the weak force into an electron, a muon neutrino, and an electron antineutrino:

$$\mu^- \to e^- + \nu_\mu + \overline{\nu}_e.$$

Much later, in the 1970s, physicists realized there was also a second generation of quarks: the **charm quark**, c, and the **strange quark**, s. This was evidence of another pattern in the Standard Model: there are as many flavors of quark as there are leptons. In Section 3.3, we will learn about the Pati–Salam model, which explains this pattern by unifying quarks and leptons.

Today, we know about three generations of fermions. Three of quarks:

Quarks by Generation					
1st Generation		2nd Generation		3rd Generation	
Name	Symbol	Name	Symbol	Name	Symbol
Up	u	Charm	c	Top	t
Down	d	Strange	s	Bottom	b

and three of leptons:

Leptons by Generation					
1st Generation		2nd Generation		3rd Generation	
Name	Symbol	Name	Symbol	Name	Symbol
Electron neutrino	ν_e	Muon neutrino	ν_μ	Tau neutrino	ν_τ
Electron	e^-	Muon	μ^-	Tau	τ^-

The second and third generations of quarks and *charged* leptons differ from the first by being more massive and able to decay into particles of the earlier generations. The various neutrinos do not decay, and for a long time it was thought they are massless, but now it is known that some and perhaps all of them are massive. This allows them to change back and forth from one type to another, a phenomenon called 'neutrino oscillation'.

The Standard Model explains all of this by something called the Higgs mechanism. Apart from how they interact with the Higgs boson, the generations are identical. For instance, as representations of G_{SM}, each generation spans another copy of F. Each generation of fermions has corresponding antifermions, spanning a copy of F^*.

All told, we thus have three copies of the Standard Model representation, $F \oplus F^*$. We will only need to discuss one generation, so we find it convenient to speak as if $F \oplus F^*$ contains particles of the first-generation. No one knows why the Standard Model is this redundant, with three sets of very similar particles. It remains a mystery.

3 Grand Unified Theories

Not all of the symmetries of G_{SM}, the gauge group of the Standard Model, are actually seen in ordinary life. This is because some of the symmetries are 'spontaneously broken'. This means that while they are symmetries of the laws of physics, they are not symmetries of the vacuum. To see these symmetries we need to do experiments at very high energies, where the asymmetry of the vacuum has less effect. So, the behavior of particles at lower energies is like a shadow of the fundamental laws of physics, cast down from on high: a cryptic clue we must struggle to interpret.

It is reasonable to ask if this process continues. Could the symmetries of the Standard Model be just a subset of all the symmetries in nature? Could they be the low energy shadows of laws still more symmetric?

A grand unified theory, or GUT, constitutes a guess at what these 'more symmetric' laws might be. It is a theory with more symmetry than the Standard Model, which reduces to the Standard Model at lower energies. It is also, therefore, an attempt to describe the physics at higher energies.

GUTs are speculative physics. The Standard Model has been tested in countless experiments. There is a lot of evidence that it is an incomplete theory, and some vague clues about what the next theory might be like, but so far there is no empirical evidence that any GUT is correct—and even some empirical evidence that some GUTs, like SU(5), are incorrect.

Nonetheless, GUTs are interesting to theoretical physicists, because they allow us to explore some very definite ideas about how to extend the Standard Model. And because they are based almost entirely on the representation theory of compact Lie groups, the underlying physical ideas provide a marvelous playground for this beautiful area of mathematics.

Amazingly, this beauty then becomes a part of the physics. The representation of G_{SM} used in the Standard Model seems ad hoc. Why this one? Why all those seemingly arbitrary hypercharges floating around, mucking up some otherwise simple representations? Why do both leptons and quarks come in left- and right-handed varieties, which transform so differently? Why do quarks come in charges which are in units $\frac{1}{3}$ times an electron's charge? Why are there the same number of quarks and leptons? GUTs can shed light on these questions, using only group representation theory.

3.1 The SU(5) GUT

The SU(5) grand unified theory appeared in a 1974 paper by Howard Georgi and Sheldon Glashow [12]. It was the first grand unified theory, and is still considered the prototypical example. As such, there are many accounts of it in the physics literature. The textbooks by Ross [31] and Mohapatra [21] both devote an entire chapter to the SU(5) theory, and a lucid summary can be found in a review article by Witten [39], which also discusses the supersymmetric generalization of this theory.

In this section, we will limit our attention to the nonsupersymmetric version of SU(5) theory, which is how it was originally proposed. Unfortunately, this theory has since been ruled out by experiment; it predicts that protons will decay faster than the current lower bound on proton lifetime [26]. Nevertheless, because of its prototypical status and intrinsic interest, we simply must talk about the SU(5) theory.

The core idea behind the SU(5) grand unified theory is that because the Standard Model representation $F \oplus F^*$ is 32-dimensional, each particle or antiparticle in the first generation of fermions can be named by a 5-bit code. Roughly speaking, these bits are the answers to five yes-or-no questions:

- Is the particle isospin up?
- Is it isospin down?

- Is it red?
- Is it green?
- Is it blue?

There are subtleties involved when we answer 'yes' to both the first two questions, or 'yes' to more than one of the last three, but let us start with an example where these issues do not arise: the bit string 01100. This names a particle that is down and red. So, it refers to a red quark whose isospin is down, meaning $-\frac{1}{2}$. Glancing at Table 1, we see just one particle meeting this description: the red left-handed down quark, d_L^r.

We can flesh out this scheme by demanding that the operation of taking antiparticles correspond to switching 0's for 1's in the code. So the code for the antiparticle of d_L^r, the 'antired right-handed antidown antiquark', is 10011. This is cute: it means that being antidown is the same as being up, while being antired is the same as being both green and blue.

Furthermore, in this scheme all antileptons are 'black' (the particles with no color, ending in 000), while leptons are 'white' (the particles with every color, ending in 111). Quarks have exactly one color, and antiquarks have exactly two.

We are slowly working our way to the SU(5) theory. Next let us bring Hilbert spaces into the game. We can take the basic properties of being up, down, red, green or blue, and treat them as basis vectors for $\mathbb{C}^5$. Let us call these vectors u, d, r, g, b. The exterior algebra $\Lambda\mathbb{C}^5$ has a basis given by wedge products of these 5 vectors. This exterior algebra is 32-dimensional, and it has a basis labelled by 5-bit strings. For example, the bit string 01100 corresponds to the basis vector $d \wedge r$, while the bit string 10011 corresponds to $u \wedge g \wedge b$.

Next we bring in representation theory. The group SU(5) has an obvious representation on $\mathbb{C}^5$. And since the operation of taking exterior algebras is functorial, this group also has a representation on $\Lambda\mathbb{C}^5$. In the SU(5) grand unified theory, this is the representation we use to describe a single generation of fermions and their antiparticles.

Just by our wording, though, we are picking out a splitting of $\mathbb{C}^5$ into $\mathbb{C}^2 \oplus \mathbb{C}^3$: the isospin and color parts, respectively. Choosing such a splitting of $\mathbb{C}^5$ picks out a subgroup of SU(5), the set of all group elements that preserve this splitting. This subgroup consists of block diagonal matrices with a 2×2 block and a 3×3 block, both unitary, such that the determinant of the whole matrix is 1. Let us denote this subgroup as $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$.

Now for the miracle: the subgroup $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ is isomorphic to the Standard Model gauge group (at least modulo a finite subgroup). And, when we restrict the representation of SU(5) on $\Lambda\mathbb{C}^5$ to $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$, we get the Standard Model representation!

There are two great things about this. The first is that it gives a concise and mathematically elegant description of the Standard Model representation. The second is that the seemingly ad hoc hypercharges in the Standard Model *must be exactly what they are* for this description to work. So, physicists say the SU(5) theory explains the fractional charges of quarks: the fact that quark charges come in units $\frac{1}{3}$ the size of electron charge pops right out of this theory.

With this foretaste of the fruits the SU(5) theory will bear, let us get to work and sow the seeds. Our work will have two parts. First we need to check that

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \cong G_{\mathrm{SM}}/N$$

where N is some finite normal subgroup that acts trivially on $F \oplus F^*$. Then we need to check that indeed

$$\Lambda\mathbb{C}^5 \cong F \oplus F^*$$

as representations of $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$.

First, the group isomorphism. Naively, one might seek to build the SU(5) theory by including G_{SM} as a subgroup of SU(5). Can this be done? Clearly, we can include $\mathrm{SU}(2) \times \mathrm{SU}(3)$ as block diagonal matrices in SU(5):

$$\begin{array}{ccc} \mathrm{SU}(2) \times \mathrm{SU}(3) & \to & \mathrm{SU}(5) \\ (g, h) & \longmapsto & \left(\begin{array}{cc} g & 0 \\ 0 & h \end{array}\right). \end{array}$$

but this is not enough, because G_{SM} also has that pesky factor of U(1), related to the hypercharge. How can we fit that in?

The first clue is that elements of U(1) must commute with the elements of $\mathrm{SU}(2) \times \mathrm{SU}(3)$. But the only elements of SU(5) that commute with everybody in the $\mathrm{SU}(2) \times \mathrm{SU}(3)$ subgroup are diagonal, since they must separately commute with $\mathrm{SU}(2) \times 1$ and $1 \times \mathrm{SU}(3)$, and the only elements doing so are diagonal. Moreover, they must be scalars on each block. So, they have to look like this:

$$\left(\begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array}\right)$$

where α stands for the 2×2 identity matrix times the complex number $\alpha \in \mathrm{U}(1)$, and similarly for β in the 3×3 block. For the above matrix to lie in SU(5), it must have determinant 1, so $\alpha^2\beta^3 = 1$. This condition cuts the group of such matrices from $\mathrm{U}(1) \times \mathrm{U}(1)$ down to U(1). In fact, all such matrices are of the form

$$\left(\begin{array}{cc} \alpha^3 & 0 \\ 0 & \alpha^{-2} \end{array}\right)$$

where α runs over U(1).

So if we throw in elements of this form, do we get $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$? More precisely, does this map:

$$\begin{array}{cccc} \phi\colon & G_{\mathrm{SM}} & \to & \mathrm{SU}(5) \\ & (\alpha, g, h) & \longmapsto & \left(\begin{array}{cc} \alpha^3 g & 0 \\ 0 & \alpha^{-2} h \end{array}\right) \end{array}$$

give an isomorphism between G_{SM} and $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$? It is clearly a homomorphism. It clearly maps into G_{SM} *into* the subgroup $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$. And it is easy to check that it maps G_{SM} *onto* this subgroup. But is it one-to-one?

The answer is *no*: the map ϕ has a kernel, $\mathbb{Z}_6$. The kernel is the set of all elements of the form

$$(\alpha, \alpha^{-3}, \alpha^2) \in \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

and this is $\mathbb{Z}_6$, because scalar matrices α^{-3} and α^2 live in SU(2) and SU(3), respectively, if and only if α is a sixth root of unity. So, all we get is

$$G_{\mathrm{SM}}/\mathbb{Z}_6 \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \hookrightarrow \mathrm{SU}(5).$$

This sets up a nerve-racking test that the SU(5) theory must pass for it to have any chance of success. After all, not all representations of G_{SM} factor through

$G_{\mathrm{SM}}/\mathbb{Z}_6$, but all those coming from representations of SU(5) must do so. A representation of G_{SM} will factor through $G_{\mathrm{SM}}/\mathbb{Z}_6$ only if the $\mathbb{Z}_6$ subgroup acts trivially.

In short: the SU(5) GUT is doomed unless $\mathbb{Z}_6$ acts trivially on every fermion. (And antifermion, but that amounts to the same thing.) For this to be true, some nontrivial relations between hypercharge, isospin and color must hold.

For example, consider the left-handed electron

$$e_L^- \in \mathbb{C}_{-1} \otimes \mathbb{C}^2 \otimes \mathbb{C}.$$

For any sixth root of unity α, we need

$$(\alpha, \alpha^{-3}, \alpha^2) \in \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

to act trivially on this particle. Let us see how it acts. Note that:

- $\alpha \in \mathrm{U}(1)$ acts on $\mathbb{C}_{-1}$ as multiplication by α^{-3};
- $\alpha^{-3} \in \mathrm{SU}(2)$ acts on $\mathbb{C}^2$ as multiplication by α^{-3};
- $\alpha^2 \in \mathrm{SU}(3)$ acts trivially on $\mathbb{C}$.

So, we have

$$(\alpha, \alpha^{-3}, \alpha^2) \cdot e_L^- = \alpha^{-3}\alpha^{-3} e_L^- = e_L^-.$$

The action is indeed trivial—precisely because α is a sixth root of unity.

Or, consider the right-handed d quark:

$$d_R \in \mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3.$$

How does $(\alpha, \alpha^{-3}, \alpha^2)$ act on this? We note:

- $\alpha \in \mathrm{U}(1)$ acts on $\mathbb{C}_{-\frac{2}{3}}$ as multiplication by α^{-2};
- $\alpha^{-3} \in \mathrm{SU}(2)$ acts trivially on the trivial representation $\mathbb{C}$;
- $\alpha^2 \in \mathrm{SU}(3)$ acts on $\mathbb{C}^3$ as multiplication by α^2.

So, we have

$$(\alpha, \alpha^{-3}, \alpha^2) \cdot d_R = \alpha^{-2}\alpha^2 d_R = d_R.$$

Again, the action is trivial.

For SU(5) to work, though, $\mathbb{Z}_6$ has to act trivially on *every* fermion. There are 16 cases to check, and it is an awful lot to demand that hypercharge, the most erratic part of the Standard Model representation, satisfies 16 relations.

Or is it? In general, for a fermion with hypercharge Y, there are four distinct possibilities:

Hypercharge relations

Case		Representation		Relation
Nontrivial SU(2), nontrivial SU(3)	$\Rightarrow$	$\mathbb{C}_Y \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$	$\Rightarrow$	$\alpha^{3Y-3+2} = 1$
Nontrivial SU(2), trivial SU(3)	$\Rightarrow$	$\mathbb{C}_Y \otimes \mathbb{C}^2 \otimes \mathbb{C}$	$\Rightarrow$	$\alpha^{3Y-3} = 1$
Trivial SU(2), nontrivial SU(3)	$\Rightarrow$	$\mathbb{C}_Y \otimes \mathbb{C} \otimes \mathbb{C}^3$	$\Rightarrow$	$\alpha^{3Y+2} = 1$
Trivial SU(2), trivial SU(3)	$\Rightarrow$	$\mathbb{C}_Y \otimes \mathbb{C} \otimes \mathbb{C}$	$\Rightarrow$	$\alpha^{3Y} = 1$

Better yet, say it like a physicist!

Hypercharge relations				
Case		Representation		Relation
Left-handed quark	$\Rightarrow$	$\mathbb{C}_Y \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$	$\Rightarrow$	$\alpha^{3Y-3+2} = 1$
Left-handed lepton	$\Rightarrow$	$\mathbb{C}_Y \otimes \mathbb{C}^2 \otimes \mathbb{C}$	$\Rightarrow$	$\alpha^{3Y-3} = 1$
Right-handed quark	$\Rightarrow$	$\mathbb{C}_Y \otimes \mathbb{C} \otimes \mathbb{C}^3$	$\Rightarrow$	$\alpha^{3Y+2} = 1$
Right-handed lepton	$\Rightarrow$	$\mathbb{C}_Y \otimes \mathbb{C} \otimes \mathbb{C}$	$\Rightarrow$	$\alpha^{3Y} = 1$

But α is sixth root of unity, so all this really says is that those exponents are multiples of six:

Hypercharge relations		
Case		Relation
Left-handed quark	$\Rightarrow$	$3Y - 3 + 2 \in 6\mathbb{Z}$
Left-handed lepton	$\Rightarrow$	$3Y - 3 \in 6\mathbb{Z}$
Right-handed quark	$\Rightarrow$	$3Y + 2 \in 6\mathbb{Z}$
Right-handed lepton	$\Rightarrow$	$3Y \in 6\mathbb{Z}$

Dividing by 3 and doing a little work, it is easy to see these are just saying:

Hypercharge relations	
Case	Hypercharge
Left-handed quark	$Y =$ even integer $+\frac{1}{3}$
Left-handed lepton	$Y =$ odd integer
Right-handed quark	$Y =$ odd integer $+\frac{1}{3}$
Right-handed lepton	$Y =$ even integer

Table 3: Hypercharge relations

Now it is easy to check this indeed holds for every fermion in the standard model. SU(5) passes the test, not despite the bizarre pattern followed by hypercharges, but *because of it!*

By this analysis, we have shown that $\mathbb{Z}_6$ acts trivially on the Standard Model rep, so it is contained in the kernel of this rep. It is better than just a containment though: $\mathbb{Z}_6$ is the entire kernel. Because of this, we could say that $G_{\mathrm{SM}}/\mathbb{Z}_6$ is the 'true' gauge group of the Standard Model. And because we now know that

$$G_{\mathrm{SM}}/\mathbb{Z}_6 \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \hookrightarrow \mathrm{SU}(5),$$

it is almost as though this $\mathbb{Z}_6$ kernel, lurking inside G_{SM} this whole time, was a cryptic hint to try the SU(5) theory.

Of course, we still need to find a representation of SU(5) that extends the Standard Model representation. Luckily, there is a very beautiful choice that works: the exterior algebra $\Lambda\mathbb{C}^5$. Since SU(5) acts on $\mathbb{C}^5$, it has a representation on $\Lambda\mathbb{C}^5$. Our next goal is to check that pulling back this representation from SU(5) to G_{SM} using ϕ, we obtain the Stadard model representation $F \oplus F^*$.

As we do this, we will see another fruit of the SU(5) theory ripen. The triviality of $\mathbb{Z}_6$ already imposed some structure on hypercharges, as outlined in above in Table 3. As we fit the fermions into $\Lambda\mathbb{C}^5$, we will see this is no accident: the hypercharges have to be *exactly what they are* for the SU(5) theory to work.

To get started, our strategy will be to use the fact that, being representations of compact Lie groups, both the fermions $F \oplus F^*$ and the exterior algebra $\Lambda\mathbb{C}^5$ are completely reducible, so they can be written as a direct sum of irreps. We will then match up these irreps one at a time.

The fermions are already written as a direct sum of irreps, so we need to work on $\Lambda\mathbb{C}^5$. Now, any element $g \in \mathrm{SU}(5)$ acts as automorphisms of the exterior algebra $\Lambda\mathbb{C}^5$:

$$g(v \wedge w) = gv \wedge gw$$

where $v, w \in \Lambda\mathbb{C}^5$. Since we know how g acts on the vectors in $\mathbb{C}^5$, and these generate $\Lambda\mathbb{C}^5$, this rule is enough to tell us how g acts on all of $\mathbb{C}^5$. This action respects grades in $\Lambda\mathbb{C}^5$, so each exterior power in

$$\Lambda\mathbb{C}^5 \cong \Lambda^0\mathbb{C}^5 \oplus \Lambda^1\mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5 \oplus \Lambda^3\mathbb{C}^5 \oplus \Lambda^4\mathbb{C}^5 \oplus \Lambda^5\mathbb{C}^5$$

is a subrepresentation. In fact, these are all irreducible, so this is how $\Lambda\mathbb{C}^5$ breaks up into irreps of SU(5). Upon restriction to G_{SM}, some of these summands break apart further into irreps of G_{SM}.

Let us see how this works, starting with the easiest cases. $\Lambda^0\mathbb{C}^5$ and $\Lambda^5\mathbb{C}^5$ are both trivial irreps of G_{SM}. There are two trivial irreps in the Standard Model representation, namely $\langle \nu_R \rangle$ and its dual $\langle \overline{\nu}_L \rangle$, where we use angle brackets to stand for the Hilbert space spanned by a vector or collection of vectors. So, we could select $\Lambda^0\mathbb{C}^5 = \langle \overline{\nu}_L \rangle$ and $\Lambda^5\mathbb{C}^5 = \langle \nu_R \rangle$, or vice versa. At this juncture, we have no reason to prefer one choice to the other.

Next let us chew on the next piece: the first exterior power, $\Lambda^1\mathbb{C}^5$. We have

$$\Lambda^1\mathbb{C}^5 \cong \mathbb{C}^5$$

as vector spaces, and as representations of G_{SM}. But what is $\mathbb{C}^5$ as a representation of G_{SM}? The Standard Model gauge group acts on $\mathbb{C}^5$ via the map

$$\phi\colon (\alpha, g, h) \longmapsto \begin{pmatrix} \alpha^3 g & 0 \\ 0 & \alpha^{-2} h \end{pmatrix}$$

Clearly, this action preserves the splitting into the 'isospin part' and the 'color part' of $\mathbb{C}^5$:

$$\mathbb{C}^5 \cong \mathbb{C}^2 \oplus \mathbb{C}^3.$$

So, let us examine these two subrepresentations in turn:

- The $\mathbb{C}^2$ part transforms in the hypercharge 1 rep of U(1): that is, α acts as multiplication by α^3. It transforms according to the fundamental representation of SU(2), and the trivial representation of SU(3). This seems to describe a left-handed lepton with hypercharge 1.

- The $\mathbb{C}^3$ part transforms in the hypercharge $-\frac{2}{3}$ rep of U(1): that is, α acts as multiplication by α^{-2}. It transforms trivially under SU(2), and according to the fundamental representation of SU(3). Table 1 shows that these are the features of a right-handed down quark.

In short, as a rep of G_{SM}, we have

$$\mathbb{C}^5 \quad \cong \quad \mathbb{C}_1 \otimes \mathbb{C}^2 \otimes \mathbb{C} \quad \oplus \quad \mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3$$

and we have already guessed which particles these correspond to. The first summand looks like a left-handed lepton with hypercharge 1, while the second is a right-handed quark with hypercharge $-\frac{2}{3}$.

Now this is problematic, because another glance at Table 1 reveals that there is no left-handed lepton with hypercharge 1. The only particles with hypercharge 1 are the right-handed antileptons, which span the representation

$$\left\langle \begin{array}{c} e^+_R \\ \overline{\nu}_R \end{array} \right\rangle \cong \mathbb{C}_1 \otimes \mathbb{C}^{2*} \otimes \mathbb{C}.$$

But wait! SU(2) is unique among the SU(n)'s in that its fundamental rep $\mathbb{C}^2$ is self-dual:

$$\mathbb{C}^2 \cong \mathbb{C}^{2*}.$$

This saves the day. As a rep of G_{SM}, $\mathbb{C}^5$ becomes

$$\mathbb{C}^5 \quad \cong \quad \mathbb{C}_1 \otimes \mathbb{C}^{2*} \otimes \mathbb{C} \quad \oplus \quad \mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3$$

so it describes the right-handed antileptons with hypercharge 1 and the right-handed quarks with hypercharge $-\frac{2}{3}$. In other words:

$$\Lambda^1\mathbb{C}^5 \cong \mathbb{C}^5 \cong \left\langle \begin{array}{c} e^+_R \\ \overline{\nu}_R \end{array} \right\rangle \oplus \langle d_R \rangle$$

where we have omitted the color label on d_R to save space. Take heed of this: $\langle d_R \rangle$ is short for the vector space $\langle d^r_R, d^g_R, d^b_R \rangle$, and it is three-dimensional.

Now we can use our knowledge of the first exterior power to compute the second exterior power, by applying the formula

$$\Lambda^2(V \oplus W) \quad \cong \quad \Lambda^2 V \;\oplus\; (V \otimes W) \;\oplus\; \Lambda^2 W.$$

So, let us calculate! As reps of G_{SM} we have

$$\Lambda^2\mathbb{C}^5 \quad \cong \quad \Lambda^2(\mathbb{C}_1 \otimes \mathbb{C}^2 \otimes \mathbb{C} \quad \oplus \quad \mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C}^2 \otimes \mathbb{C}^3)$$

$$\cong \quad \Lambda^2(\mathbb{C}_1 \otimes \mathbb{C}^2 \otimes \mathbb{C}) \quad \oplus \quad (\mathbb{C}_1 \otimes \mathbb{C}^2 \otimes \mathbb{C}) \otimes (\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3) \quad \oplus \quad \Lambda^2(\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3).$$

Consider the first summand, $\Lambda^2(\mathbb{C}_1 \otimes \mathbb{C}^2 \otimes \mathbb{C})$. As a rep of SU(2) this space is just $\Lambda^2\mathbb{C}^2$, which is the one-dimensional trivial rep, $\mathbb{C}$. As a rep of SU(3) it is also trivial. But as a rep of U(1), it is nontrivial. Inside it we are juxtaposing two particles with hypercharge 1. Hypercharges add, just like charges, so the composite particle, which consists of one particle *and* the other, has hypercharge 2. So, as a representation of the Standard Model gauge group we have

$$\Lambda^2(\mathbb{C}_1 \otimes \mathbb{C}^2 \otimes \mathbb{C}) \cong \mathbb{C}_2 \otimes \mathbb{C} \otimes \mathbb{C}.$$

Glancing at Table 1 we see this matches the left-handed positron, e^+_L. Note that the hypercharges are becoming useful now, since they uniquely identify all the fermion and antifermion representations, except for neutrinos.

Next consider the second summand:

$$(\mathbb{C}_1 \otimes \mathbb{C}^2 \otimes \mathbb{C}) \otimes (\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3).$$

Again, we can add hypercharges, so this representation of G_{SM} is isomorphic to

$$\mathbb{C}_{\frac{1}{3}} \otimes \mathbb{C}^2 \otimes \mathbb{C}^3.$$

This is the space for left-handed quarks of hypercharge $\frac{1}{3}$, which from Table 1 is:

$$\left\langle \begin{array}{c} u_L \\ d_L \end{array} \right\rangle$$

where once again we have suppressed the label for colors.

Finally, the third summand in $\Lambda^2\mathbb{C}^5$ is

$$\Lambda^2(\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3).$$

This has isospin $-\frac{4}{3}$, so by Table 1 it had better correspond to the left-handed antiup antiquark, which lives in the representation

$$\mathbb{C}_{-\frac{4}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^{3*}.$$

Let us check. The rep $\Lambda^2(\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3)$ is trivial under SU(2). As a rep of SU(3) it is the same as $\Lambda^2\mathbb{C}^3$. But because SU(3) preserves the volume form on $\mathbb{C}^3$, taking Hodge duals gives an isomorphism

$$\Lambda^p\mathbb{C}^3 \cong (\Lambda^{3-p}\mathbb{C}^3)^*$$

so we have

$$\Lambda^2\mathbb{C}^3 \cong (\Lambda^1\mathbb{C}^3)^* \cong \mathbb{C}^{3*}$$

which is just what we need to show

$$\Lambda^2(\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C}^3) \cong \mathbb{C}_{-\frac{4}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^{3*} \cong \langle \overline{u}_L \rangle.$$

In summary, the following pieces of the Standard Model rep sit inside $\Lambda^2\mathbb{C}^5$:

$$\Lambda^2\mathbb{C}^5 \cong \langle e_L^+ \rangle \oplus \left\langle \begin{array}{c} u_L \\ d_L \end{array} \right\rangle \oplus \langle \overline{u}_L \rangle$$

We are almost done. Because SU(5) preserves the canonical volume form on $\mathbb{C}^5$, taking Hodge duals gives an isomorphism between

$$\Lambda^p\mathbb{C}^5 \cong (\Lambda^{5-p}\mathbb{C}^5)^*$$

as representations of SU(5). Thus given our results so far:

$$\begin{array}{rcl} \Lambda^0\mathbb{C}^5 & \cong & \langle \overline{\nu}_L \rangle \\ \Lambda^1\mathbb{C}^5 & \cong & \left\langle \begin{array}{c} e_R^+ \\ \overline{\nu}_R \end{array} \right\rangle \oplus \langle d_R \rangle \\ \Lambda^2\mathbb{C}^5 & \cong & \langle e_L^+ \rangle \oplus \left\langle \begin{array}{c} u_L \\ d_L \end{array} \right\rangle \oplus \langle \overline{u}_L \rangle \end{array}$$

we automatically get the antiparticles of these upon taking Hodge duals,

$$\begin{array}{rcl} \Lambda^3\mathbb{C}^5 & \cong & \langle e_R^- \rangle \oplus \left\langle \begin{array}{c} \overline{d}_R \\ \overline{u}_R \end{array} \right\rangle \oplus \langle u_R \rangle \\ \Lambda^4\mathbb{C}^5 & \cong & \left\langle \begin{array}{c} \nu_L \\ e_L^- \end{array} \right\rangle \oplus \langle \overline{d}_L \rangle \\ \Lambda^5\mathbb{C}^5 & \cong & \langle \nu_R \rangle. \end{array}$$

So $\Lambda\mathbb{C}^5 \cong F \oplus F^*$, as desired.

How does all this look in terms of the promised binary code? Remember, a 5-bit code is short for a wedge product of basis vectors $u, d, r, g, b \in \mathbb{C}^5$. For example, 01101 corresponds to $d \wedge r \wedge b$. And now that we have found an isomorphism $\Lambda\mathbb{C}^5 \cong F \oplus F^*$, each of these wedge products corresponds to a fermion or antifermion. How does this correspondence go, exactly?

First consider the grade-one part $\Lambda^1\mathbb{C}^5 \cong \mathbb{C}^2 \oplus \mathbb{C}^3$. This has basis vectors called u, d, r, g, and b. We have seen that the subspace $\mathbb{C}^2$, spanned by u and d, corresponds to

$$\left\langle \begin{array}{c} e_R^+ \\ \overline{\nu}_R \end{array} \right\rangle .$$

The top particle here has isospin up, while the bottom one has isospin down, so we must have $e_R^+ = u$ and $\overline{\nu}_R = d$. Likewise, the subspace $\mathbb{C}^3$ spanned by r, g and b corresponds to

$$\langle d_R \rangle = \langle d_R^r, d_R^g, d_R^b \rangle .$$

Thus we must have $d_R^c = c$, where c runs over the colors r, g, b.

Next consider the grade-two part:

$$\Lambda^2\mathbb{C}^5 \cong \langle e_L^+ \rangle \oplus \left\langle \begin{array}{c} u_L \\ d_L \end{array} \right\rangle \oplus \langle \overline{u}_L \rangle .$$

Here e_L^+ lives in the one-dimensional $\Lambda^2\mathbb{C}^2$ rep of SU(2), which is spanned by the vector $u \wedge d$. Thus, $e_L^+ = u \wedge d$. The left-handed quarks live in the $\mathbb{C}^2 \otimes \mathbb{C}^3$ rep of $\mathrm{SU}(2) \times \mathrm{SU}(3)$, which is spanned by vectors that consist of one isospin and one color. We must have $u_L^c = u \wedge c$ and $d_L^c = d \wedge c$, where again c runs over all the colors r, g, b. And now for the tricky part: the $\overline{u}_L$ quarks live in the $\Lambda^2\mathbb{C}^3$ rep of SU(3), but this is isomorphic to the fundamental representation of SU(3) on $\mathbb{C}^{3*}$, which is spanned by antired, antired and antiblue:

$$\overline{r} = g \wedge b, \qquad \overline{g} = b \wedge r, \qquad \overline{b} = r \wedge g .$$

These vectors form the basis of $\Lambda^2\mathbb{C}^3$ that is dual to r, g, and b under Hodge duality in $\Lambda\mathbb{C}^3$. So we must have

$$u_L^{\overline{c}} = \overline{c}$$

where $\overline{c}$ can be any anticolor. Take heed of the fact that $\overline{c}$ is grade 2, even though it may look like grade 1.

To work out the other grades, note that Hodge duality corresponds to switching 0's and 1's in our binary code. For instance, the dual of 01101 is 10010: or written in terms of basis vectors, the dual of $d \wedge r \wedge b$ is $u \wedge g$. Thus given the binary codes for the first few exterior powers:

$\Lambda^0\mathbb{C}^5$	$\Lambda^1\mathbb{C}^5$	$\Lambda^2\mathbb{C}^5$
$\overline{\nu}_L = 1$	$e_R^+ = u$	$e_L^+ = u \wedge d$
	$\overline{\nu}_R = d$	$u_L^c = u \wedge c$
	$d_R^c = c$	$d_L^c = d \wedge c$
		$\overline{u}_L^{\overline{c}} = \overline{c}$

taking Hodge duals gives the binary codes for the the rest:

$\Lambda^3\mathbb{C}^5$	$\Lambda^4\mathbb{C}^4$	$\Lambda^5\mathbb{C}^5$
$e_R^- = r \wedge g \wedge b$	$e_L^- = d \wedge r \wedge g \wedge b$	$\nu_R = u \wedge d \wedge r \wedge g \wedge b$
$\overline{u}_R^c = d \wedge \overline{c}$	$\nu_L = u \wedge r \wedge g \wedge b$	
$\overline{d}_R^c = u \wedge \overline{c}$	$\overline{d}_L^c = u \wedge d \wedge \overline{c}$	
$u_R^c = u \wedge d \wedge c$		

Putting these together, we get the binary code for every particle and antiparticle in the first generation of fermions. To save space, let us omit the wedge product symbols:

The Binary Code for SU(5)					
$\Lambda^0\mathbb{C}^5$	$\Lambda^1\mathbb{C}^5$	$\Lambda^2\mathbb{C}^5$	$\Lambda^3\mathbb{C}^5$	$\Lambda^4\mathbb{C}^4$	$\Lambda^5\mathbb{C}^5$
$\overline{\nu}_L = 1$	$e_R^+ = u$	$e_L^+ = ud$	$e_R^- = rgb$	$e_L^- = drgb$	$\nu_R = udrgb$
	$\overline{\nu}_R = d$	$u_L^c = uc$	$\overline{u}_R^c = d\overline{c}$	$\nu_L = urgb$	
	$d_R^c = c$	$d_L^c = dc$	$\overline{d}_R^c = u\overline{c}$	$\overline{d}_L^c = ud\overline{c}$	
		$\overline{u}_L^c = \overline{c}$	$u_R^c = udc$		

Table 4: Binary code for first-generation fermions, where $c = r, g, b$ and $\overline{c} = gb, br, rg$

Now we can see a good, though not decisive, reason to choose $\Lambda^0\mathbb{C}^5 \cong \overline{\nu}_L$. With this choice, and not the other, we get left-handed particles in the even grades, and right-handed particles in the odd grades. We *choose* to have this pattern now, but later on we need it.

Table 4 defines a linear isomorphism $f\colon F \oplus F^* \to \Lambda\mathbb{C}^5$ in terms of the basis vectors, so the equations in this table are a bit of a exaggeration. When we write say, $e_R^+ = u$, we really mean $f(e_R^+) = u$. This map f is an isomorphism between representations of G_{SM}. It tells us how these representations are the 'same'.

More precisely, we mean these representations are the same when we identify $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ with $G_{\mathrm{SM}}/\mathbb{Z}_6$ using the isomorphism induced by ϕ. In general, we can think of a unitary representation as a Lie group homomorphism

$$G \to \mathrm{U}(V)$$

where V is a finite-dimensional Hilbert space and $\mathrm{U}(V)$ is the Lie group of unitary operators on V. In this section we have been comparing two unitary representations: an ugly, complicated representation of G_{SM}:

$$G_{\mathrm{SM}} \to \mathrm{U}(F \oplus F^*)$$

and a nice, beautiful representation of SU(5):

$$\mathrm{SU}(5) \to \mathrm{U}(\Lambda\mathbb{C}^5).$$

We built a homomorphism

$$\phi\colon G_{\mathrm{SM}} \to \mathrm{SU}(5)$$

so it is natural to wonder if is there a fourth homomorphism

$$\mathrm{U}(F \oplus F^*) \to \mathrm{U}(\Lambda\mathbb{C}^5)$$

such that this square:

$$\begin{tikzcd}
G_{\mathrm{SM}} \arrow[r, "\phi"] \arrow[d] & \mathrm{SU}(5) \arrow[d] \\
\mathrm{U}(F \oplus F^*) \arrow[r] & \mathrm{U}(\Lambda\mathbb{C}^5)
\end{tikzcd}$$

commutes.

Indeed, we just showed this! We have seen there exists a unitary operator from the Standard Model rep to $\Lambda\mathbb{C}^5$, say

$$f \colon F \oplus F^* \xrightarrow{\sim} \Lambda\mathbb{C}^5,$$

such that the induced isomorphism of the unitary groups,

$$\mathrm{U}(f) \colon \mathrm{U}(F \oplus F^*) \xrightarrow{\sim} \mathrm{U}(\Lambda\mathbb{C}^5),$$

makes the above square commute. So, let us summarize this result as a theorem:

Theorem 1. *The following square commutes:*

$$\begin{tikzcd}
G_{\mathrm{SM}} \arrow[r, "\phi"] \arrow[d] & \mathrm{SU}(5) \arrow[d] \\
\mathrm{U}(F \oplus F^*) \arrow[r, "\mathrm{U}(f)"] & \mathrm{U}(\Lambda\mathbb{C}^5)
\end{tikzcd}$$

where the left vertical arrow is the Standard Model representation and the right one is the natural representation of $\mathrm{SU}(5)$ *on the exterior algebra of* $\mathbb{C}^5$.

3.2 The Spin(10) Theory

We now turn our attention to another grand unified theory. Physicists call it the 'SO(10) theory', but we shall call it the Spin(10) theory, because the Lie group involved is really Spin(10), the double cover of SO(10). This theory appeared in a 1974 paper by Georgi [10], shortly after his paper with Glashow on the SU(5) theory. However, Georgi has said that he conceived of the Spin(10) theory first. See Zee [40], Chapter VII.7, for a concise and readable account.

The SU(5) GUT has helped us explain the pattern of hypercharges in the Standard Model, and thanks to the use of the exterior algebra, $\Lambda\mathbb{C}^5$, we can interpret it in terms of a binary code. This binary code explains another curious fact about the Standard Model. Specifically, why is the number of fermions a power of 2? There are 16 fermions, and 16 antifermions, which makes the Standard Model rep have dimension

$$\dim(F \oplus F^*) = 2^5 = 32.$$

With the binary code interpretation, it could not be any other way.

In actuality, however, the existence of a right-handed neutrino (or its antiparticle, the left-handed antineutrino) has been controversial. Because it transforms trivially in the Standard Model, it does not interact with anything except perhaps the Higgs.

The right-handed neutrino certainly improves the aesthetics of the SU(5) theory. When we include this particle (and its antiparticle), we obtain the rep

$$\Lambda^0\mathbb{C}^5 \oplus \Lambda^1\mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5 \oplus \Lambda^3\mathbb{C}^5 \oplus \Lambda^4\mathbb{C}^5 \oplus \Lambda^5\mathbb{C}^5$$

which is all of $\Lambda\mathbb{C}^5$, whereas without this particle we would just have

$$\Lambda^1\mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5 \oplus \Lambda^3\mathbb{C}^5 \oplus \Lambda^4\mathbb{C}^5$$

which is much less appealing—it *wants* to be $\Lambda\mathbb{C}^5$, but it comes up short.

More importantly, there is increasing indirect evidence from experimental particle physics that right-handed neutrinos *do* exist. For details, see Pati [27]. If this is true, the number of fermions really could be 16, and we have a ready-made explanation for that number in the binary code.

However, this creates a new mystery. The SU(5) works nicely with the representation $\Lambda\mathbb{C}^5$, but SU(5) does not *require* this. It works just fine with the smaller rep

$$\Lambda^1\mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5 \oplus \Lambda^3\mathbb{C}^5 \oplus \Lambda^4\mathbb{C}^5.$$

It would be nicer to have a theory that *required* us to use all of $\Lambda\mathbb{C}^5$. Better yet, if our new GUT were an *extension* of SU(5), the beautiful explanation of hypercharges would live on in our new theory. With luck, we might even get away with using the same underlying vector space, $\Lambda\mathbb{C}^5$. Could it be that the SU(5) GUT is only the beginning of the story? Could unification go on, with a grand unified theory that extends SU(5) just as SU(5) extended the Standard Model?

Let us look for a group that extends SU(5) and has an irrep whose dimension is some power of 2. The dimension is a big clue. What representations have dimensions that are powers of 2? *Spinors.*

What are spinors? They are certain representations of $\mathrm{Spin}(n)$, the double cover of the rotation group in n dimensions, which do not factor through the quotient $\mathrm{SO}(n)$. Their dimensions are always a power of two. We build them by exhibiting $\mathrm{Spin}(n)$ as a subgroup of a Clifford algebra. Recall that the **Clifford algebra** Cliff_n is the associative algebra freely generated by $\mathbb{R}^n$ with relations

$$vw + wv = -2\langle v, w\rangle.$$

If we take products of pairs of unit vectors in $\mathbb{R}^n$ inside this algebra, these generate the group $\mathrm{Spin}(n)$: multiplication in this group coincides with multiplication in the Clifford algebra. Using this fact, we can get representations of $\mathrm{Spin}(2n)$ from modules of Cliff_n.

We can use this method to get a rep of Spin(10) on $\Lambda\mathbb{C}^5$ that extends the rep of SU(5) on this space. In fact, quite generally Cliff_{2n} acts on $\Lambda\mathbb{C}^n$. Then, because

$$\mathrm{Spin}(2n) \hookrightarrow \mathrm{Cliff}_{2n},$$

$\Lambda\mathbb{C}^n$ becomes a representation of $\mathrm{Spin}(2n)$, called the **Dirac spinor representation**.

To see this, we use operators on $\Lambda\mathbb{C}^n$ called 'creation and annihilation operators'. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{C}^n$. Each of these gives a **creation operator**:

$$\begin{array}{rcl} a_j^*\colon \Lambda\mathbb{C}^n & \to & \Lambda\mathbb{C}^n \\ \psi & \mapsto & e_j \wedge \psi. \end{array}$$

We use the notation a_j^* because $\Lambda\mathbb{C}^n$ is a Hilbert space, so a_j^* is the adjoint of some other operator

$$a_j \colon \Lambda\mathbb{C}^n \to \Lambda\mathbb{C}^n,$$

which is called an **annihilation operator**.

In physics, we can think of the basis vectors e_j as particles. For example, in the binary code approach to the SU(5) theory we imagine five particles from which the observed particles in the Standard Model are composed: *up, down, red, green* and *blue*. Taking the wedge product with e_j 'creates a particle' of type j, while the adjoint 'annihilates a particle' of type j.

It may seem odd that creation is the adjoint of annihilation, rather than its inverse. One reason for this is that the creation operator, a_j^*, has no inverse. In some sense, its adjoint a_j is the best substitute.

This adjoint *does* do what want, which is to delete any particle of type j. Explicitly, it deletes the 'first' occurence of e_j from any basis element, bringing out any minus signs we need to make this respect the antisymmetry of the wedge product:

$$a_j(e_{i_1} \wedge \cdots \wedge e_{i_p}) = (-1)^{k+1} e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \cdots \wedge e_{i_p}, \text{ if } j = i_k.$$

And if no particle of type j appears, we get zero.

Now, whenever we have an inner product space like $\mathbb{C}^n$, we get an inner product on $\Lambda\mathbb{C}^n$. The fastest, if not most elegant, route to this inner product is to remember that, given an orthonormal basis $e_1, \ldots, e_n$ for $\mathbb{C}^n$, the induced basis, consisting of elements of the form $e_{i_1} \wedge \cdots \wedge e_{i_p}$, should be orthonormal in $\Lambda\mathbb{C}^n$. But choosing an orthonormal basis defines an inner product, and in this case it defines an inner product on the whole exterior algebra, one that reduces to the usual one for the grade one elements, $\Lambda^1\mathbb{C}^n \cong \mathbb{C}^n$.

It is with respect to this inner product on $\Lambda\mathbb{C}^n$ that a_j and a_j^* are adjoint. That is, they satisfy

$$\langle v, a_j w\rangle = \langle a_j^* v, w\rangle$$

for any elements $v, w \in \Lambda\mathbb{C}^n$. Showing this from the definitions we have given is a straightforward calculation, which we leave to the reader.

These operators satisfy the following relations:

$$\begin{aligned} \{a_j, a_k\} &= 0 \\ \{a_j^*, a_k^*\} &= 0 \\ \{a_j, a_k^*\} &= \delta_{jk} \end{aligned}$$

where curly brackets denote the **anticommutator** of two linear operators, namely $\{a, b\} = ab + ba$.

As an algebra, Cliff_{2n} is generated by the standard basis vectors of $\mathbb{R}^{2n}$. Let us call the elements of Cliff_{2n} corresponding to these basis vectors $\gamma_1, \ldots, \gamma_{2n}$. From the definition of the Clifford algebra, is easy to check that

$$\{\gamma_k, \gamma_\ell\} = -2\delta_{k\ell}.$$

In other words, the elements γ_k are anticommuting square roots of -1. So, we can turn $\Lambda\mathbb{C}^n$ into a Cliff_{2n}-module by finding $2n$ linear operators on $\Lambda\mathbb{C}^n$ that anticommute and square to -1. We build these from the raw material provided by a_j and a_j^*. Indeed, it is easy to see that

$$\begin{aligned} \phi_j &= i(a_j + a_j^*) \\ \pi_j &= a_j - a_j^* \end{aligned}$$

do the trick. Now we can map $\gamma_1, \ldots, \gamma_{2n}$ to these operators, in any order, and $\Lambda\mathbb{C}^n$ becomes a Cliff_{2n}-module as promised.

Now for $n > 1$ we may define $\mathrm{Spin}(2n)$ to be the universal cover of $\mathrm{SO}(2n)$, with group structure making the covering map

$$\begin{array}{c} \mathrm{Spin}(2n) \\ \big\downarrow {\scriptstyle p} \\ \mathrm{SO}(2n) \end{array}$$

into a homomorphism. This universal cover is a double cover, because the fundamental group of $\mathrm{SO}(2n)$ is $\mathbb{Z}_2$ for n in this range.

This construction of $\mathrm{Spin}(2n)$ is fairly abstract. Luckily, we can realize $\mathrm{Spin}(2n)$ as the multiplicative group in Cliff_{2n} generated by products of pairs of unit vectors. This gives us the inclusion

$$\mathrm{Spin}(2n) \hookrightarrow \mathrm{Cliff}_{2n}$$

we need to make $\Lambda\mathbb{C}^n$ into a representation of $\mathrm{Spin}(2n)$. From this, one can show that the Lie algebra $\mathfrak{so}(2n)$ is generated by the commutators of the γ_j. Because we know how to map each γ_j to an operator on $\Lambda\mathbb{C}^n$, this gives us an explicit formula for the action of $\mathfrak{so}(2n)$ on $\Lambda\mathbb{C}^n$. Each γ_j changes the parity of the grades, and their commutators do this twice, restoring grade parity. Thus, $\mathfrak{so}(2n)$ preserves the parity of the grading on $\Lambda\mathbb{C}^n$, and $\mathrm{Spin}(2n)$ does the same. This breaks $\Lambda\mathbb{C}^n$ into two subrepresentations:

$$\Lambda\mathbb{C}^n = \Lambda^{\mathrm{ev}}\mathbb{C}^n \oplus \Lambda^{\mathrm{odd}}\mathbb{C}^n$$

where $\Lambda^{\mathrm{ev}}\mathbb{C}^n$ is the direct sum of the even-graded parts:

$$\Lambda^{\mathrm{ev}}\mathbb{C}^n = \Lambda^0\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n \oplus \cdots$$

while $\Lambda^{\mathrm{odd}}\mathbb{C}^n$ is the sum of the odd-graded parts:

$$\Lambda^{\mathrm{odd}}\mathbb{C}^n = \Lambda^1\mathbb{C}^n \oplus \Lambda^3\mathbb{C}^n \oplus \cdots.$$

In fact, both these representations of $\mathrm{Spin}(2n)$ are irreducible, and $\mathrm{Spin}(2n)$ acts faithfully on their direct sum $\Lambda\mathbb{C}^n$. Elements of these two irreps of $\mathrm{Spin}(2n)$ are called **left- and right-handed Weyl spinors**, respectively, while elements of $\Lambda\mathbb{C}^n$ are called **Dirac spinors**.

All this works for any n, but we are especially interested in the case $n = 5$. The big question is: does the Dirac spinor representation of $\mathrm{Spin}(10)$ extend the obvious representation of $\mathrm{SU}(5)$ on $\Lambda\mathbb{C}^5$? Or, more generally, does the Dirac spinor representation of $\mathrm{Spin}(2n)$ extend the representation of $\mathrm{SU}(n)$ on $\Lambda\mathbb{C}^n$?

Remember, we can think of a unitary representation as a group homomorphism

$$G \to \mathrm{U}(V)$$

where V is the Hilbert space on which G acts as unitary operators. Here we are concerned with two representations. One of them is the familiar representation of $\mathrm{SU}(n)$ on $\Lambda\mathbb{C}^n$:

$$\rho\colon \mathrm{SU}(n) \to \mathrm{U}(\Lambda\mathbb{C}^n),$$

which acts as the fundamental rep on $\Lambda^1\mathbb{C}^n \cong \mathbb{C}^n$ and respects wedge products. The other is the representation of $\mathrm{Spin}(2n)$ on the Dirac spinors, which happen to form the same vector space $\Lambda\mathbb{C}^n$:

$$\rho'\colon \mathrm{Spin}(2n) \to \mathrm{U}(\Lambda\mathbb{C}^n).$$

Our big question is answered affirmatively by this theorem, which can be found in a classic paper by Atiyah, Bott and Shapiro [2]:

Theorem 2. *There exists a Lie group homomorphism ψ that makes this triangle commute:*

$$\begin{tikzcd} \mathrm{SU}(n) \arrow[r, "\psi"] \arrow[dr, "\rho"'] & \mathrm{Spin}(2n) \arrow[d, "\rho'"] \\ & \mathrm{U}(\Lambda\mathbb{C}^n) \end{tikzcd}$$

Proof. The complex vector space $\mathbb{C}^n$ has an underlying real vector space of dimension $2n$, and the real part of the usual inner product on $\mathbb{C}^n$ gives an inner product on this underlying real vector space, so we have an inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{O}(2n)$. The connected component of the identity in $\mathrm{O}(2n)$ is $\mathrm{SO}(2n)$, and $\mathrm{U}(n)$ is connected, so this gives an inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n)$ and thus $\mathrm{SU}(n) \hookrightarrow \mathrm{SO}(2n)$. Passing to Lie algebras, we obtain an inclusion $\mathfrak{su}(n) \hookrightarrow \mathfrak{so}(2n)$. A homomorphism of Lie algebras gives a homomorphism of the corresponding simply-connected Lie groups, so this in turn gives the desired map $\psi\colon \mathrm{SU}(n) \to \mathrm{Spin}(2n)$.

Next we must check that ψ makes the above triangle commute. Since all the groups involved are connected, it suffices to check that this diagram

$$\begin{tikzcd} \mathfrak{su}(n) \arrow[r, "d\psi"] \arrow[dr, "d\rho"'] & \mathfrak{so}(2n) \arrow[d, "d\rho'"] \\ & \mathfrak{u}(\Lambda\mathbb{C}^n) \end{tikzcd}$$

commutes. Since the Dirac representation $d\rho'$ is defined in terms of creation and annihilation operators, we should try to express $d\rho$ this way. To do so, we will need a good basis for $\mathfrak{su}(n)$. Remember,

$$\mathfrak{su}(n) = \{n \times n \text{ traceless skew-adjoint matrices over } \mathbb{C}\}.$$

If E_{jk} denotes the matrix with 1 in the jkth entry and 0 everywhere else, then the traceless skew-adjoint matrices have this basis:

$$\begin{array}{ll} E_{jk} - E_{kj} & j > k \\ i(E_{jk} + E_{kj}) & j > k \\ i(E_{jj} - E_{j+1,j+1}) & j = 1, \ldots, n-1. \end{array}$$

For example, $\mathfrak{su}(2)$ has the basis

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and our basis for $\mathfrak{su}(n)$ simply generalizes this.

Now, it is easy to guess a formula for $d\rho$ in terms of creation and annihilation operators. After all, the elementary matrix E_{jk} satisfies

$$E_{jk}(e_\ell) = \begin{cases} e_j & \text{if } \ell = k \\ 0 & \text{if } \ell \neq k \end{cases}$$

and $a_j^* a_k$ acts the same way on $\Lambda^1\mathbb{C}^n \subseteq \Lambda\mathbb{C}^n$. So, we certainly have

$$\begin{array}{rcl} d\rho\,(E_{jk} - E_{kj}) & = & a_j^* a_k - a_k^* a_j \\ d\rho\,(i(E_{jk} + E_{kj})) & = & i(a_j^* a_k + a_k^* a_j) \\ d\rho\,(i(E_{jj} - E_{j+1,j+1})) & = & i(a_j^* a_j - a_{j+1}^* a_{j+1}) \end{array}$$

on the subspace $\Lambda^1\mathbb{C}^n$. But do these operators agree on the rest of $\Lambda\mathbb{C}^n$? Remember, ρ preserves wedge products:

$$\rho(x)(v \wedge w) = \rho(x)v \wedge \rho(x)w$$

for all $x \in \mathrm{SU}(n)$. Differentiating this condition, we see that $\mathfrak{su}(n)$ must act as derivations:

$$d\rho(X)(v \wedge w) = d\rho(X)v \wedge w + v \wedge d\rho(X)w$$

for all $X \in \mathfrak{su}(n)$. Derivations of $\Lambda\mathbb{C}^n$ are determined by their action on $\Lambda^1\mathbb{C}^n$. So, $d\rho$ will be given on all $\Lambda\mathbb{C}^n$ by the above formulas if we can show that

$$a_j^* a_k - a_k^* a_j, \quad i(a_j^* a_k + a_k^* a_j), \quad \text{and} \quad i(a_j^* a_j - a_{j+1}^* a_{j+1})$$

are derivations.

Now, the annihilation operators are a lot like derivations: they are **antiderivations**. That is, if $v \in \Lambda^p\mathbb{C}^n$ and $w \in \Lambda^q\mathbb{C}^n$, then

$$a_j(v \wedge w) = a_j v \wedge w + (-1)^p v \wedge a_j w.$$

However, the creation operators are nothing like derivations. They satisfy

$$a_j^*(v \wedge w) = a_j^* v \wedge w = (-1)^p v \wedge a_j^* w,$$

because a_j^* acts by wedging with e_j, and moving this through v introduces p minus signs. Luckily, this relation combines with the previous one to make the composites $a_j^* a_k$ into derivations for every j and k. We leave this for the reader to check.

So, $d\rho$ can really be expressed in terms of annihilation and creation operators as above. Checking that

$$\begin{tikzcd} \mathfrak{su}(n) \arrow[r, "d\psi"] \arrow[dr, "d\rho"'] & \mathfrak{so}(2n) \arrow[d, "d\rho'"] \\ & \mathfrak{u}(\Lambda\mathbb{C}^n) \end{tikzcd}$$

commutes is now a straightforward but somewhat tedious job, which we leave to the dedicated reader. □

This theorem had a counterpart for the SU(5) GUT—namely, Theorem 1. There we saw a homomorphism ϕ that showed us how to extend the Standard Model group G_{SM} to SU(5), and made this square commute:

$$\begin{tikzcd} G_{\mathrm{SM}} \arrow[r, "\phi"] \arrow[d] & \mathrm{SU}(5) \arrow[d] \\ \mathrm{U}(F \oplus F^*) \arrow[r, "\mathrm{U}(f)"] & \mathrm{U}(\Lambda\mathbb{C}^5) \end{tikzcd}$$

Now ψ says how to extend SU(5) further to Spin(10), and makes this square commute:

$$\begin{tikzcd} \mathrm{SU}(5) \arrow[r, "\psi"] \arrow[d, "\rho"'] & \mathrm{Spin}(10) \arrow[d, "\rho'"] \\ \mathrm{U}(\Lambda\mathbb{C}^5) \arrow[r, "1"] & \mathrm{U}(\Lambda\mathbb{C}^5) \end{tikzcd}$$

We can put these squares together to get this commutative diagram:

$$\begin{tikzcd} G_{\mathrm{SM}} \arrow[r, "\psi\phi"] \arrow[d] & \mathrm{Spin}(10) \arrow[d] \\ \mathrm{U}(F \oplus F^*) \arrow[r, "\mathrm{U}(f)"] & \mathrm{U}(\Lambda\mathbb{C}^5) \end{tikzcd}$$

This diagram simply says that Spin(10) is a GUT: it extends the Standard Model group G_{SM} in a way that is compatible with the Standard Model representation, $F \oplus F^*$. In Section 3.1, all the hard work lay in showing the representations $F \oplus F^*$ and $\Lambda\mathbb{C}^5$ of G_{SM} were the same. Here, we do not have to do that. We just showed that Spin(10) extends SU(5). Since SU(5) already extended G_{SM}, Spin(10) extends that, too.

3.3 The Pati–Salam Model

Next we turn to a unified theory that is not so 'grand': its gauge group is not a simple Lie group, as it was for the SU(5) and Spin(10) theories. This theory is called the Pati–Salam model, after its inventors [28]; it has gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(4)$, which is merely semisimple.

We might imagine the SU(5) theory as an answer to this question:

> Why are the hypercharges in the Standard Model what they are?

The answer it provides is something like this:

> Because SU(5) is the actual gauge group of the world, acting on the representation $\Lambda\mathbb{C}^5$.

But there are other intriguing patterns in the Standard Model that SU(5) does *not* explain—and these lead us in different directions.

First, there is a strange similarity between quarks and leptons. Each generation of fermions in the Standard Model has two quarks and two leptons. For example, in the first generation we have the quarks u and d, and the leptons ν and e^-. The quarks come in three 'colors': this is a picturesque way of saying that they transform in the fundamental representation of SU(3) on $\mathbb{C}^3$. The leptons, on the other hand, are 'white': they transform in the trivial representation of SU(3) on $\mathbb{C}$.

Representations of SU(3)	
Particle	Representation
Quark	$\mathbb{C}^3$
Lepton	$\mathbb{C}$

Could the lepton secretly be a fourth color of quark? Maybe it could in a theory where the SU(3) color symmetry of the Standard Model is extended to SU(4).

Of course this larger symmetry would need to be broken to explain the very real *difference* between leptons and quarks.

Second, there is a strange difference between left- and right-handed fermions. The left-handed ones participate in the weak force governed by SU(2), while the right-handed ones do not. Mathematically speaking, the left-handed ones live in a nontrivial representation of SU(2), while the right-handed ones live in a trivial one. The nontrivial one is $\mathbb{C}^2$, while the trivial one is $\mathbb{C} \oplus \mathbb{C}$:

Representations of SU(2)	
Particle	Representation
Left-handed fermion	$\mathbb{C}^2$
Right-handed fermion	$\mathbb{C} \oplus \mathbb{C}$

But there is a suspicious similarity between $\mathbb{C}^2$ and $\mathbb{C} \oplus \mathbb{C}$. Could there be another copy of SU(2) that acts on the right-handed particles? Again, this 'right-handed' SU(2) would need to be broken, to explain why we do not see a 'right-handed' version of the weak force that acts on right-handed particles.

Following Pati and Salam, let us try to sculpt a theory that makes these ideas precise. In the last two sections, we saw some of the ingredients we need to make a grand unified theory: we need to extend the symmetry group G_{SM} to a larger group G using an inclusion

$$G_{\mathrm{SM}} \hookrightarrow G$$

(up to some discrete kernel), and we need a representation V of G which reduces to the Standard Model representation when restricted to G_{SM}:

$$F \oplus F^* \cong V.$$

We can put all these ingredients together into a diagram

$$\begin{tikzcd} G_{\mathrm{SM}} \arrow[r] \arrow[d] & G \arrow[d] \\ \mathrm{U}(F \oplus F^*) \arrow[r, "\sim"] & \mathrm{U}(V) \end{tikzcd}$$

which commutes only when our G theory works out.

We now use the same methods to chip away at our current challenge. We asked if leptons correspond to a fourth color. We already know that every quark comes in three colors, r, g, and b, which form a basis for the vector space $\mathbb{C}^3$. This is the fundamental representation of SU(3), the color symmetry group of the Standard Model. If leptons correspond to a fourth color, say 'white', then we should use the colors r, g, b and w, as a basis for the vector space $\mathbb{C}^4$. This is the fundamental representation of SU(4), so let us take that group to describe color symmetries in our new GUT.

Now SU(3) has an obvious inclusion into SU(4), using block diagonal matrices:

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

When restricted to this subgroup, the fundamental representation $\mathbb{C}^4$ breaks into a direct sum of irreps:

$$\mathbb{C}^4 \cong \mathbb{C}^3 \oplus \mathbb{C}.$$

These are precisely the irreps of SU(3) that describe quarks and leptons. For antiquarks and antileptons we can use

$$\mathbb{C}^{4*} \cong \mathbb{C}^{3*} \oplus \mathbb{C}.$$

It looks like we are on the right track.

We can do even better if we *start* with the splitting

$$\mathbb{C}^4 \cong \mathbb{C}^3 \oplus \mathbb{C}.$$

Remember that when we studied SU(5), the splitting

$$\mathbb{C}^5 \cong \mathbb{C}^2 \oplus \mathbb{C}^3$$

had the remarkable effect of introducing U(1), and thus hypercharge, into SU(5) theory. This was because the subgroup of SU(5) that preserves this splitting is larger than $\mathrm{SU}(2) \times \mathrm{SU}(3)$, roughly by a factor of U(1):

$$(\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3))/\mathbb{Z}_6 \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

It was this factor of U(1) that made SU(5) theory so fruitful.

So, if we choose a splitting $\mathbb{C}^4 \cong \mathbb{C}^3 \oplus \mathbb{C}$, we should again look at the subgroup that preserves this splitting. Namely:

$$\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(1)) \subseteq \mathrm{SU}(4).$$

Just as in the SU(5) case, this group is bigger than $\mathrm{SU}(3) \times \mathrm{SU}(1)$, roughly by a factor of U(1). And again, this factor of U(1) is related to hypercharge!

This works very much as it did for SU(5). We want a map

$$\mathrm{U}(1) \times \mathrm{SU}(3) \to \mathrm{SU}(4)$$

and we already have one that works for the SU(3) part:

$$\begin{array}{ccc} \mathrm{SU}(3) & \to & \mathrm{SU}(4) \\ h & \mapsto & \left(\begin{array}{cc} h & 0 \\ 0 & 1 \end{array} \right) \end{array}$$

So, we just need to include a factor of U(1) that commutes with everything in the SU(3) subgroup. Elements of SU(4) that do this are of the form

$$\left(\begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array} \right)$$

where α stands for the 3×3 identity matrix times the complex number $\alpha \in \mathrm{U}(1)$, and similarly for β in the 1×1 block. For the above matrix to lie in SU(4), it must have determinant 1, so $\alpha^3\beta = 1$. Thus we must include U(1) using matrices of this form:

$$\left(\begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{-3} \end{array} \right).$$

This gives our map:

$$\begin{array}{ccc} \mathrm{U}(1) \times \mathrm{SU}(3) & \to & \mathrm{SU}(4) \\ (\alpha, h) & \mapsto & \left(\begin{array}{cc} \alpha h & 0 \\ 0 & \alpha^{-3} \end{array} \right). \end{array}$$

If we let $\mathrm{U}(1) \times \mathrm{SU}(3)$ act on $\mathbb{C}^4 \cong \mathbb{C}^3 \oplus \mathbb{C}$ via this map, the 'quark part' $\mathbb{C}^3$ transforms as though it has hypercharge $\frac{1}{3}$: that is, it gets multiplied by a factor of α. Meanwhile, the 'lepton part' $\mathbb{C}$ transforms as though it has hypercharge -1, getting multiplied by a factor of α^{-3}. So, as a representation of $\mathrm{U}(1) \times \mathrm{SU}(3)$, we have

$$\mathbb{C}^4 \quad \cong \quad \mathbb{C}_{\frac{1}{3}} \otimes \mathbb{C}^3 \quad \oplus \quad \mathbb{C}_{-1} \otimes \mathbb{C}.$$

A peek at Table 1 reveals something nice. This exactly how the left-handed quarks and leptons in the Standard Model transform under $\mathrm{U}(1) \times \mathrm{SU}(3)$!

The right-handed leptons do not work this way. That is a problem we need to address. But this brings us to our second question, which was about the strange difference between left- and right-handed particles.

Remember that in the Standard Model, the left-handed particles live in the fundamental rep of $\mathrm{SU}(2)$ on $\mathbb{C}^2$, while the right-handed ones live in the trivial rep on $\mathbb{C} \oplus \mathbb{C}$. Physicists write this by grouping left-handed particles into 'doublets', while leaving the right-handed particles as 'singlets':

$$\begin{pmatrix} \nu_L \\ e^-_L \end{pmatrix} \qquad \begin{matrix} \nu_R \\ e^-_R \end{matrix} \; .$$

But there is a suspicious similarity between $\mathbb{C}^2$ and $\mathbb{C} \oplus \mathbb{C}$. Could there be another copy of $\mathrm{SU}(2)$ that acts on the right-handed particles? Physically speaking, this would mean that the left- and right-handed particles both form doublets:

$$\begin{pmatrix} \nu_L \\ e^-_L \end{pmatrix} \qquad \begin{pmatrix} \nu_R \\ e^-_R \end{pmatrix}$$

but under the actions of different $\mathrm{SU}(2)$'s. Mathematically, this would amount to extending the representations of the 'left-handed' $\mathrm{SU}(2)$:

$$\mathbb{C}^2 \qquad \mathbb{C} \oplus \mathbb{C}$$

to representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$:

$$\mathbb{C}^2 \otimes \mathbb{C} \qquad \mathbb{C} \otimes \mathbb{C}^2$$

where the first copy of $\mathrm{SU}(2)$ acts on the first factor in these tensor products, while the second copy acts on the second factor. The first copy of $\mathrm{SU}(2)$ is the 'left-handed' one familiar from the Standard Model. The second copy is a new 'right-handed' one.

If we restrict these representations to the 'left-handed' $\mathrm{SU}(2)$ subgroup, we obtain:

$$\begin{aligned} \mathbb{C}^2 \otimes \mathbb{C} &\cong \mathbb{C}^2 \\ \mathbb{C} \otimes \mathbb{C}^2 &\cong \mathbb{C} \oplus \mathbb{C}. \end{aligned}$$

These are exactly the representations of $\mathrm{SU}(2)$ that appear in the Standard Model. It looks like we are on the right track!

Now let us try to combine these ideas into a theory with symmetry group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$. We have seen that letting $\mathrm{SU}(4)$ act on $\mathbb{C}^4$ is a good way to unify our treatment of color for all the left-handed fermions. Similarly, the dual representation on $\mathbb{C}^{4*}$ is good for their antiparticles. So, we will tackle color by letting $\mathrm{SU}(4)$ act on the direct sum $\mathbb{C}^4 \oplus \mathbb{C}^{4*}$. This space is 8-dimensional. We have also seen that letting $\mathrm{SU}(2) \times \mathrm{SU}(2)$ act on $\mathbb{C}^2 \otimes \mathbb{C} \oplus \mathbb{C} \otimes \mathbb{C}^2$ is a good way to unify our treatment of isospin for left- and right-handed fermions. This space is 4-dimensional.

Since $8 \times 4 = 32$, and the Standard Model representation is 32-dimensional, let us take the tensor product

$$V = \left((\mathbb{C}^2 \otimes \mathbb{C}) \;\oplus\; (\mathbb{C} \otimes \mathbb{C}^2)\right) \;\otimes\; \left(\mathbb{C}^4 \;\oplus\; \mathbb{C}^{4*}\right).$$

This becomes a representation of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$, which we call the **Pati–Salam representation**. To obtain a theory that extends the Standard Model, we also need a way to map G_{SM} to $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$, such that pulling back V to a representation of G_{SM} gives the Standard model representation.

How can we map G_{SM} to $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$? There are several possibilities. Our work so far suggests this option:

$$\begin{array}{ccc} \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) & \to & \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \\ (\alpha, g, h) & \mapsto & \left(g,\ 1,\ \begin{pmatrix} \alpha h & 0 \\ 0 & \alpha^{-3} \end{pmatrix}\right) \end{array}$$

Let us see what this gives. The Pati–Salam representation of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ is a direct sum of four irreducibles:

$$V \quad \cong \quad \mathbb{C}^2 \otimes \mathbb{C} \otimes \mathbb{C}^4 \quad \oplus \quad \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^4 \quad \oplus \quad \mathbb{C}^2 \otimes \mathbb{C} \otimes \mathbb{C}^{4*} \quad \oplus \quad \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{4*}.$$

We hope the first two will describe left- and right-handed fermions, so let us give them names that suggest this:

$$F_L = \mathbb{C}^2 \otimes \mathbb{C} \otimes \mathbb{C}^4,$$

$$F_R = \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^4.$$

The other two are the duals of the first two, since the 2-dimensional irrep of $\mathrm{SU}(2)$ is its own dual:

$$F_L^* = \mathbb{C}^2 \otimes \mathbb{C} \otimes \mathbb{C}^{4*},$$
$$F_R^* = \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{4*}.$$

Given our chosen map from G_{SM} to $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$, we can work out which representations of the G_{SM} these four spaces give. For example, consider F_L. We have already seen that under our chosen map,

$$\mathbb{C}^4 \quad \cong \quad \mathbb{C}_{\frac{1}{3}} \otimes \mathbb{C}^3 \quad \oplus \quad \mathbb{C}_{-1} \otimes \mathbb{C}$$

as representations of $\mathrm{U}(1) \times \mathrm{SU}(3)$, while

$$\mathbb{C}^2 \otimes \mathbb{C} \cong \mathbb{C}^2$$

as representations of the left-handed $\mathrm{SU}(2)$. So, as representations of G_{SM} we have

$$F_L \quad \cong \quad \mathbb{C}_{\frac{1}{3}} \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \quad \oplus \quad \mathbb{C}_{-1} \otimes \mathbb{C}^2 \otimes \mathbb{C}.$$

Table 1 shows that these indeed match the left-handed fermions.

If we go ahead and do the other four cases, we see that everything works *except for the hypercharges of the right-handed particles*—and their antiparticles. Here we just show results for the particles:

The Pati–Salam Model — First Try		
Particle	Hypercharge: predicted	Hypercharge: actual
$\begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$	-1	-1
$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\frac{1}{3}$	$\frac{1}{3}$
ν_R	-1	0
e_R^-	-1	-2
u_R	$\frac{1}{3}$	$\frac{4}{3}$
d_R	$\frac{1}{3}$	$-\frac{2}{3}$

The problem is that the right-handed particles are getting the same hypercharges as their left-handed brethren. To fix this problem, we need a more clever map from G_{SM} to $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(4)$. This map must behave differently on the U(1) factor of G_{SM}, so the hypercharges come out differently. And it must take advantage of the right-handed copy of SU(2), which acts nontrivially only on the right-handed particles. For example, we can try this map:

$$\begin{array}{ccc} \mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3) & \to & \mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(4) \\ (\alpha, g, h) & \mapsto & \left(g, \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^{-k} \end{pmatrix}, \begin{pmatrix} \alpha h & 0 \\ 0 & \alpha^{-3} \end{pmatrix}\right) \end{array}$$

for any integer k. This will not affect the above table except for the hypercharges of right-handed particles. It will add $k/3$ to the hypercharges of the 'up' particles in right-handed doublets (ν_R and u_R), and subtract $k/3$ from the 'down' ones (e_R^- and d_R). So, we obtain these results:

The Pati–Salam Model — Second Try		
Particle	Hypercharge: predicted	Hypercharge: actual
$\begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}$	-1	-1
$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\frac{1}{3}$	$\frac{1}{3}$
ν_R	$-1+\frac{k}{3}$	0
e_R^-	$-1-\frac{k}{3}$	-2
u_R	$\frac{1}{3}+\frac{k}{3}$	$\frac{4}{3}$
d_R	$\frac{1}{3}-\frac{k}{3}$	$-\frac{2}{3}$

Miraculously, all the hypercharges match if we choose $k = 3$. So, let us use this map:

$$\begin{array}{rcl} \beta\colon \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) & \to & \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \\ (\alpha, g, h) & \mapsto & \left(g, \left(\begin{array}{cc} \alpha^3 & 0 \\ 0 & \alpha^{-3} \end{array} \right), \left(\begin{array}{cc} \alpha h & 0 \\ 0 & \alpha^{-3} \end{array} \right) \right). \end{array}$$

When we take the Pati–Salam representation of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ and pull it back along this map β, we obtain the Standard Model representation. As in Section 3.1, we use complete reducibility to see this, but we can be more concrete. We saw in Table 4 how we can specify the intertwining map precisely by using a specific basis, which for $\Lambda\mathbb{C}^5$ results in the binary code.

Similarly, we can create a kind of 'Pati–Salam code' to specify an isomorphism of Hilbert spaces

$$\ell\colon F \oplus F^* \to \big((\mathbb{C}^2 \otimes \mathbb{C}) \ \oplus \ (\mathbb{C} \otimes \mathbb{C}^2) \big) \ \otimes \ \big(\mathbb{C}^4 \ \oplus \ \mathbb{C}^{4*} \big),$$

and doing this provides a nice summary of the ideas behind in the Pati–Salam model. We take the space $\mathbb{C}^2 \otimes \mathbb{C}$ to be spanned by u_L and d_L, the **left-isospin up** and **left-isospin down** states. Similarly, the space $\mathbb{C} \otimes \mathbb{C}^2$ has basis u_R and d_R, called **right-isospin up** and **right-isospin down**. Take care not to confuse these with the similarly named quarks. These have no color, and only correspond to isospin.

The color comes from $\mathbb{C}^4$ of course, which we already decreed to be spanned by r, g, b and w. For antiparticles, we also require anticolors, which we take to do be the dual basis $\overline{r}$, $\overline{g}$, $\overline{b}$ and $\overline{w}$, spanning $\mathbb{C}^{4*}$.

It is now easy, with our knowledge of how the Pati–Salam model is to work, to construct this code. Naturally, the left-handed quark doublet corresponds to the left-isospin up and down states, which come in all three colors $c = r, g, b$:

$$u^c_L = u_L \otimes c \qquad d^c_L = d_L \otimes c.$$

The corresponding doublet of left-handed leptons is just the 'white' version of this:

$$\nu_L = u_L \otimes w \qquad e^-_L = d_L \otimes w.$$

The right-handed fermions are the same, but with R's instead of L's. Thus we get the Pati–Salam code for the fermions:

F_L	F_R
$\nu_L = u_L \otimes w$	$\nu_R = u_R \otimes w$
$e^-_L = d_L \otimes w$	$e^-_R = d_R \otimes w$
$u^c_L = u_L \otimes c$	$u^c_R = u_R \otimes c$
$d^c_L = d_L \otimes c$	$d^c_R = d_R \otimes c$

The result is very similar for the antifermions in F^*_L and F^*_R, but watch out: taking antiparticles *swaps up and down*, and also *swaps left and right*, so the particles in F^*_L are right-handed, despite the subscript L, while those in F^*_R are left-handed. This is because it is the right-handed antiparticles that feel the weak force, which in terms of representation theory means they are nontrivial under the left $\mathrm{SU}(2)$. So, the Pati–Salam code for the antifermions is this:

F_L^*	F_R^*
$e_R^+ = u_L \otimes \overline{w}$	$e_L^+ = u_R \otimes \overline{w}$
$\overline{\nu}_R = d_L \otimes \overline{w}$	$\overline{\nu}_L = d_R \otimes \overline{w}$
$\overline{d}_R^{\overline{c}} = u_L \otimes \overline{c}$	$\overline{d}_L^{\overline{c}} = u_R \otimes \overline{c}$
$\overline{u}_R^{\overline{c}} = d_L \otimes \overline{c}$	$\overline{u}_L^{\overline{c}} = d_R \otimes \overline{c}$

Putting these together we get the full Pati–Salam code:

The Pati–Salam Code			
F_L	F_R	F_L^*	F_R^*
$\nu_L = u_L \otimes w$	$\nu_R = u_R \otimes w$	$e_R^+ = u_L \otimes \overline{w}$	$e_L^+ = u_R \otimes \overline{w}$
$e_L^- = d_L \otimes w$	$e_R^- = d_R \otimes w$	$\overline{\nu}_R = d_L \otimes \overline{w}$	$\overline{\nu}_L = d_R \otimes \overline{w}$
$u_L^c = u_L \otimes c$	$u_R^c = u_R \otimes c$	$\overline{d}_R^{\overline{c}} = u_L \otimes \overline{c}$	$\overline{d}_L^{\overline{c}} = u_R \otimes \overline{c}$
$d_L^c = d_L \otimes c$	$d_R^c = d_R \otimes c$	$\overline{u}_R^{\overline{c}} = d_L \otimes \overline{c}$	$\overline{u}_L^{\overline{c}} = d_R \otimes \overline{c}$

Table 5: Pati–Salam code for first-generation fermions, where $c = r, g, b$ and $\overline{c} = \overline{r}, \overline{b}, \overline{g}$.

This table defines an isomorphism of Hilbert spaces

$$\ell\colon F \oplus F^* \to \big((\mathbb{C}^2 \otimes \mathbb{C}) \ \oplus\ (\mathbb{C} \otimes \mathbb{C}^2)\big) \ \otimes\ \big(\mathbb{C}^4 \ \oplus\ \mathbb{C}^{4*}\big)$$

so where it says, for example, $\nu_L = u_L \otimes w$, that is just short for $\ell(\nu_L) = u_L \otimes w$. This map ℓ is also an isomorphism between representations of G_{SM}. It tells us how these representations are the 'same', just as the map h did for $F \oplus F^*$ and $\Lambda\mathbb{C}^5$ at the end of Section 3.1.

As with SU(5) and Spin(10), we can summarize all the results of this section in a commutative square:

Theorem 3. *The following square commutes:*

$$\begin{tikzcd}
G_{\mathrm{SM}} \arrow[r, "\beta"] \arrow[d] & \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \arrow[d] \\
\mathrm{U}(F \oplus F^*) \arrow[r, "U(\ell)"] & \mathrm{U}\left(\big((\mathbb{C}^2 \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes \mathbb{C}^2)\big) \otimes \big(\mathbb{C}^4 \oplus \mathbb{C}^{4*}\big)\right)
\end{tikzcd}$$

where the left vertical arrow is the Standard Model representation and the right one is the Pati–Salam representation.

The Pati–Salam representation and especially the homomorphism β look less natural than the representation of SU(5) on $\Lambda\mathbb{C}^5$ and the homomorphism $\phi\colon G_{\mathrm{SM}} \to \mathrm{SU}(5)$. But appearances can be deceiving: in the next section we shall see a more elegant way to describe them.

3.4 The Route to Spin(10) via Pati–Salam

In the last section, we showed how the Pati–Salam model answers two questions about the Standard Model:

> Why are quarks and leptons so similar? Why are left and right so different?

We were able to describe leptons as a fourth color of quark, 'white', and treat right-handed and left-handed particles on a more equal footing. Neither of these ideas worked on its own, but together, they made a full-fledged extension of the Standard Model, much like SU(5) and Spin(10), but based on seemingly different principles.

Yet thinking of leptons as 'white' should be strangely familiar, not just from the Pati–Salam perspective, but from the binary code that underlies both the SU(5) and the Spin(10) theories. There, leptons were indeed white: they all have color $r \wedge g \wedge b \in \Lambda\mathbb{C}^5$.

Alas, while SU(5) hints that leptons might be a fourth color, it does not deliver on this. The quark colors

$$r, g, b \in \Lambda^1\mathbb{C}^5$$

lie in a different irrep of SU(5) than does $r \wedge g \wedge b \in \Lambda^3\mathbb{C}^5$. So, leptons in the SU(5) theory are white, but unlike the Pati–Salam model, this theory does not unify leptons with quarks.

Yet SU(5) theory is not the only game in town when it comes to the binary code. We also have Spin(10), which acts on the same vector space as SU(5). As a representation of Spin(10), $\Lambda\mathbb{C}^5$ breaks up into just two irreps: the even grades, $\Lambda^{\mathrm{ev}}\mathbb{C}^5$, which contain the left-handed particles and antiparticles:

$$\Lambda^{\mathrm{ev}}\mathbb{C}^5 \cong \langle \overline{\nu}_L \rangle \oplus \langle e^+_L \rangle \oplus \left\langle \begin{array}{c} u_L \\ d_L \end{array} \right\rangle \oplus \langle \overline{u}_L \rangle \oplus \left\langle \begin{array}{c} \nu_L \\ e^-_L \end{array} \right\rangle \oplus \langle \overline{d}_L \rangle$$

and the odd grades $\Lambda^{\mathrm{odd}}\mathbb{C}^5$, which contain the right handed particles and antiparticles:

$$\Lambda^{\mathrm{odd}}\mathbb{C}^5 \cong \langle \nu_R \rangle \oplus \langle e^-_R \rangle \oplus \left\langle \begin{array}{c} \overline{d}_R \\ \overline{u}_R \end{array} \right\rangle \oplus \langle u_R \rangle \oplus \left\langle \begin{array}{c} e^+_R \\ \overline{\nu}_R \end{array} \right\rangle \oplus \langle d_R \rangle.$$

Unlike SU(5), the Spin(10) GUT really does unify $r \wedge g \wedge b$ with the colors r, g and b, because they both live in the irrep $\Lambda^{\mathrm{odd}}\mathbb{C}^5$.

In short, it seems that the Spin(10) GUT, which we built as an extension of the SU(5) GUT, somehow managed to pick up this feature of the Pati–Salam model. How does Spin(10) relate to Pati–Salam's gauge group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$, exactly? In general, we only know there is a map $\mathrm{SU}(n) \to \mathrm{Spin}(2n)$, but in low dimensions, there is much more, because some groups coincide:

$$\begin{array}{rcl} \mathrm{Spin}(3) & \cong & \mathrm{SU}(2) \\ \mathrm{Spin}(4) & \cong & \mathrm{SU}(2) \times \mathrm{SU}(2) \\ \mathrm{Spin}(5) & \cong & \mathrm{Sp}(2) \\ \mathrm{Spin}(6) & \cong & \mathrm{SU}(4) \end{array}$$

What really stands out is this:

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \cong \mathrm{Spin}(4) \times \mathrm{Spin}(6).$$

This brings out an obvious relationship between the Pati–Salam model and the Spin(10) theory, because the inclusion $\mathrm{SO}(4) \times \mathrm{SO}(6) \hookrightarrow \mathrm{SO}(10)$ lifts to the universal covers, so we get a homomorphism

$$\eta \colon \mathrm{Spin}(4) \times \mathrm{Spin}(6) \to \mathrm{Spin}(10).$$

A word of caution is needed here. While η is the lift of an inclusion, it is not an inclusion itself: it is two-to-one. This is because the universal cover $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ of $\mathrm{SO}(4) \times \mathrm{SO}(6)$ is a four-fold cover, being a double cover on each factor.

So we can try to extend the symmetries $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ to $\mathrm{Spin}(10)$, though this can only work if the kernel of η acts trivially on the Pati–Salam representation. What is this representation like? There is an obvious representation of $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ that extends to a $\mathrm{Spin}(10)$ rep. Both $\mathrm{Spin}(4)$ and $\mathrm{Spin}(6)$ have Dirac spinor representations, so their product $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ has a representation on $\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$. And in fact, the obvious map

$$g \colon \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3 \to \Lambda\mathbb{C}^5$$

given by

$$v \otimes w \mapsto v \wedge w$$

is an isomorphism compatible with the actions of $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ on these two spaces. More concisely, this square:

$$\begin{tikzcd}
\mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[r, "\eta"] \arrow[d] & \mathrm{Spin}(10) \arrow[d] \\
\mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \arrow[r, "\mathrm{U}(g)"] & \mathrm{U}(\Lambda\mathbb{C}^5)
\end{tikzcd}$$

commutes.

We will prove this in a moment. First though, we must check that this representation of $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ is secretly just another name for the Pati–Salam representation of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ on the space we discussed in Section 3.3:

$$\big((\mathbb{C}^2 \otimes \mathbb{C}) \ \oplus\ (\mathbb{C} \otimes \mathbb{C}^2)\big) \ \otimes\ \big(\mathbb{C}^4 \ \oplus\ \mathbb{C}^{4*}\big).$$

Checking this involves choosing an isomorphism between $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ and $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$. Luckily, we can choose one that works:

Theorem 4. *There exists an isomorphism of Lie groups*

$$\alpha \colon \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \to \mathrm{Spin}(4) \times \mathrm{Spin}(6)$$

and a unitary operator

$$k \colon \big((\mathbb{C}^2 \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes \mathbb{C}^2)\big) \otimes \big(\mathbb{C}^4 \oplus \mathbb{C}^{4*}\big) \ \ \to\ \ \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$$

that make this square commute:

$$\begin{tikzcd}
\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[d] \\
\mathrm{U}\left(\big((\mathbb{C}^2 \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes \mathbb{C}^2)\big) \otimes \big(\mathbb{C}^4 \oplus \mathbb{C}^{4*}\big)\right) \arrow[r, "\mathrm{U}(k)"] & \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3)
\end{tikzcd}$$

where the left vertical arrow is the Pati–Salam representation and the right one is a tensor product of Dirac spinor representations.

Proof. We can prove this in pieces, by separately finding a unitary operator

$$(\mathbb{C}^2 \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes \mathbb{C}^2) \cong \Lambda\mathbb{C}^2$$

that makes this square commute:

$$\begin{CD} \mathrm{SU}(2)\times\mathrm{SU}(2) @>{\sim}>> \mathrm{Spin}(4) \\ @VVV @VVV \\ \mathrm{U}(\mathbb{C}^2\otimes\mathbb{C}\oplus\mathbb{C}\otimes\mathbb{C}^2) @>{\sim}>> \mathrm{U}(\Lambda\mathbb{C}^2) \end{CD}$$

and a unitary operator

$$\mathbb{C}^4\oplus\mathbb{C}^{4*}\cong\Lambda\mathbb{C}^3$$

that make this square:

$$\begin{CD} \mathrm{SU}(4) @>{\sim}>> \mathrm{Spin}(6) \\ @VVV @VVV \\ \mathrm{U}(\mathbb{C}^4\oplus\mathbb{C}^{4*}) @>{\sim}>> \mathrm{U}(\Lambda\mathbb{C}^3) \end{CD}$$

commute.

First, the Spin(6) piece. It suffices to show that the Dirac spinor rep of $\mathrm{Spin}(6)\cong\mathrm{SU}(4)$ on $\Lambda\mathbb{C}^3$ is isomorphic to $\mathbb{C}^4\oplus\mathbb{C}^{4*}$ as a rep of SU(4). We start with the action of Spin(6) on $\Lambda\mathbb{C}^3$. This breaks up into irreps:

$$\Lambda\mathbb{C}^3\cong\Lambda^{\mathrm{ev}}\mathbb{C}^3\oplus\Lambda^{\mathrm{odd}}\mathbb{C}^3$$

called the **left-handed** and **right-handed Weyl spinors**, and these are dual to each other because $6=2 \bmod 4$, by a theorem that can be found in Adams' lectures [1]. Call these representations

$$\rho_{\mathrm{ev}}\colon\mathrm{Spin}(6)\to\mathrm{U}(\Lambda^{\mathrm{ev}}\mathbb{C}^3),\qquad \rho_{\mathrm{odd}}\colon\mathrm{Spin}(6)\to\mathrm{U}(\Lambda^{\mathrm{odd}}\mathbb{C}^3).$$

Since these reps are dual, it suffices just to consider one of them, say ρ_{odd}.

Passing to Lie algebras, we have a homomorphism

$$d\rho_{\mathrm{odd}}\colon\mathfrak{so}(6)\to\mathfrak{u}(\Lambda^{\mathrm{odd}}\mathbb{C}^3)\cong\mathfrak{u}(4)\cong\mathfrak{u}(1)\oplus\mathfrak{su}(4).$$

Homomorphic images of semisimple Lie algebras are semisimple, so the image of $\mathfrak{so}(6)$ must lie entirely in $\mathfrak{su}(4)$. In fact $\mathfrak{so}(6)$ is simple, so this nontrivial map must be an injection

$$d\rho_{\mathrm{odd}}\colon\mathfrak{so}(6)\to\mathfrak{su}(4),$$

and because the dimension is 15 on both sides, this map is also onto. Thus $d\rho_{\mathrm{odd}}$ is an isomorphism of Lie algebras, so ρ_{odd} is an isomorphism of the simply connected Lie groups Spin(6) and SU(4):

$$\rho_{\mathrm{odd}}\colon\mathrm{Spin}(6)\to\mathrm{SU}(\Lambda^{\mathrm{odd}}\mathbb{C}^3)\cong\mathrm{SU}(4).$$

Furthermore, under this isomorphism

$$\Lambda^{\mathrm{odd}}\mathbb{C}^3\cong\mathbb{C}^4$$

as a representation of $\mathrm{SU}(\Lambda^{\mathrm{odd}}\mathbb{C}^3)\cong\mathrm{SU}(4)$. Taking duals, we obtain an isomorphism

$$\Lambda^{\mathrm{ev}}\mathbb{C}^3\cong\mathbb{C}^{4*}$$

Putting these together, we get an isomorphism $\mathbb{C}^4 \oplus \mathbb{C}^{4*} \cong \Lambda\mathbb{C}^3$ that makes this square commute:

$$\begin{tikzcd}
\mathrm{Spin}(6) \arrow[r, "\rho_{\mathrm{odd}}"] \arrow[d] & \mathrm{SU}(4) \arrow[d] \\
\mathrm{U}(\Lambda\mathbb{C}^3) \arrow[r] & \mathrm{U}(\mathbb{C}^4 \oplus \mathbb{C}^{4*})
\end{tikzcd}$$

which completes the proof for Spin(6).

Next, the Spin(4) piece. It suffices to show that the spinor rep $\Lambda\mathbb{C}^2$ of $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ is isomorphic to $\mathbb{C}^2 \otimes \mathbb{C} \ \oplus \ \mathbb{C} \otimes \mathbb{C}^2$ as a rep of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. We start with the action of Spin(4) on $\Lambda\mathbb{C}^2$. This again breaks up into irreps:

$$\Lambda\mathbb{C}^2 \cong \Lambda^{\mathrm{odd}}\mathbb{C}^2 \oplus \Lambda^{\mathrm{ev}}\mathbb{C}^2.$$

Again we call these representations

$$\rho_{\mathrm{ev}} \colon \mathrm{Spin}(4) \to \mathrm{U}(\Lambda^{\mathrm{ev}}\mathbb{C}^2), \qquad \rho_{\mathrm{odd}} \colon \mathrm{Spin}(4) \to \mathrm{U}(\Lambda^{\mathrm{odd}}\mathbb{C}^2).$$

First consider ρ_{ev}. Passing to Lie algebras, this gives a homomorphism

$$d\rho_{\mathrm{ev}} \colon \mathfrak{so}(4) \to \mathfrak{u}(\Lambda^{\mathrm{ev}}\mathbb{C}^2) \cong \mathfrak{u}(2) \cong \mathfrak{u}(1) \oplus \mathfrak{su}(2)$$

Homomorphic images of semisimple Lie algebras are semisimple, so the image of $\mathfrak{so}(4)$ must lie entirely in $\mathfrak{su}(2)$. Similarly, $d\rho_{\mathrm{odd}}$ also takes $\mathfrak{so}(4)$ to $\mathfrak{su}(2)$:

$$d\rho_{\mathrm{odd}} \colon \mathfrak{so}(4) \to \mathfrak{su}(2)$$

and we can combine these maps to get

$$d\rho_{\mathrm{odd}} \oplus d\rho_{\mathrm{ev}} \colon \mathfrak{so}(4) \to \mathfrak{su}(2) \oplus \mathfrak{su}(2),$$

which is just the derivative of Spin(4)'s representation on $\Lambda\mathbb{C}^2$. Since this representation is faithful, the map $d\rho_{\mathrm{odd}} \oplus d\rho_{\mathrm{ev}}$ of Lie algebras is injective. But the dimensions of $\mathfrak{so}(4)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ agree, so $d\rho_{\mathrm{odd}} \oplus d\rho_{\mathrm{ev}}$ is also onto. Thus it is an isomorphism of Lie algebras. This implies that $\rho_{\mathrm{odd}} \oplus \rho_{\mathrm{ev}}$ is an isomorphism of of the simply connected Lie groups Spin(4) and $\mathrm{SU}(2) \times \mathrm{SU}(2)$

$$\rho_{\mathrm{odd}} \oplus \rho_{\mathrm{ev}} \colon \mathrm{Spin}(4) \to \mathrm{SU}(\Lambda^{\mathrm{odd}}\mathbb{C}^2) \times \mathrm{SU}(\Lambda^{\mathrm{ev}}\mathbb{C}^2) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$$

under which $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on $\Lambda^{\mathrm{odd}}\mathbb{C}^2 \oplus \Lambda^{\mathrm{ev}}\mathbb{C}^2$. The left factor of SU(2) acts irreducibly on $\Lambda^{\mathrm{odd}}\mathbb{C}^2$, which the second factor is trivial on. Thus $\Lambda^{\mathrm{odd}}\mathbb{C}^2 \cong \mathbb{C}^2 \otimes \mathbb{C}$ as a rep of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Similarly, $\Lambda^{\mathrm{ev}}\mathbb{C}^2 \cong \mathbb{C} \otimes \mathbb{C}^2$ as a rep of this group. Putting these together, we get an isomorphism $\mathbb{C}^2 \otimes \mathbb{C} \oplus \mathbb{C} \otimes \mathbb{C}^2 \cong \Lambda\mathbb{C}^2$ that makes this square commute:

$$\begin{tikzcd}
\mathrm{Spin}(4) \arrow[r, "\rho_{\mathrm{ev}} \oplus \rho_{\mathrm{odd}}"] \arrow[d] & \mathrm{SU}(2) \times \mathrm{SU}(2) \arrow[d] \\
\mathrm{U}(\Lambda\mathbb{C}^2) \arrow[r] & \mathrm{U}(\mathbb{C}^2 \otimes \mathbb{C} \oplus \mathbb{C} \otimes \mathbb{C}^2)
\end{tikzcd}$$

which completes the proof for Spin(4). $\square$

In the proof of the preceding theorem, we merely showed that there *exists* an isomorphism

$$k \colon \big((\mathbb{C}^2 \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes \mathbb{C}^2)\big) \otimes \big(\mathbb{C}^4 \oplus \mathbb{C}^{4*}\big) \to \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$$

making the square commute. We did not say exactly what k was. In the proof, we built it after quietly choosing three unitary operators, giving these isomorphisms:

$$\mathbb{C}^2 \otimes \mathbb{C} \cong \Lambda^{\mathrm{odd}}\mathbb{C}^2, \quad \mathbb{C} \otimes \mathbb{C}^2 \cong \Lambda^{\mathrm{ev}}\mathbb{C}^2, \quad \mathbb{C}^4 \cong \Lambda^{\mathrm{odd}}\mathbb{C}^3.$$

Since the remaining map $\Lambda^{\mathrm{ev}}\mathbb{C}^3 \cong \mathbb{C}^{4*}$ is determined by duality, these three operators determine k, and they also determine the Lie group isomorphism α via the construction in our proof.

There is, however, a specific choice for these unitary operators that we prefer, because this choice makes the particles in the Pati–Salam representation $\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$ look almost exactly like those in the SU(5) representation $\Lambda\mathbb{C}^5$.

First, since $\mathbb{C}^2 = \Lambda^1\mathbb{C}^2 = \Lambda^{\mathrm{odd}}\mathbb{C}^2$ is spanned by u and d, the (left-handed) isospin states of the Standard Model, we really ought to identify the left-isospin states u_L and d_L of the Pati–Salam model with these. So, we should use this unitary operator:

$$\begin{array}{ccc} \mathbb{C}^2 \otimes \mathbb{C} & \xrightarrow{\sim} & \Lambda^{\mathrm{odd}}\mathbb{C}^2 \\ u_L & \mapsto & u \\ d_L & \mapsto & d. \end{array}$$

Next, we should use this unitary operator for right-isospin states:

$$\begin{array}{ccc} \mathbb{C} \otimes \mathbb{C}^2 & \xrightarrow{\sim} & \Lambda^{\mathrm{ev}}\mathbb{C}^2 \\ u_R & \mapsto & u \wedge d \\ d_R & \mapsto & 1. \end{array}$$

Why? Because the right-isospin up particle is the right-handed neutrino ν_R, which corresponds to $u \wedge d \wedge r \wedge g \wedge b$ in the SU(5) theory, but $u_R \otimes w$ in the Pati–Salam model. This suggests that $u \wedge d$ and u_R should be identified.

Finally, because $\mathbb{C}^3$ is spanned by the colors r, g and b, while $\mathbb{C}^4$ is spanned by the colors r, g, b and w, we really ought to use this unitary operator:

$$\begin{array}{ccc} \mathbb{C}^4 & \xrightarrow{\sim} & \Lambda^{\mathrm{odd}}\mathbb{C}^3 \\ r & \mapsto & r \\ g & \mapsto & g \\ b & \mapsto & b \\ w & \mapsto & r \wedge g \wedge b \end{array}$$

Dualizing this, we get the unitary operator

$$\begin{array}{ccc} \mathbb{C}^{4*} & \xrightarrow{\sim} & \Lambda^{\mathrm{ev}}\mathbb{C}^3 \\ \overline{r} & \mapsto & g \wedge b \\ \overline{g} & \mapsto & b \wedge r \\ \overline{b} & \mapsto & r \wedge g \\ \overline{w} & \mapsto & 1 \end{array}$$

These choices determine the unitary operator

$$k \colon \big((\mathbb{C}^2 \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes \mathbb{C}^2)\big) \otimes \big(\mathbb{C}^4 \oplus \mathbb{C}^{4*}\big) \to \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3.$$

With this specific choice of k, we can combine the commutative squares built in Theorems 3 and 4:

$$\begin{tikzcd} G_{\mathrm{SM}} \arrow[r, "\beta"] \arrow[d] & \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[d] \\ \mathrm{U}(F \oplus F^*) \arrow[r, "U(\ell)"] & \mathrm{U}\left(\big((\mathbb{C}^2 \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes \mathbb{C}^2)\big) \otimes \big(\mathbb{C}^4 \oplus \mathbb{C}^{4*}\big)\right) \arrow[r, "\mathrm{U}(k)"] & \mathrm{U}(\Lambda\mathbb{C}^5) \end{tikzcd}$$

to obtain the following result:

Theorem 5. *Taking $\theta = \alpha\beta$ and $h = k\ell$, the following square commutes:*

$$\begin{array}{ccc} G_{\mathrm{SM}} & \xrightarrow{\theta} & \mathrm{Spin}(4) \times \mathrm{Spin}(6) \\ \downarrow & & \downarrow \\ \mathrm{U}(F \oplus F^*) & \xrightarrow{\mathrm{U}(h)} & \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \end{array}$$

where the left vertical arrow is the Standard Model representation and the right one is a tensor product of Dirac spinor representations.

The map h, which tells us how to identify $F \oplus F^*$ and $\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$, is given by applying k to the 'Pati–Salam code' in Table 5. This gives a binary code for the Pati–Salam model:

The Binary Code for Pati–Salam			
$\Lambda^{\mathrm{odd}}\mathbb{C}^2 \otimes \Lambda^{\mathrm{odd}}\mathbb{C}^3$	$\Lambda^{\mathrm{ev}}\mathbb{C}^2 \otimes \Lambda^{\mathrm{odd}}\mathbb{C}^3$	$\Lambda^{\mathrm{odd}}\mathbb{C}^2 \otimes \Lambda^{\mathrm{ev}}\mathbb{C}^3$	$\Lambda^{\mathrm{ev}}\mathbb{C}^2 \otimes \Lambda^{\mathrm{ev}}\mathbb{C}^3$
$\nu_L = u \otimes rgb$	$\nu_R = ud \otimes rgb$	$e^+_R = u \otimes 1$	$e^+_L = ud \otimes 1$
$e^-_L = d \otimes rgb$	$e^-_R = 1 \otimes rgb$	$\overline{\nu}_R = d \otimes 1$	$\overline{\nu}_L = 1 \otimes 1$
$u^c_L = u \otimes c$	$u^c_R = ud \otimes c$	$\overline{d}^{\overline{c}}_R = u \otimes \overline{c}$	$\overline{d}^{\overline{c}}_L = ud \otimes \overline{c}$
$d^c_L = d \otimes c$	$d^c_R = 1 \otimes c$	$\overline{u}^{\overline{c}}_R = d \otimes \overline{c}$	$\overline{u}^{\overline{c}}_L = 1 \otimes \overline{c}$

Table 6: Pati–Salam binary code for first-generation fermions, where $c = r, g, b$ and $\overline{c} = gb, br, rg$.

We have omitted wedge product symbols to save space. Note that if we apply the obvious isomorphism

$$g \colon \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3 \to \Lambda\mathbb{C}^5$$

given by

$$v \otimes w \mapsto v \wedge w$$

then the above table does more than merely resemble Table 4, which gives the binary code for the SU(5) theory. The two tables become *identical!*

This fact is quite intriguing. We will explore its meaning in the next section. But first, let us start by relating the Pati–Salam model to the Spin(10) theory:

Theorem 6. *The following square commutes:*

$$\begin{array}{ccc} \mathrm{Spin}(4) \times \mathrm{Spin}(6) & \xrightarrow{\eta} & \mathrm{Spin}(10) \\ \downarrow & & \downarrow \\ \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) & \xrightarrow{\mathrm{U}(g)} & \mathrm{U}(\Lambda\mathbb{C}^5) \end{array}$$

where the right vertical arrow is the Dirac spinor representation, the left one is the tensor product of Dirac spinor representations, and

$$\eta \colon \mathrm{Spin}(4) \times \mathrm{Spin}(6) \to \mathrm{Spin}(10)$$

is the homomorphism lifting the inclusion of $\mathrm{SO}(4) \times \mathrm{SO}(6)$ *in* $\mathrm{SO}(10)$.

Proof. At the Lie algebra level, we have the inclusion

$$\mathfrak{so}(4) \oplus \mathfrak{so}(6) \hookrightarrow \mathfrak{so}(10)$$

by block diagonals, which is also just the differential of the inclusion $\mathrm{SO}(4) \times \mathrm{SO}(6) \hookrightarrow \mathrm{SO}(10)$ at the Lie group level. Given how the spinor reps are defined in terms of creation and annihilation operators, it is easy to see that

$$\begin{tikzcd} \mathfrak{so}(4) \oplus \mathfrak{so}(6) \arrow[r, hook] \arrow[d] & \mathfrak{so}(10) \arrow[d] \\ \mathfrak{gl}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \arrow[r, "\mathfrak{gl}(g)"] & \mathfrak{gl}(\Lambda\mathbb{C}^5) \end{tikzcd}$$

commutes, because g is an intertwining operator between representations of $\mathfrak{so}(4) \oplus \mathfrak{so}(6)$. That is because the $\mathfrak{so}(4)$ part only acts on $\Lambda\mathbb{C}^2$, while the $\mathfrak{so}(6)$ part only acts on $\Lambda\mathbb{C}^3$.

But these Lie algebras act by skew-adjoint operators, so really

$$\begin{tikzcd} \mathfrak{so}(4) \oplus \mathfrak{so}(6) \arrow[r, hook] \arrow[d] & \mathfrak{so}(10) \arrow[d] \\ \mathfrak{u}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \arrow[r, "\mathfrak{u}(g)"] & \mathfrak{u}(\Lambda\mathbb{C}^5) \end{tikzcd}$$

commutes. Since the $\mathfrak{so}(n)$'s and their direct sums are semisimple, so are their images. Therefore, their images live in the semisimple part of the unitary Lie algebras, which is just another way of saying the special unitary Lie algebras. We get that

$$\begin{tikzcd} \mathfrak{so}(4) \oplus \mathfrak{so}(6) \arrow[r, hook] \arrow[d] & \mathfrak{so}(10) \arrow[d] \\ \mathfrak{su}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \arrow[r, "\mathfrak{su}(g)"] & \mathfrak{su}(\Lambda\mathbb{C}^5) \end{tikzcd}$$

commutes, and this gives a commutative square in the world of simply connected Lie groups:

$$\begin{tikzcd} \mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[r, "\eta"] \arrow[d] & \mathrm{Spin}(10) \arrow[d] \\ \mathrm{SU}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \arrow[r, "\mathrm{SU}(g)"] & \mathrm{SU}(\Lambda\mathbb{C}^5) \end{tikzcd}$$

This completes the proof. □

This result shows us how to reach the Spin(10) theory, not through the SU(5) theory, but through the Pati–Salam model. For physics texts that treat this issue, see for example Zee [40] and Ross [31].

3.5 The Question of Compatibility

We now have two routes to the Spin(10) theory. In Section 3.2 we saw how to reach it via the SU(5) theory:

$$
\begin{tikzcd}
G_{\mathrm{SM}} \arrow[r, "\phi"] \arrow[d] & \mathrm{SU}(5) \arrow[r, "\psi"] \arrow[d] & \mathrm{Spin}(10) \arrow[d] \\
\mathrm{U}(F \oplus F^*) \arrow[r, "\mathrm{U}(f)"] & \mathrm{U}(\Lambda\mathbb{C}^5) \arrow[r, "1"] & \mathrm{U}(\Lambda\mathbb{C}^5)
\end{tikzcd}
$$

More $\overset{\rightsquigarrow}{\text{Unification}}$

Our work in that section and in Section 3.1 showed that this diagram commutes, which is a way of saying that the Spin(10) theory extends the Standard Model.

In Section 3.4 we saw another route to the Spin(10) theory, which goes through $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$:

$$
\begin{tikzcd}
G_{\mathrm{SM}} \arrow[r, "\theta"] \arrow[d] & \mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[r, "\eta"] \arrow[d] & \mathrm{Spin}(10) \arrow[d] \\
\mathrm{U}(F \oplus F^*) \arrow[r, "\mathrm{U}(h)"] & \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \arrow[r, "\mathrm{U}(g)"] & \mathrm{U}(\Lambda\mathbb{C}^5)
\end{tikzcd}
$$

More $\overset{\rightsquigarrow}{\text{Unification}}$

Our work in that section and Section 3.3 showed that this diagram commutes as well. So, we have *another* way to extend the Standard Model and get the Spin(10) theory.

Drawing these two routes to Spin(10) together gives us a cube:

$$
\begin{tikzcd}
& G_{\mathrm{SM}} \arrow[rr, "\phi"] \arrow[dl, "\theta"'] \arrow[dd] & & \mathrm{SU}(5) \arrow[dl, "\psi"] \arrow[dd] \\
\mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[rr, crossing over, "\eta"] \arrow[dd] & & \mathrm{Spin}(10) & \\
& \mathrm{U}(F \oplus F^*) \arrow[rr, "\mathrm{U}(f)"] \arrow[dl, "\mathrm{U}(h)"'] & & \mathrm{U}(\Lambda\mathbb{C}^5) \arrow[dl, "1"] \\
\mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \arrow[rr, "\mathrm{U}(g)"] & & \mathrm{U}(\Lambda\mathbb{C}^5) \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

Are these two routes to Spin(10) theory the same? That is, does the cube commute?

Theorem 7. *The cube commutes.*

Proof. We have already seen in Sections 3.1-3.4 that the vertical faces commute. So, we are left with two questions involving the horizontal faces. First: does the top face of the cube

$$
\begin{tikzcd}
& G_{\mathrm{SM}} \arrow[rr, "\phi"] \arrow[dl, "\theta"'] & & \mathrm{SU}(5) \arrow[dl, "\psi"] \\
\mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[rr, "\eta"] & & \mathrm{Spin}(10) &
\end{tikzcd}
$$

commute? In other words: does a symmetry in G_{SM} go to the same place in Spin(10) no matter how we take it there? And second: does the bottom face of the cube commute? In other words: does this triangle:

$$\begin{tikzcd}
F \oplus F^* \arrow[r, "f"] \arrow[d, "h"'] & \Lambda\mathbb{C}^5 \\
\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3 \arrow[ur, "g"'] &
\end{tikzcd}$$

commute?

In fact they both do, and we can use our affirmative answer to the second question to settle the first. As we remarked in Section 3.4, applying the map g to the Pati–Salam binary code given in Table 6, we get the SU(5) binary code given in Table 4. Thus, the linear maps f and gh agree on a basis, so this triangle commutes:

$$\begin{tikzcd}
F \oplus F^* \arrow[r, "f"] \arrow[d, "h"'] & \Lambda\mathbb{C}^5 \\
\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3 \arrow[ur, "g"'] &
\end{tikzcd}$$

This in turn implies that the bottom face of the cube commutes, from which we see that the two maps from G_{SM} to $\mathrm{U}(\Lambda\mathbb{C}^5)$ going around the bottom face are equal:

$$\begin{tikzcd}
G_{\mathrm{SM}} \arrow[dr] & & \\
& \mathrm{U}(F \oplus F^*) \arrow[r, "\mathrm{U}(f)"] \arrow[d, "\mathrm{U}(h)"'] & \mathrm{U}(\Lambda\mathbb{C}^5) \arrow[d, "1"] \\
& \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \arrow[r, "\mathrm{U}(g)"] & \mathrm{U}(\Lambda\mathbb{C}^5)
\end{tikzcd}$$

The work of Section 3.1 through Section 3.4 showed that the vertical faces of the cube commute. We can thus conclude from diagrammatic reasoning that the two maps from G_{SM} to $\mathrm{U}(\Lambda\mathbb{C}^5)$ going around the top face are equal:

$$\begin{tikzcd}
G_{\mathrm{SM}} \arrow[r, "\phi"] \arrow[d, "\theta"'] & \mathrm{SU}(5) \arrow[d, "\psi"] & \\
\mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[r, "\eta"] & \mathrm{Spin}(10) \arrow[dr] & \\
& & \mathrm{U}(\Lambda\mathbb{C}^5)
\end{tikzcd}$$

Since the Dirac spinor representation is faithful, the map $\mathrm{Spin}(10) \to \mathrm{U}(\Lambda\mathbb{C}^5)$ is injective. This means we can drop it from the above diagram, and the remaining square commutes. But this is exactly the top face of the cube. So, the proof is done. $\square$

4 Conclusion

We have studied three different grand unified theories: the SU(5), $\mathrm{Spin}(4)\times\mathrm{Spin}(6)$ and Spin(10) theories. The SU(5) and $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ theories were based on different visions about how to extend the Standard Model. However, we saw that both of these theories can be extended to the Spin(10) theory, which therefore unites these visions.

The SU(5) theory is all about treating isospin and color on an equal footing: it combines the two isospins of $\mathbb{C}^2$ with the three colors of $\mathbb{C}^3$, and posits an SU(5) symmetry acting on the resulting $\mathbb{C}^5$. The particles and antiparticles in a single generation of fermions are described by vectors in $\Lambda\mathbb{C}^5$. So, we can describe each of these particles and antiparticles by a binary code indicating the presence or absence of *up, down, red, green* and *blue*.

In doing so, the SU(5) theory introduces unexpected relationships between matter and antimatter. The irreducible representations of SU(5)

$$\Lambda^0\mathbb{C}^5 \oplus \Lambda^1\mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5 \oplus \Lambda^3\mathbb{C}^5 \oplus \Lambda^4\mathbb{C}^5 \oplus \Lambda^5\mathbb{C}^5$$

unify some particles we normally consider to be 'matter' with some we normally consider 'antimatter', as in

$$\Lambda^1\mathbb{C}^5 \cong \left\langle \begin{matrix} e^+_R \\ \overline{\nu}_R \end{matrix} \right\rangle \oplus \langle d_R \rangle.$$

In the Standard Model representation, we can think of the matter-antimatter distinction as a $\mathbb{Z}_2$-grading, because the Standard Model representation $F \oplus F^*$ splits into F and F^*. By failing to respect this grading, the SU(5) symmetry group fails to preserve the usual distinction between matter and antimatter.

But the Standard Model has another $\mathbb{Z}_2$-grading that SU(5) *does* respect. This is the distinction between left- and right-handedness. Remember, the left-handed particles and antiparticles live in the even grades:

$$\Lambda^{\mathrm{ev}}\mathbb{C}^5 \cong \langle \overline{\nu}_L \rangle \oplus \langle e^+_L \rangle \oplus \left\langle \begin{matrix} u_L \\ d_L \end{matrix} \right\rangle \oplus \langle \overline{u}_L \rangle \oplus \left\langle \begin{matrix} \nu_L \\ e^-_L \end{matrix} \right\rangle \oplus \langle \overline{d}_L \rangle$$

while the right-handed ones live in the odd grades:

$$\Lambda^{\mathrm{odd}}\mathbb{C}^5 \cong \langle \nu_R \rangle \oplus \langle e^-_R \rangle \oplus \left\langle \begin{matrix} \overline{d}_R \\ \overline{u}_R \end{matrix} \right\rangle \oplus \langle u_R \rangle \oplus \left\langle \begin{matrix} e^+_R \\ \overline{\nu}_R \end{matrix} \right\rangle \oplus \langle d_R \rangle.$$

The action of SU(5) automatically preserves this $\mathbb{Z}_2$-grading, because it comes from the $\mathbb{Z}$-grading on $\Lambda\mathbb{C}^5$, which SU(5) already respects.

This characteristic of the SU(5) theory lives on in its extension to Spin(10). There, the distinction between left and right is the only distinction among particles and antiparticles that Spin(10) knows about, because $\Lambda^{\mathrm{ev}}\mathbb{C}^5$ and $\Lambda^{\mathrm{odd}}\mathbb{C}^5$ are irreducible. This says the Spin(10) theory unifies *all* left-handed particles and antiparticles, and *all* right-handed particles and antiparticles.

In contrast, the $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ theory was all about adding a fourth 'color', w, to represent leptons, and restoring a kind of symmetry between left and right by introducing a right-handed SU(2) that treats right-handed particles like the left-handed SU(2) treats left-handed particles.

Unlike the SU(5) theory, the $\mathrm{Spin}(4)\times\mathrm{Spin}(6)$ theory respects *both* $\mathbb{Z}_2$-gradings in the Standard Model: the matter-antimatter grading, and the right-left grading.

The reason is that $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ respects the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on $\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$, and we have:

$$\begin{array}{ccccc} F_L & \cong & \Lambda^{\mathrm{odd}}\mathbb{C}^2 & \otimes & \Lambda^{\mathrm{odd}}\mathbb{C}^3 \\ F_R & \cong & \Lambda^{\mathrm{ev}}\mathbb{C}^2 & \otimes & \Lambda^{\mathrm{odd}}\mathbb{C}^3 \\ F_L^* & \cong & \Lambda^{\mathrm{odd}}\mathbb{C}^2 & \otimes & \Lambda^{\mathrm{ev}}\mathbb{C}^3 \\ F_R^* & \cong & \Lambda^{\mathrm{ev}}\mathbb{C}^2 & \otimes & \Lambda^{\mathrm{ev}}\mathbb{C}^3 \end{array}$$

Moreover, the matter-antimatter grading and the right-left grading are *all* that $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ respects, since each of the four spaces listed is an irrep of this group.

When we extend $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ to the $\mathrm{Spin}(10)$ theory, we identify $\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$ with $\Lambda\mathbb{C}^5$. Then the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on $\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$ gives the $\mathbb{Z}_2$-grading on $\Lambda\mathbb{C}^5$ using addition in $\mathbb{Z}_2$. This sounds rather technical, but it is as simple as "even + odd = odd":

$$\Lambda^{\mathrm{odd}}\mathbb{C}^5 \quad \cong \quad \left(\Lambda^{\mathrm{ev}}\mathbb{C}^2 \otimes \Lambda^{\mathrm{odd}}\mathbb{C}^3\right) \oplus \left(\Lambda^{\mathrm{odd}}\mathbb{C}^2 \otimes \Lambda^{\mathrm{ev}}\mathbb{C}^3\right) \quad \cong \quad F_R \oplus F_L^*$$

and "odd + odd = even", "even + even = even":

$$\Lambda^{\mathrm{ev}}\mathbb{C}^5 \quad \cong \quad \left(\Lambda^{\mathrm{odd}}\mathbb{C}^2 \otimes \Lambda^{\mathrm{odd}}\mathbb{C}^3\right) \oplus \left(\Lambda^{\mathrm{ev}}\mathbb{C}^2 \otimes \Lambda^{\mathrm{ev}}\mathbb{C}^3\right) \quad \cong \quad F_L \oplus F_R^*.$$

Recall that $F_R \oplus F_L^*$ consists of all the fermions and antifermions that are *right-handed*, while $F_L \oplus F_R^*$ consists of the *left-handed* ones.

Furthermore, the two routes to the $\mathrm{Spin}(10)$ theory that we have described, one going through $\mathrm{SU}(5)$ and the other through $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$, are compatible. In other words, this cube commutes:

$$\begin{tikzcd}
 & G_{\mathrm{SM}} \arrow[rr, "\phi"] \arrow[dl, "\theta"'] \arrow[dd] & & \mathrm{SU}(5) \arrow[dl, "\psi"] \arrow[dd] \\
\mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[rr, "\eta"] \arrow[dd] & & \mathrm{Spin}(10) \arrow[dd] & \\
 & \mathrm{U}(F \oplus F^*) \arrow[rr, "U(f)"] \arrow[dl, "\mathrm{U}(h)"'] & & \mathrm{U}(\Lambda\mathbb{C}^5) \arrow[dl, "1"] \\
\mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \arrow[rr, "\mathrm{U}(g)"] & & \mathrm{U}(\Lambda\mathbb{C}^5) &
\end{tikzcd}$$

So, all four theories fit together in an elegant algebraic pattern. What this means for physics—if anything—remains unknown. Yet we cannot resist feeling that it means something, and we cannot resist venturing a guess: *the Standard Model is exactly the theory that reconciles the visions built into the* $\mathrm{SU}(5)$ *and* $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ *theories.*

What this might mean is not yet precise, but since all these theories involve symmetries and representations, the 'reconciliation' must take place at both those levels—and we can see *this* in a precise way. First, at the level of symmetries, our

Lie groups are related by the commutative square of homomorphisms:

$$\begin{tikzcd} G_{\mathrm{SM}} \arrow[r, "\phi"] \arrow[d, "\theta"'] & \mathrm{SU}(5) \arrow[d, "\psi"] \\ \mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[r, "\eta"] & \mathrm{Spin}(10) \end{tikzcd}$$

Because this commutes, the image of G_{SM} lies in the intersection of the images of $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ and $\mathrm{SU}(5)$ inside $\mathrm{Spin}(10)$. But we claim it is *precisely* that intersection!

To see this, first recall that the image of a group under a homomorphism is just the quotient group formed by modding out the kernel of that homomorphism. If we do this for each of our homomorphisms above, we get a commutative square of inclusions:

$$\begin{tikzcd} G_{\mathrm{SM}}/\mathbb{Z}_6 \arrow[r, hook] \arrow[d, hook] & \mathrm{SU}(5) \arrow[d, hook] \\ \dfrac{\mathrm{Spin}(4) \times \mathrm{Spin}(6)}{\mathbb{Z}_2} \arrow[r, hook] & \mathrm{Spin}(10) \end{tikzcd}$$

This implies that

$$G_{\mathrm{SM}}/\mathbb{Z}_6 \subseteq \mathrm{SU}(5) \cap \left(\frac{\mathrm{Spin}(4) \times \mathrm{Spin}(6)}{\mathbb{Z}_2} \right)$$

as subgroups of $\mathrm{Spin}(10)$. To make good on our claim, we must show these subgroups are equal:

$$G_{\mathrm{SM}}/\mathbb{Z}_6 = \mathrm{SU}(5) \cap \left(\frac{\mathrm{Spin}(4) \times \mathrm{Spin}(6)}{\mathbb{Z}_2} \right).$$

In other words, our commutative square of inclusions is a 'pullback square'.

As a step towards showing this, first consider what happens when we pass from the spin groups to the rotation groups. We can accomplish this by modding out by an additional $\mathbb{Z}_2$ above. We get another commutative square of inclusions:

$$\begin{tikzcd} G_{\mathrm{SM}}/\mathbb{Z}_6 \arrow[r, hook] \arrow[d, hook] & \mathrm{SU}(5) \arrow[d, hook] \\ \mathrm{SO}(4) \times \mathrm{SO}(6) \arrow[r, hook] & \mathrm{SO}(10) \end{tikzcd}$$

Here the reader may wonder why we could quotient $(\mathrm{Spin}(4) \times \mathrm{Spin}(6))/\mathbb{Z}_2$ and $\mathrm{Spin}(10)$ by $\mathbb{Z}_2$ without having to do the same for their respective subgroups, $G_{\mathrm{SM}}/\mathbb{Z}_6$ and $\mathrm{SU}(5)$. It is because $\mathbb{Z}_2$ intersects both of those subgroups trivially. We can see this for $\mathrm{SU}(5)$ because we know the inclusion $\mathrm{SU}(5) \hookrightarrow \mathrm{Spin}(10)$ is just the lift of the inclusion $\mathrm{SU}(5) \hookrightarrow \mathrm{SO}(10)$ to universal covers, so it makes this diagram commute:

$$\begin{tikzcd} \mathrm{SU}(5) \arrow[r, hook] \arrow[dr, hook] & \mathrm{Spin}(10) \arrow[d, "p"] \\ & \mathrm{SO}(10) \end{tikzcd}$$

But this means that SU(5) intersects $\mathbb{Z}_2 = \ker p$ in only the identity. The subgroup $G_{\mathrm{SM}}/\mathbb{Z}_6$ therefore intersects $\mathbb{Z}_2$ trivially as well.

Now, let us show:

Theorem 8. $G_{\mathrm{SM}}/\mathbb{Z}_6 = \mathrm{SU}(5) \cap (\mathrm{SO}(4) \times \mathrm{SO}(6)) \subseteq \mathrm{SO}(10)$.

Proof. We can prove this in the same manner that we showed, in Section 3.1, that

$$G_{\mathrm{SM}}/\mathbb{Z}_6 \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \subseteq \mathrm{SU}(5)$$

is precisely the subgroup of SU(5) that preserves the 2+3 splitting of $\mathbb{C}^5 \cong \mathbb{C}^2 \oplus \mathbb{C}^3$.

To begin with, the group SO(10) is the group of orientation-preserving symmetries of the 10-dimensional real inner product space $\mathbb{R}^{10}$. But $\mathbb{R}^{10}$ is suspiciously like $\mathbb{C}^5$, a 5-dimensional complex inner product space. Indeed, if we forget the complex structure on $\mathbb{C}^5$, we get an isomorphism $\mathbb{C}^5 \cong \mathbb{R}^{10}$, a real inner product space with symmetries SO(10). We can consider the subgroup of SO(10) that preserves the original complex structure. This is $\mathrm{U}(5) \subseteq \mathrm{SO}(10)$. If we further pick a volume form on $\mathbb{C}^5$, i.e. a nonzero element of $\Lambda^5\mathbb{C}^5$, and look at the symmetries fixing that volume form, we get a copy of $\mathrm{SU}(5) \subseteq \mathrm{SO}(10)$.

Then we can pick a $2+3$ splitting on $\mathbb{C}^5 \cong \mathbb{C}^2 \oplus \mathbb{C}^3$. The subgroup of SU(5) that also preserves this is

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \hookrightarrow \mathrm{SU}(5) \hookrightarrow \mathrm{SO}(10).$$

These inclusions form the top and right sides of our square:

$$\begin{tikzcd}
G_{\mathrm{SM}}/\mathbb{Z}_6 \arrow[r, hook] \arrow[d, hook] & \mathrm{SU}(5) \arrow[d, hook] \\
\mathrm{SO}(4) \times \mathrm{SO}(6) \arrow[r, hook] & \mathrm{SO}(10)
\end{tikzcd}$$

We can also reverse the order of these processes. Imposing a $2+3$ splitting on $\mathbb{C}^5$ yields a $4+6$ splitting on the underlying real vector space, $\mathbb{R}^{10} \cong \mathbb{R}^4 \oplus \mathbb{R}^6$. The subgroup of SO(10) that preserves this splitting is $\mathrm{S}(\mathrm{O}(4) \times \mathrm{O}(6))$: the block diagonal matrices with 4×4 and 6×6 orthogonal blocks and overall determinant 1. The connected component of this subgroup is $\mathrm{SO}(4) \times \mathrm{SO}(6)$.

The direct summands in $\mathbb{R}^4 \oplus \mathbb{R}^6$ came from forgetting the complex structure on $\mathbb{C}^2 \oplus \mathbb{C}^3$. The subgroup of $\mathrm{S}(\mathrm{O}(4) \times \mathrm{O}(6))$ preserving the original complex structure is $\mathrm{U}(2) \times \mathrm{U}(3)$, and the subgroup of this that also fixes a volume form on $\mathbb{C}^5$ is $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$. This group is connected, so it must lie entirely in the connected component of the identity, and we get the inclusions:

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)) \hookrightarrow \mathrm{SO}(4) \times \mathrm{SO}(6) \hookrightarrow \mathrm{SO}(10).$$

These maps form the left and bottom sides of our square.

It follows that $G_{\mathrm{SM}}/\mathbb{Z}_6$ is precisely the subgroup of SO(10) that preserves a complex structure on $\mathbb{R}^{10}$, a chosen volume form on the resulting complex vector space, and a $2+3$ splitting on this space. But this $2+3$ splitting is the same as a *compatible* $4+6$ splitting of $\mathbb{R}^{10}$, one in which each summand is a complex vector subspace as well as a real subspace. This means that

$$G_{\mathrm{SM}}/\mathbb{Z}_6 = \mathrm{SU}(5) \cap \mathrm{S}(\mathrm{O}(4) \times \mathrm{O}(6)) \subseteq \mathrm{SO}(10),$$

and since $G_{\text{SM}}/\mathbb{Z}_6$ is connected,

$$G_{\text{SM}}/\mathbb{Z}_6 = \text{SU}(5) \cap (\text{SO}(4) \times \text{SO}(6)) \subseteq \text{SO}(10)$$

as desired. $\square$

From this, a little diagram chase proves our earlier claim:

Theorem 9. $G_{\text{SM}}/\mathbb{Z}_6 = \text{SU}(5) \cap (\text{Spin}(4) \times \text{Spin}(6))/\mathbb{Z}_2 \subseteq \text{Spin}(10)$.

Proof. By now we have built the following commutative diagram:

$$\begin{tikzcd}
G_{\text{SM}}/\mathbb{Z}_6 \arrow[r, hook, "\tilde{\phi}"] \arrow[d, hook, "\tilde{\theta}"'] & \text{SU}(5) \arrow[d, hook, "\psi"] \\
(\text{Spin}(4) \times \text{Spin}(6))/\mathbb{Z}_2 \arrow[r, hook, "\tilde{\eta}"] \arrow[d, "q"'] & \text{Spin}(10) \arrow[d, "p"] \\
\text{SO}(4) \times \text{SO}(6) \arrow[r, hook, "i"] & \text{SO}(10)
\end{tikzcd}$$

where both the bottom vertical arrows are two-to-one, but the composite vertical maps $q\tilde{\theta}$ and $p\psi$ are one-to-one. Our previous theorem says that the big square with $q\tilde{\theta}$ and $p\psi$ as vertical sides is a pullback. Now we must show that the upper square is also a pullback. So, suppose we are given $g \in (\text{Spin}(4) \times \text{Spin}(6))/\mathbb{Z}_2$ and $g' \in \text{SU}(5)$ with

$$\tilde{\eta}(g) = \psi(g').$$

We need to show there exists $x \in G_{\text{SM}}/\mathbb{Z}_6$ such that

$$\tilde{\theta}(x) = g, \qquad \tilde{\phi}(x) = g'.$$

Now, we know that

$$iq(g) = p\tilde{\eta}(g) = p\psi(g')$$

so since the big square is a pullback, there exists $x \in G_{\text{SM}}/\mathbb{Z}_6$ with

$$q\tilde{\theta}(x) = q(g), \qquad \tilde{\phi}(x) = g'.$$

The second equation is half of what we need to show. So, we only need to check that the first equation implies $\tilde{\theta}(x) = g$.

The kernel of q consists of two elements, which we will simply call ± 1. Since $q\tilde{\theta}(x) = q(g)$, we know

$$\pm\tilde{\theta}(x) = g.$$

Since $\tilde{\eta}(g) = \psi(g')$, we thus have

$$\tilde{\eta}(\pm\tilde{\theta}(x)) = \psi(g') = \psi\tilde{\phi}(x).$$

The one-to-one map $\tilde{\eta}$ sends the kernel of q to the kernel of p, which consists of two elements that we may again call ± 1. So, $\pm\tilde{\eta}\tilde{\theta}(x) = \psi\tilde{\phi}(x)$. On the other hand, since the top square commutes we know $\tilde{\eta}\tilde{\theta}(x) = \psi\tilde{\phi}(x)$. Thus the element ± 1 must actually be 1, so $g = \tilde{\theta}(x)$ as desired. $\square$

In short, the Standard Model has precisely the symmetries shared by both the SU(5) theory and the $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ theory. Now let us see what this means for the Standard Model representation.

We can 'break the symmetry' of the Spin(10) theory in two different ways. In the first way, we start by picking the subgroup of Spin(10) that preserves the $\mathbb{Z}$-grading and volume form in $\Lambda\mathbb{C}^5$. This is SU(5). Then we pick the subgroup of SU(5) that respects the splitting of $\mathbb{C}^5$ into $\mathbb{C}^2 \oplus \mathbb{C}^3$. This subgroup is the Standard Model gauge group, modulo a discrete subgroup, and its representation on $\Lambda\mathbb{C}^5$ is the Standard Model representation.

We can draw this symmetry breaking process in the following diagram:

$$\begin{tikzcd}
G_{\mathrm{SM}} \arrow[r, "\phi"] \arrow[d] & \mathrm{SU}(5) \arrow[r, "\psi"] \arrow[d] & \mathrm{Spin}(10) \arrow[d] \\
\mathrm{U}(F \oplus F^*) \arrow[r, "\mathrm{U}(f)"] & \mathrm{U}(\Lambda\mathbb{C}^5) \arrow[r, "1"] & \mathrm{U}(\Lambda\mathbb{C}^5)
\end{tikzcd}$$

$$\begin{tikzcd}
{} & {} \arrow[l, squiggly, "\text{splitting}"'] & {} \arrow[l, squiggly, "\text{grading and}"', "\text{volume form}"]
\end{tikzcd}$$

The SU(5) theory shows up as a 'halfway house' here.

We can also break the symmetry of Spin(10) in a way that uses the $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ theory as a halfway house. We do essentially the same two steps as before, but in the reverse order! This time we start by picking the subgroup of Spin(10) that respects the splitting of $\mathbb{R}^{10}$ as $\mathbb{R}^4 \oplus \mathbb{R}^6$. This subgroup is $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ modulo a discrete subgroup. The two factors in this subgroup act separately on the factors of $\Lambda\mathbb{C}^5 \cong \Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$. Then we pick the subgroup of $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ that respects the $\mathbb{Z}$-grading and volume form on $\Lambda\mathbb{C}^5$. This subgroup is the Standard Model gauge group, modulo a discrete subgroup, and its representation on $\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3$ is the Standard Model representation.

We can draw this alternate symmetry breaking process in the following diagram:

$$\begin{tikzcd}
G_{\mathrm{SM}} \arrow[r, "\theta"] \arrow[d] & \mathrm{Spin}(4) \times \mathrm{Spin}(6) \arrow[r, "\eta"] \arrow[d] & \mathrm{Spin}(10) \arrow[d] \\
\mathrm{U}(F \oplus F^*) \arrow[r, "\mathrm{U}(h)"] & \mathrm{U}(\Lambda\mathbb{C}^2 \otimes \Lambda\mathbb{C}^3) \arrow[r, "\mathrm{U}(g)"] & \mathrm{U}(\Lambda\mathbb{C}^5)
\end{tikzcd}$$

$$\begin{tikzcd}
{} & {} \arrow[l, squiggly, "\text{grading and}"', "\text{volume form}"] & {} \arrow[l, squiggly, "\text{splitting}"']
\end{tikzcd}$$

Will these tantalizing patterns help us understand physics beyond the Standard Model? Only time will tell.

Acknowledgements

We thank the denizens of the n-Category Café for catching mistakes and suggesting stylistic improvements. This research was supported by a grant from the Foundational Questions Institute.

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