Higher-Dimensional Algebra IV: 2-Tangles

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Abstract

Just as knots and links can be algebraically described as certain morphisms in the category of tangles in 3 dimensions, compact surfaces smoothly embedded in $\mathbb{R}^4$ can be described as certain 2-morphisms in the 2-category of '2-tangles in 4 dimensions'. Using the work of Carter, Rieger and Saito, we prove that this 2-category is the 'free semistrict braided monoidal 2-category with duals on one unframed self-dual object'. By this universal property, any unframed self-dual object in a braided monoidal 2-category with duals determines an invariant of 2-tangles in 4 dimensions.

1 Introduction

One of the most exciting aspects of higher-dimensional algebra is how weak n-categories seem to provide precisely the right mathematics for algebraic topology. From one point of view, weak n-categories are purely algebraic structures consisting of objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, and so on up to n-morphisms, together with various composition operations, satisfying laws that arise naturally from algebraic considerations [2]. But time and time again, the mathematics of weak n-categories has turned out to be perfectly suited to n-dimensional topology.

Until the late 1980's, the most striking instances of this phenomenon came from homotopy theory. By now there is a large body of evidence supporting a conjecture that would completely explain the relation between n-categories and homotopy theory [5]. In rough terms, this conjecture states that spaces with vanishing homotopy groups above dimension n are equivalent to a certain class of weak n-categories, the 'weak n-groupoids'. A weak n-groupoid is a weak n-category where every j-morphism has a 'weak inverse'. For $j = n$, a weak inverse for the j-morphism $f: x \to y$ is just an inverse in the usual sense, while for $j < n$, a weak inverse for f is recursively defined to be a j-morphism $g: x \to y$ such that fg and gf are the identity *up to a weakly invertible* $(j + 1)$*-morphism.*

Starting with the discovery of the Jones polynomial and a family of related knot invariants, a new branch of algebraic topology has emerged in the last decade. It is often called 'quantum topology' because of its close ties to quantum field theory. Its relation to more traditional forms of algebraic topology based on homotopy theory was initially very mysterious. Now it appears that quantum topology goes beyond homotopy theory precisely by exploiting a larger class of n-categories, the 'n-categories with duals'. These are poorly understood except in some low-dimensional cases, but some of their essential features are already clear. Most importantly, while every j-morphism $f: x \to y$ has a 'dual' $f^*: y \to x$, this dual need not be a weak inverse of f. One important example of a category with duals is the category of Hilbert spaces, where the dual of a linear operator is its Hilbert space adjoint. Another is the category of tangles in 3-dimensional space, where the dual of a tangle is obtained by reflecting it to switch its source and target.

The category of tangles in 3 dimensions is especially important, because it has a beautiful algebraic characterization in terms of a universal property. This was initially developed by Turaev [28], Freyd–Yetter [18, 29], and Joyal–Street [19], and it reached a highly polished form in the work of Shum [25]. In our language [1, 3], her result is that isotopy classes of framed oriented tangles in 3 dimensions are the morphisms of the 'free braided monoidal category with duals on one object'. Using this universal property, we can easily obtain functors from this category to other braided monoidal categories with duals, such as categories of representations of quantum groups. Any such functor gives an invariant of tangles, and in particular, a knot invariant. This is the easiest way to understand the Jones polynomial and its relatives [24].

The 'tangle hypothesis' [3] suggests a vast generalization of this result, applicable to n-dimensional surfaces embedded in $(n + k)$-dimensional space for all n and k. This generalization involves the notion of a 'k-tuply monoidal n-category'. A k-tuply monoidal n-category is an $(n + k)$-category that has only trivial j-morphisms for $j < k$. By reindexing we can think of this as an n-category with extra structure and properties. Some low-dimensional special cases are shown in Table 1 below.

Briefly put, the tangle hypothesis says that framed oriented n-tangles in $(n + k)$ dimensions are the n-morphisms of the 'free weak k-tuply monoidal n-category with duals on one object'. In this n-category, the objects correspond to collections of points embedded in $[0, 1]^k$. The 1-morphisms correspond to compact 1-manifolds with boundary embedded in $[0, 1]^{k+1}$ going from one such object to another. Similarly, the 2-morphisms correspond to compact 2-manifolds with corners embedded in $[0, 1]^{k+2}$ going from one 1-morphism $f: x \to y$ to another 1-morphism $g: x \to y$, and so on. Finally, the n-morphisms correspond to isotopy classes of n-manifolds with corners embedded in $[0, 1]^{n+k}$. We call these 'n-tangles in $n + k$ dimensions'.

	$n=0$	$n=1$	$n=2$
$k=0$	sets	categories	2-categories
$k=1$	monoids	monoidal categories	monoidal 2-categories
$k=2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories
$k=3$	‘’	symmetric monoidal categories	weakly involutory monoidal 2-categories
$k=4$	‘’	‘’	strongly involutory monoidal 2-categories
$k=5$	‘’	‘’	‘’

Table 1. k-tuply monoidal n-categories

Unfortunately the tangle hypothesis involves concepts from topology and n-category theory that presently have only been made precise in certain low-dimensional cases. As a kind of warmup, we wish to prove a version of this hypothesis in the case $n = k = 2$. So far we have only completed work on the unframed, unoriented case, which allows us to take maximal advantage of the recent work of Carter, Rieger and Saito [9]. Since the theory of k-tuply monoidal weak n-categories is not yet well developed for $n = k = 2$, we use the better-understood ‘semistrict’ ones as a kind of stopgap. These are also known as ‘semistrict braided monoidal 2-categories’. Our result is thus that the 2-category of unframed unoriented 2-tangles in 4 dimensions is the ‘free semistrict braided monoidal 2-category with duals on one unframed self-dual object’.

This result is closely related to the fact that the category of unframed unoriented tangles in 3 dimensions is the free braided monoidal category with duals on one unframed self-dual object. In particular, the Reidemeister moves, which arise as *equations* between morphisms in the category of tangles in 3 dimensions, arise as *2-isomorphisms* in our context. For this reason we say that our result is a ‘categorification’ of the 3-dimensional one. For more on categorification and how it relates to the tangle hypothesis, see our previous papers [3, 5].

The study of duality in n-categories is only beginning, so an important part of this paper consists of finding an appropriate definition of a braided monoidal 2-category ‘with duals’. Given this, we simply define a ‘self-dual’ object x to be one for which $x = x^*$. On the other hand, the notion of an ‘unframed’ object really takes advantage of categorification. In the category of tangles, a twist in the framing corresponds to a morphism called the ‘balancing’. In our situation, an ‘unframed object’ is not one for which the balancing *equals* the identity, but one for which the balancing is *isomorphic* to the identity via a certain 2-isomorphism. This 2-isomorphism, corresponding to the Reidemeister I move, satisfies a highly nontrivial equation of its own.

The study of universal properties for n-categories is also just beginning, so we must clarify what is meant by the 'free' braided monoidal 2-category with duals on one unframed self-dual object. Finally, since there is presently no general construction of the k-tuply monoidal n-category of k-tangles in n dimensions, we must construct this 'by hand' in the case $n = k = 2$ before proving our result. To obtain a semistrict braided monoidal 2-category, we cannot let the objects be simply collections of points embedded in $[0, 1]^2$. Instead, we must introduce an equivalence relation on such collections, and take objects to be equivalence classes. Similarly, the morphisms in our 2-category are certain equivalence classes of tangles.

We must choose these equivalence relations carefully, in order to avoid the errors present in Fischer's attempt [17] to define a 2-category of 2-tangles. Kharlamov and Turaev [21] have shown that composition of 2-morphisms is not well-defined if, as Fischer did, we take isotopy classes of tangles as our 1-morphisms. Kharlamov and Turaev showed how to avoid this problem by introducing a 'height function' on $[0, 1]^3$ and saying that two tangles define the same 1-morphism only if they differ by an isotopy that preserves the order of the heights of local extrema of this function. Our work is based on Carter, Rieger and Saito's recent combinatorial description of 2-tangles [9], which places a somewhat stronger restriction on the isotopies: they must preserve the order of heights of local maxima, local minima, and crossings relative to a specified projection.

The plan of the paper is as follows. In Section 2 we give a topological description of a 2-category $\mathcal{T}$ of unframed unoriented 2-tangles in 4 dimensions. We define duality for monoidal and braided monoidal 2-categories, and show that $\mathcal{T}$ is a braided monoidal 2-category with duals. We also define the notion of an 'unframed self-dual object' in a braided monoidal 2-category with duals, and we show that $\mathcal{T}$ has an unframed self-dual object Z corresponding to a single point in the unit square. In Section 3 we give an alternate, purely combinatorial description of a 2-category of 2-tangles, which we denote by $\mathcal{C}$. Using the work of Carter, Rieger and Saito, we then show that $\mathcal{T}$ and $\mathcal{C}$ are isomorphic. In Section 4 we use this isomorphism to show that $\mathcal{T}$ is generated, as a braided monoidal 2-category with duals, by the unframed self-dual object Z. Given a strict monoidal 2-functor $F\colon \mathcal{T} \to \mathcal{B}$, we define what it means for F to 'preserve braiding and duals semistrictly on the generator'. Finally, we show that for any braided monoidal 2-category with duals $\mathcal{B}$ containing an unframed self-dual object B, there is a unique strict monoidal 2-functor $F\colon \mathcal{T} \to \mathcal{B}$ with $F(Z) = B$ that preserves braiding and duals semistrictly on the generator. This is the precise sense in which the 2-category of 2-tangles is the free braided monoidal 2-category with duals on an unframed self-dual object.

This paper is based upon the second author's Ph.D. thesis. A summary of the results here can be found in a previous paper of ours [6], and also in the magnificently illustrated book by Carter and Saito [11]. In the Errata section at the end of this paper we correct some errors in our previous one. Also, as promised, we treat the universal property of $\mathcal{T}$ more carefully here, allowing us to omit the conditions $\tilde{R}_{(A|A,A)} = 1$ and $\tilde{R}_{(A|A,A)} = 1$ which previously appeared in the definition of an unframed self-dual object A.

We refer to the paper in which the tangle hypothesis was first stated as HDA0 [3], and refer to the earlier papers in this series as HDA1 [7], HDA2 [1], and HDA3 [4].

2 A Topological Description of 2-Tangles

In this section we describe the 2-category $\mathcal{T}$ of 2-tangles using the language of differential topology, and prove that $\mathcal{T}$ is a braided monoidal 2-category with duals. First we carefully describe the objects, 1-morphisms, and 2-morphisms of $\mathcal{T}$, and show that $\mathcal{T}$ has the structure of a 2-category. Then we show that $\mathcal{T}$ actually has the structure of a braided monoidal 2-category with duals.

In what follows, all manifolds are assumed to be compact and smooth, but possibly with corners shaped like the subset $\{(x_1, \ldots, x_n)\colon\ x_1, \ldots, x_i \geq 0\}$. All maps between them are assumed to be smooth in the obvious sense, but not necessarily mapping corners to corners. A diffeomorphism, defined as a smooth map with smooth inverse, automatically maps corners to corners. An *ambient isotopy* of a manifold M is defined to be a map $H\colon M \times [0,1] \to M$ such that $H(\cdot, s)$ is a diffeomorphism for each $s \in [0,1]$. We define a *fiber isotopy* of a bundle $p\colon E \to B$ to be a ambient isotopy H of the total space such that $H(\cdot, s)$ maps fibers to (possibly different) fibers for each $s \in [0,1]$.

2.1 The 2-Category $\mathcal{T}$

We shall think of 2-tangles as lying in the 4-cube $I_1 \times I_2 \times I_3 \times I_4$, where $I_i = [0,1]$. We take the coordinates of this 4-cube to be x, y, z, and t, respectively. We refer to points as being *behind* or *in front* if they have greater or smaller x values, to the *left* or *right* if they have smaller or greater y values, *above* or *below* for smaller or greater z values, and *before* or *after* if they have smaller or greater t values. We refer to the function z as the *height*. Sometimes a 2-tangle will be illustrated by what Carter, Rieger and Saito [9] call a 'movie': a finite sequence of tangles that are cross-sections of the 2-tangle at successive values of t. We draw tangles in the cube, with axes x, y and z as shown on the left of Fig. 1. Another way we illustrate a 2-tangle is to draw a generic projection of it into the cube with axes y, z, and t as shown on the right of Fig. 1.

2.1.1 Objects

The objects of $\mathcal{T}$ are in one-to-one correspondence with the natural numbers $0, 1, 2, \ldots$. Technically, we define objects to be equivalence classes of generic finite subsets of the

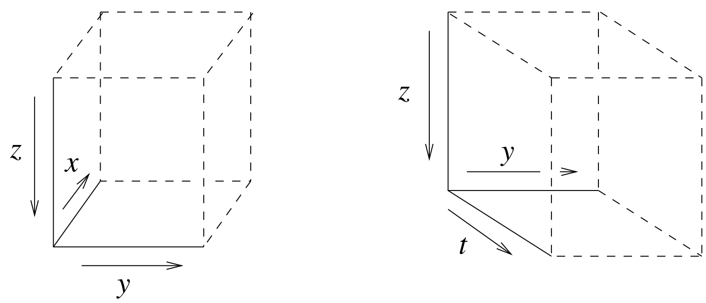


Figure 1: Orientation of illustrations

square. We say a finite subset of the square $I_1 \times I_2$ is *generic* if it lies in the interior of the square and no two points in the set have the same y coordinate. Two such sets A and B are said to be *equivalent* if there is a level-preserving ambient isotopy H of the square such that $H(A, 0) = A$ and $H(A, 1) = B$, where we say H is *level preserving* if it is a fiber isotopy of the bundle $p: I_1 \times I_2 \to I_2$. We call such an isotopy an *equivalence isotopy* between generic finite subsets of the square. This defines an equivalence relation under which any two such sets with the same number of elements are equivalent.

2.1.2 1-Morphisms

The 1-morphisms of $\mathcal{T}$ are equivalence classes of generic tangles. More precisely:

Definition 1. *A* tangle *is a 1-dimensional manifold T with boundary embedded in* $\mathrm{int}(I_1 \times I_2) \times I_3$ *such that:*

1. *The boundary points of T lie in* $\mathrm{int}(I_1 \times I_2) \times \{0, 1\}$.
2. *T has a* product structure *near the top and bottom; that is, there exists $\epsilon > 0$ such that $(x, y, z) \in T$ if and only if $(x, y, z_0) \in T$ when z is within ϵ of $z_0 = 0$ or $z_0 = 1$.*

We say a tangle T is generic *if its projection to $I_2 \times I_3$ is an embedding except for finitely many crossings (transverse double points), the critical points of the height function on T are all nondegenerate local extrema, and all the crossings and critical points occur at different heights.*

Note that any tangle is ambient isotopic to a generic tangle. This allows us to restrict attention to generic tangles without loss of generality.

Definition 2. *Two generic tangles are* equivalent *if there is an ambient isotopy H carrying one to the other such that:*

1. *H is* level preserving*, meaning that it is a fiber isotopy for the bundle* $p\colon I_1 \times I_2 \times I_3 \to I_2 \times I_3$ *and also for the bundle* $\pi\colon I_1 \times I_2 \times I_3 \to I_3$.

2. *H has a* product structure *in a neighborhood of* $I_1 \times I_2 \times \partial I_3$*, meaning that for some* $\epsilon > 0$*,* $H(x, y, z, s)$ *is of the form*

$$(X(x, y, z_0, s), Y(x, y, z_0, s), z)$$

if z *is within* ϵ *of* $z_0 = 0$ *or* $z_0 = 1$.

We call such an isotopy an equivalence isotopy *between generic tangles.*

The level-preserving properties of this equivalence relation imply that generic tangles whose projections onto the square differ by Reidemeister moves or by changing the order of heights of crossings or local extrema are not equivalent. We may thus represent 1-morphisms of $\mathcal{T}$ by planar diagrams of tangles for which the crossings and local extrema of z are ordered by their height.

Given a 1-morphism f in $\mathcal{T}$ represented by a generic tangle T, we define its *source* to be the object represented by to the set $T \cap (I_1 \times I_2 \times \{0\})$. Similarly, we define its *target* to be the object corresponding to $T \cap (I_1 \times I_2 \times \{1\})$. The restriction of an equivalence isotopy between generic tangles to $I_1 \times I_2 \times \{0\}$ or $I_1 \times I_2 \times \{1\}$ is an equivalence isotopy between generic finite subsets of the square, so the source and target of a 1-morphism are well-defined objects.

2.1.3 2-Morphisms

Similarly, the 2-morphisms of $\mathcal{T}$ are equivalence classes of generic 2-tangles.

Definition 3. *A* 2-tangle *is a 2-manifold S with corners embedded in* $I_1 \times I_2 \times I_3 \times I_4$ *such that:*

1. *The boundary of S is embedded in* $I_1 \times I_2 \times \partial(I_3 \times I_4)$*, the intersection* $S \cap (I_1 \times I_2 \times I_3 \times \partial I_4)$ *is a pair of tangles, and* $S \cap (I_1 \times I_2 \times \partial I_3 \times I_4)$ *consists of finitely many straight lines of the form* $(x, y, z) \times I_4$.

2. *S has a* product structure *near the boundary. That is, there exists* $\epsilon > 0$ *such that if* $(x, y, z, t) \in S$ *then* $(x, y, z', t) \in S$ *if both z and z' are within ϵ of either 0 or 1, and* $(x, y, z, t') \in S$ *if both t and $'$ are within ϵ of either 0 or 1.*

We say a 2-tangle S is generic *if its intersection with the hyperplanes of constant t are generic tangles except for finitely many values of t at which one of the 'full set of elementary string interactions' occurs. Briefly, these are: the three Reidemeister moves, the birth or death of an unknotted circle, a saddle point of the function t on S, a cusp on a fold line, a double point arc crossing a fold line, and moves that interchange the relative height of two crossings and/or local extrema.*

Just as for tangles, any 2-tangle is ambient isotopic to a generic one. For a proof of this and a more detailed description of the full set of elementary string interactions, see Carter, Saito and Rieger [9].

Definition 4. *Two generic 2-tangles are* equivalent *if there is an ambient isotopy*

$$H\colon I_1 \times I_2 \times I_3 \times I_4 \times [0,1] \to I_1 \times I_2 \times I_3 \times I_4$$

carrying one to the other such that:

1. *The restriction of H to $I_1 \times I_2 \times I_3 \times \{t_0\}$ for $t_0 = 0$ or 1 is an equivalence of generic tangles.*
2. *The restriction of H to $I_1 \times I_2 \times \{z_0\} \times I_4$ for $z_0 = 0$ or 1 is* level preserving *in the sense that it is a fiber isotopy for the bundle $p'\colon I_1 \times I_2 \times I_4 \to I_2 \times I_4$, and also for the bundle $\pi'\colon I_1 \times I_2 \times I_4 \to I_4$.*
3. *H has a* product structure *near $I_1 \times I_2 \times \partial(I_3 \times I_4)$; specifically, there is an $\epsilon > 0$ such that H is of the form*

   $$H(x,y,z,t,s) = (X(x,y,z,t_0,s), Y(x,y,z,t_0,s), Z(x,y,z,t_0,s), t)$$

   if t is within ϵ of $t_0 = 0$ or $t_0 = 1$, and of the form

   $$H(x,y,z,t,s) = (X(x,y,z_0,t,s), Y(x,y,z_0,t,s), z, T(x,y,z_0,t,s))$$

   if z is within ϵ of $z_0 = 0$ or $z_0 = 1$.

We call such an isotopy an equivalence isotopy *between generic 2-tangles.*

Given a 2-morphism of $\mathcal{T}$ represented by a generic 2-tangle S, we define its *source* to be the 1-morphism represented by the generic tangle $S \cap (I_1 \times I_2 \times I_3 \times \{0\})$. We define its *target* to be the 1-morphism represented by $S \cap (I_1 \times I_2 \times I_3 \times \{1\})$. The restriction of an equivalence isotopy between generic 2-tangles to $I_1 \times I_2 \times I_3 \times \{0\}$ or $I_1 \times I_2 \times I_3 \times \{1\}$ gives an equivalence isotopy between generic tangles, so the source and target of a 2-morphism are well-defined 1-morphisms.

Note that generic 2-tangles without boundary are equivalent if and only if they are ambient isotopic. It follows that 2-morphisms in $\mathcal{T}$ having the 1-morphism corresponding to the empty tangle as both source and target are the same thing as ambient isotopy classes of closed 2-dimensional submanifolds of $[0,1]^4$, or equivalently, $\mathbb{R}^4$. This is the precise sense in which our algebraic characterization of $\mathcal{T}$ gives an algebraic description of knotted surfaces in 4-dimensional space.

In what follows we often use the same notation for an object (resp. morphism, 2-morphism) of $\mathcal{T}$ and a subset of the square (resp. generic tangle, generic 2-tangle) representing it. The difference should be clear from the context.

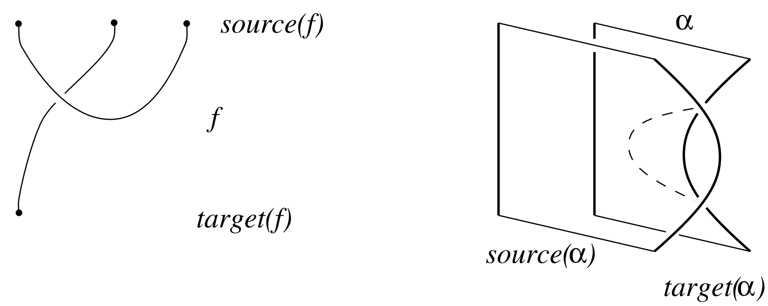


Figure 2: Morphisms and 2-morphisms

2.1.4 Composition of 1-morphisms and 2-morphisms

We now describe how to compose 1-morphisms and 2-morphisms in $\mathcal{T}$. We use the same conventions and notation regarding 2-categories as in HDA1. In particular, composition of 1-morphisms, horizontal composition of a 1-morphism and a 2-morphism in either order, and horizontal composition of 2-morphisms is denoted by juxtaposition. Vertical composition of 2-morphisms is denoted by $\cdot$. We use the ordering in which, for example, the composite of $f\colon A \to B$ and $g\colon B \to C$ is written as fg. We treat composition of 1-morphisms, horizontal composition of 2-morphisms, and vertical composition of 2-morphisms as fundamental, and define horizontal composition of a 1-morphism and a 2-morphism in terms of these in the usual way:

$$f\alpha := 1_f\alpha, \qquad \alpha f := \alpha 1_f.$$

Composition of 1-morphisms corresponds to gluing tangles along their source and target sets. Since we require that tangles be straight near their source and target, the resulting tangle is indeed a smooth submanifold. Specifically, let $f\colon A \to B$ and $g\colon B \to C$ be 1-morphisms in $\mathcal{T}$. Then the composite $fg\colon A \to C$ is defined as follows. Choose generic tangles representing f and g, which by abuse of language we also call f and g. Assume that the set representing the target of f equals the set representing the source of g. By abuse of language we call this set B. Also assume that f has a product structure in the z direction for $z \in [1/4, 1]$ — i.e., $f \cap I_1 \times I_2 \times \{z\} = B$ for such z — and that g has a product structure in the z direction for $z \in [0, 3/4]$. Then fg is the 1-morphism given by the tangle

$$(f \cap I_1 \times I_2 \times [0, 1/2]) \cup (g \cap I_1 \times I_2 \times [1/2, 1]).$$

The tangles f and g agree on $I_1 \times I_2 \times [1/4, 3/4]$, so this indeed gives a tangle, and in fact a generic tangle, as shown in Fig. 3.

Horizontal composition of 2-morphisms corresponds to gluing generic 2-tangles along the strands at their top and bottom. Since the top and bottom of a 2-tangle consists of finitely many straight strands, there is a unique way to glue them together,

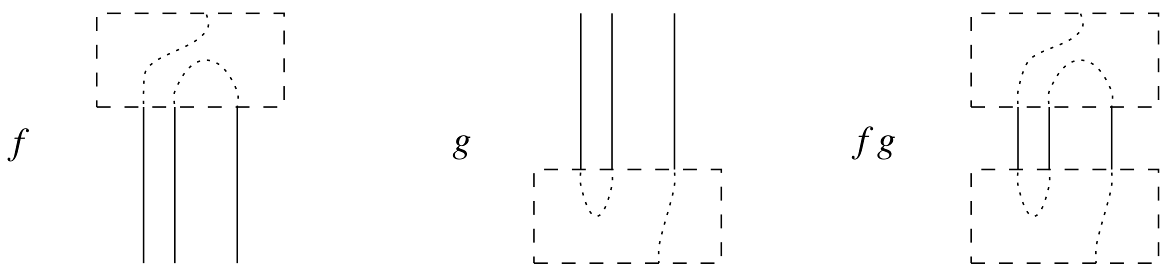


Figure 3: Composition of 1-morphisms

and since we require 2-tangles to have a product structure near their top and bottom, this gluing indeed results in a 2-tangle, as shown in Fig. 4.

More precisely, suppose we have 2-morphisms $\alpha: f \Rightarrow f'$ and $\beta: g \Rightarrow g'$ in $\mathcal{T}$ such that $f, f': A \to B$ and $g, g': B \to C$. Then we define their horizontal composite $\alpha\beta$ as follows. We choose generic 2-tangles representing these 2-morphisms, which by abuse of language we also call α and β, such that

$$\alpha \cap I_1 \times I_2 \times \{1\} \times I_4 = \beta \cap I_1 \times I_2 \times \{0\} \times I_4,$$

α has a product structure in the z direction for $z \in [1/4, 1]$, and β has a product structure in the z direction for $z \in [0, 3/4]$. Then $\alpha\beta$ is the 2-morphism represented by

$$(\alpha \cap I_1 \times I_2 \times [0, 1/2] \times I_4) \cup (\beta \cap I_1 \times I_2 \times [1/2, 1] \times I_4).$$

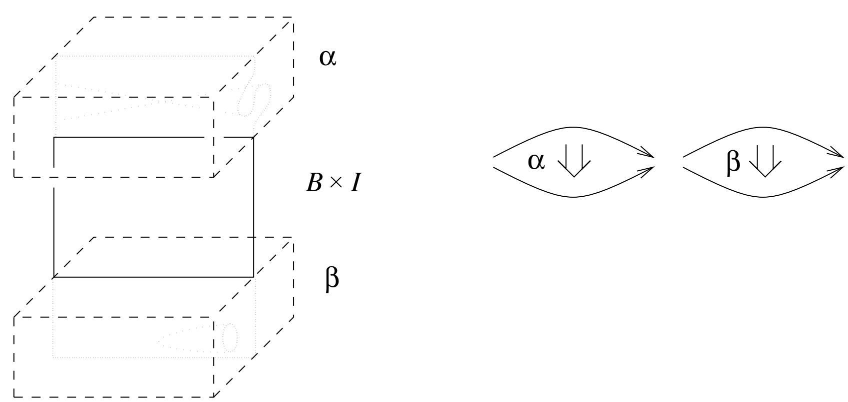


Figure 4: Horizontal composition of 2-morphisms

Vertical composition of 2-morphisms $\alpha: f \Rightarrow g$ and $\beta: g \Rightarrow h$ corresponds to gluing together 2-tangles representing α and β along a tangle representing g. For vertical composition to be well-defined, it is crucial that tangles with different height orderings of their local extrema cannot represent the same 1-morphism. This is built into Definition 2. Since we require 2-tangles to have a product structure near their source and target, this gluing indeed gives a smooth 2-tangle, as shown in Fig. 5.

More precisely, given 2-morphisms $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$ in $\mathcal{T}$, we define the vertical composite $\alpha \cdot \beta$ as follows. We choose representatives of these 2-morphisms, which we also call α and β, such that

$$\alpha \cap I_1 \times I_2 \times I_3 \times \{1\} = \beta \cap I_1 \times I_2 \times I_3 \times \{0\},$$

α has a product structure in the t direction for $t \in [1/4, 1]$, and β has a product structure in the t direction for $t \in [0, 3/4]$. Then $\alpha \cdot \beta$ is the 2-morphism represented by

$$(\alpha \cap I_1 \times I_2 \times I_3 \times [0, 1/2]) \cup (\beta \cap I_1 \times I_2 \times I_3 \times [1/2, 1]).$$

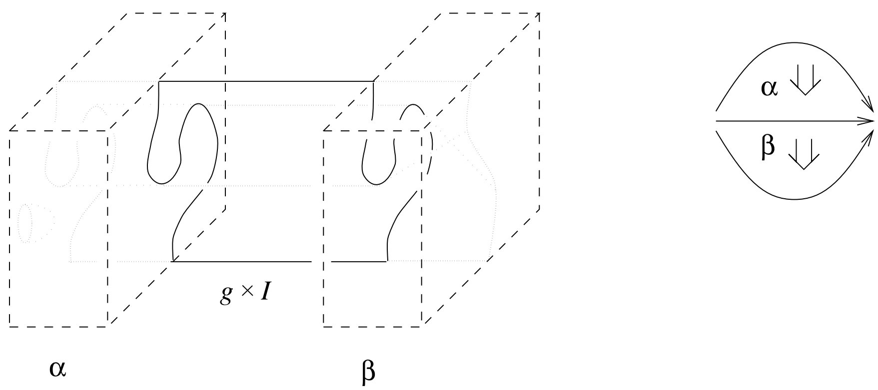


Figure 5: Vertical composition of 2-morphisms

Lemma 5. *Composition of 1-morphisms and horizontal and vertical composition of 2-morphisms are well defined.*

Proof - Suppose we have a composable pair of 1-morphisms. To check that their composite is well defined, choose a pair of generic tangles f, g representing these 1-morphisms and satisfying the conditions in the definition of composition, and also choose another pair, say f' and g'. Let $H(x, y, z, s)$ be an equivalence isotopy between f and f' with a product structure in the z direction for $z \in [1/4, 1]$ — by which we mean that $H(x, y, z, s) = (X(x, y, s), Y(x, y, s), z)$ for $z \in [1/4, 1]$. Similarly, let $J(x, y, z, s)$ be an equivalence isotopy between g and g' with a product structure in the z direction for $z \in [0, 3/4]$. Glue these together as follows:

$$K(x, y, z, s) = (1 - \phi(z))H(x, y, z, s) + \phi(z)J(x, y, z, s)$$

where $\phi\colon [0, 1] \to [0, 1]$ is a smooth monotone function that equals 0 on $[0, 1/4 + \epsilon]$ and 1 on $[3/4 - \epsilon, 1]$. We claim that K is an equivalence isotopy between the tangles

$$(f \cap I_1 \times I_2 \times [0, 1/2]) \cup (g \cap I_1 \times I_2 \times [1/2, 1])$$

and

$$(f' \cap I_1 \times I_2 \times [0,1/2]) \cup (g' \cap I_1 \times I_2 \times [1/2,1]).$$

Clearly K is a homotopy carrying the first tangle to the second. The fact that K has a product structure in a neighborhood of $I_1 \times I_2 \times \partial(I_3)$ follows from the same properties for H and J. Similarly, one can show that K is level preserving using the fact that restricted to any given value of z, K is a convex linear combination of the level-preserving maps H and J.

Finally we check that K is really an ambient isotopy. Note that for each fixed s, the z coordinate of $H(x,y,z,s)$ depends only on z, defining a monotone increasing diffeomorphism of the interval. Similarly, for each fixed z and s, the y coordinate of $H(x,y,z,s)$ depends only on y, defining a monotone increasing diffeomorphism of the interval. Also, for each fixed y, z and s, the x coordinate of $H(x,y,z,s)$ depends on x, defining a monotone increasing diffeomorphism of the interval. The same properties also hold for J. Since K is a convex linear combination of H and J for each fixed z, it follows that K has these properties as well, so $K(\cdot, s)$ is a diffeomorphism for each s.

We can construct similar equivalence isotopies to show that horizontal and vertical composition of 2-morphisms are well defined. Since this is where the height function for tangles is used, we give a fairly complete sketch of the proof for the more interesting case of vertical composition.

Suppose we have a vertically composable pair of 2-morphisms. To show that their composite is well defined, choose a pair of generic 2-tangles α, β representing them and satisfying the conditions in the definition of composition, and also choose another such pair, α' and β'. In particular, α and α' must have a product structure in the t direction for $t \in [1/4, 1]$, while β and β' have a product structure in the t direction for $t \in [0, 3/4]$. Let H be an equivalence isotopy between α and α' that has a product structure in the t direction for $t \in [1/4, 1]$, meaning that $H(x,y,z,t,s) = (X(x,y,z,s), Y(x,y,z,s), Y(x,y,z,s), t)$ for t in this interval. Similarly, let J be an equivalence isotopy between β and β' with a product structure in the t direction for $t \in [0, 3/4]$. Then we glue H and J to get a homotopy

$$K(x,y,z,t,s) = (1 - \phi(t))H(x,y,z,s) + \phi(t)J(x,y,z,s)$$

where $\phi\colon [0,1] \to [0,1]$ is a smooth monotone function that equals 0 on $[0, 1/4 + \epsilon]$ and 1 on $[3/4 - \epsilon, 1]$.

We claim that K is an equivalence isotopy between the 2-tangle formed by gluing α and β together and that formed by gluing α' and β' together. Clearly K is a homotopy carrying the first 2-tangle to the second. The conditions for K to be an equivalence isotopy are trivially satisfied for t outside $[1/4, 3/4]$. For t in this interval, both H and J have a product structure in the t direction, so we may define a homotopy K_t to be the restriction of K to a specific value of t in this interval. Since ϕ depends only t, this homotopy K_t is a convex linear combination of the corresponding isotopies H_t and J_t. Since H and J have a product structure for t in this interval, and H_1 and J_0 are equivalences of generic tangles, the isotopies H_t and J_t are also equivalences of generic tangles, and in particular they are level preserving. Being a convex linear combination of level-preserving ambient isotopies, K_t is itself a level-preserving ambient isotopy. It follows that K is an ambient isotopy and that the restriction of K to $I_1 \times I_2 \times \{z_0\} \times I_4$ is level preserving for $z_0 = 0$ and 1. Similarly, K has a product structure near $I_1 \times I_2 \times \partial(I_3 \times I_4)$ because H and J do.

To show that $\alpha\beta$ is well defined, we define a similar isotopy, pasting together isotopies of representatives of α and β along $1/4 \leq z \leq 3/4$. The product structure and level-preserving properties of these isotopies imply that the resulting map is an equivalence isotopy, using an argument similar to the previous ones. $\square$

2.1.5 Verifying the conditions

We conclude by checking that the structures described above make $\mathcal{T}$ into a 2-category.

Lemma 6. *$\mathcal{T}$ is a 2-category.*

Proof - By Lemma 5, composition of 1-morphisms and horizontal and vertical composition of 2-morphisms are well defined. One can easily check that the composites fg, $\alpha \cdot \beta$ and $\alpha\beta$ have the desired source and target. In addition, the property that a generic 2-tangle intersected with $I_1 \times I_2 \times \partial I_3 \times I_4$ consists of finitely many straight lines of the form of the form $(x, y, z) \times I_4$ implies that

$$\text{target}(\text{source}(\alpha)) = \text{target}(\text{target}(\alpha))$$

and

$$\text{source}(\text{target}(\alpha)) = \text{source}(\text{source}(\alpha))$$

for any 2-morphism α.

We define identity 1-morphisms and 2-morphisms as shown in Fig. 6. Given an object represented by the generic finite subset A of the square, we let the identity of that object be the 1-morphism represented by the generic tangle $1_A = A \times I_3$. Similarly, given a morphism represented by the generic tangle f, we let the identity of that morphism be the 2-morphism represented by the generic 2-tangle $1_f = f \times I_4$. Given an equivalence isotopy between two generic finite subsets A and A' of the square, we can take its product with I_3 to get an equivalence isotopy between 1_A and $1_{A'}$, and similarly an equivalence isotopy between generic tangles f and f' gives an equivalence isotopy between 1_f and $1_{f'}$, so the identity of an object or 1-morphism is well defined.

For any $f\colon A \to B$, the identity 1-morphisms satisfy $1_A f = f 1_B = f$, as shown in Fig. 7. More precisely, notice that we can find representatives of f that are straight on specified intervals of the form $[0, \epsilon]$ or $[\epsilon, 1]$ for any $\epsilon \in (0, 1)$; a representative that is

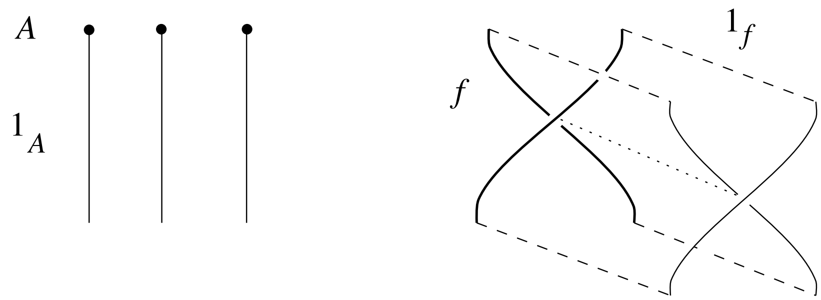


Figure 6: Identity 1-morphisms and 2-morphisms

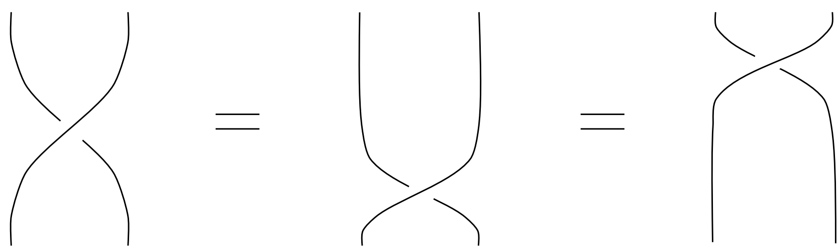


Figure 7: $f = 1_A f = f 1_A$

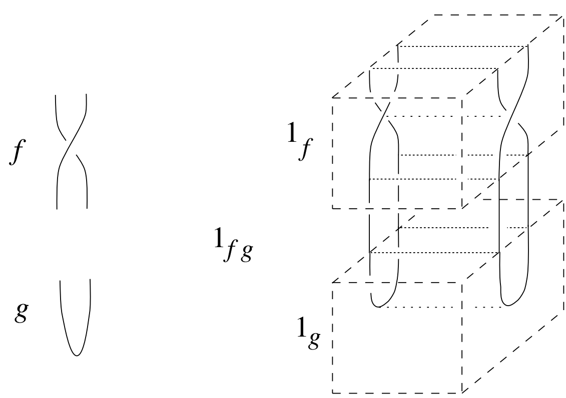


Figure 8: $1_{fg} = 1_f 1_g$

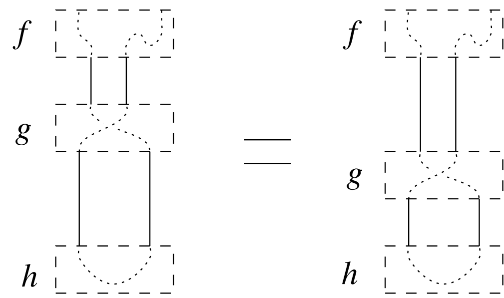


Figure 9: $(fg)h = f(gh)$

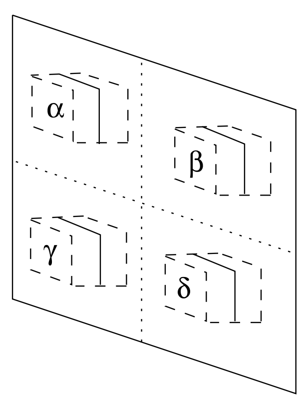


Figure 10: $(\alpha \cdot \beta)(\gamma \cdot \delta) = (\alpha\gamma) \cdot (\beta\delta)$

straight on $[0, 3/4]$ would also clearly be a representative of $1_A f$, and a representative that is straight on $[1/4, 1]$ is, likewise, clearly a representative of $f 1_B$. Similarly, for any $f, g\colon A \to B$ and $\alpha\colon f \Rightarrow g$, the identity 2-morphisms satisfy $1_f \cdot \alpha = \alpha \cdot 1_g = \alpha$ and $1_{1_A}\alpha = \alpha 1_{1_B} = \alpha$.

We also have $1_{fg} = 1_f 1_g$ whenever the composite 1-morphism fg is well-defined. To see this, recall that in the definition of composition of 1-morphisms, we take representatives of f and g that have a product structure on certain intervals in the z direction; likewise, in the definition of horizontal composition of 2-morphisms we take representatives of 1_f and 1_g that have a product structure on the same intervals in the z direction. If we use representatives of f and g to obtain representatives of 1_f and 1_g as above, then the 2-morphisms 1_{fg} and $1_f 1_g$ are represented by the same surface $(fg) \times I_4$, as shown in Fig. 8.

The operations of composition of 1-morphisms and horizontal and vertical composition of 2-morphisms are associative. This is true in each case because the equivalence relations for 1-morphisms and 2-morphisms allow for smooth changes in the coordinate directions, if small neighborhoods of the source and target are fixed. This is sufficient to transform a standard "follow the definitions" representative of one composition to a standard representative of the other, as shown in Fig. 9.

Finally we need to check the exchange identity. Let α, β, γ and δ be 2-morphisms such that the compostions below are defined. To see that

$$(\alpha \cdot \beta)(\gamma \cdot \delta) = (\alpha\gamma) \cdot (\beta\delta),$$

consider representatives of the 2-morphism $(\alpha \cdot \beta)(\gamma \cdot \delta)$ that has a product structure in the t direction outside small regions representing α, β, γ and δ, as in Fig. 10. Splitting this horizontally, it is clear that this is a representative of $(\alpha \cdot \beta)(\gamma \cdot \delta)$, but splitting it vertically, it is also clear that this is a representative of $(\alpha\gamma) \cdot (\beta\delta)$. □

2.2 $\mathcal{T}$ is a Monoidal 2-Category

By a *monoidal 2-category*, we mean a semistrict monoidal 2-category as defined by Kapranov and Voevodsky [20]. The more compact formulation later given in HDA1 can be unpacked to give precisely their definition. In the following sections we first introduce some extra structures on the 2-category $\mathcal{T}$, and then prove they make it into a monoidal 2-category. These structures are:

1. An object I, called the *unit object*.

2. For any objects A and B, an object $A \otimes B$. For any object A and 1-morphism $f\colon B \to C$, 1-morphisms

$$A \otimes f\colon A \otimes B \to A \otimes C, \qquad f \otimes A\colon B \otimes A \to C \otimes A.$$

For any object A and 2-morphism $\alpha\colon f \Rightarrow g$, 2-morphisms

$$A \otimes \alpha\colon A \otimes f \Rightarrow A \otimes g, \qquad \alpha \otimes A\colon f \otimes A \Rightarrow g \otimes A.$$

3. For any 1-morphisms $f\colon A \to A'$ and $g\colon B \to B'$, a 2-morphism

$$\bigotimes_{f,g}\colon (A \otimes g)(f \otimes B') \Rightarrow (f \otimes B)(A' \otimes g),$$

called the *tensorator*.

2.2.1 The unit object

We define I to be the object represented by the empty set.

2.2.2 Tensoring by an object

Given objects $A, B \in \mathcal{T}$, choose representatives — which by abuse of language we also call A and B — such that the y coordinate of every point in A is less than the y coordinate of every point in B. Then we define $A \otimes B \in \mathcal{T}$ to be the object represented by $A \cup B$. To simplify notation, we sometimes suppress the $\otimes$ in writing the tensor product of objects; that is, $A \otimes B$ may be written as AB.

Given an object A and a 1-morphism $f\colon B \to C$ in $\mathcal{T}$, the 1-morphism $A \otimes f\colon A \otimes B \to A \otimes C$ is represented by a tangle with 1_A to the left of f. More precisely, choose representatives such that the y coordinates of every point in A is less than the y coordinate of every point in f. Then we define $A \otimes f$ to be the 1-morphism represented by $(A \times I_3) \cup f$. Clearly the source and target of $A \otimes f$ are $A \otimes B$ and $A \otimes C$, respectively. The product $f \otimes A\colon B \otimes A \to C \otimes A$ is defined similarly using a tangle with 1_A to the right of f.

Given an object A and a 2-morphism $\alpha\colon f \Rightarrow g$ in $\mathcal{T}$, the 2-morphism $A \otimes \alpha\colon A \otimes f \to A \otimes g$ is represented by a 2-tangle with 1_{1_A} to the left of α. More precisely, choose representatives such that the y coordinate of every point of A is less than the y coordinate of every point of α. Then we define $A \otimes \alpha$ to be the 2-morphism represented by $(A \times I_3 \times I_4) \cup \alpha$. Clearly the source and target of $A \otimes \alpha$ are $A \otimes f$ and $A \otimes g$, respectively. The product $\alpha \otimes A\colon f \otimes A \Rightarrow g \otimes B$ is defined similarly using a 2-tangle with 1_{1_A} to the right of α.

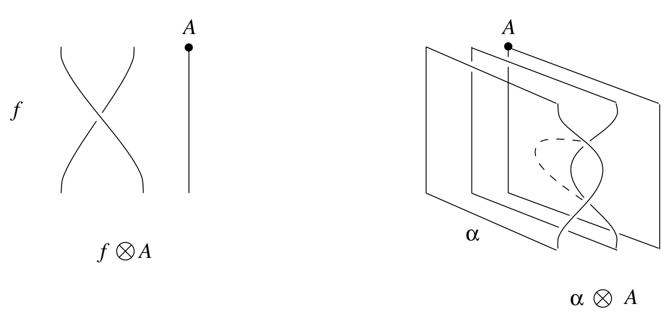


Figure 11: Tensor product of an object A with a 1-morphism f and a 2-morphism α

Lemma 7. *The tensor product of an object, 1-morphism or 2-morphism of $\mathcal{T}$ by an object of $\mathcal{T}$ is well defined.*

Proof - We outline the proof for the tensor product $A \otimes \alpha$ of an object A and a 2-morphism α; the other results follow from similar arguments. Let A, A' be representatives of the object A and let α, α' be representatives of the 2-morphism α, such that the y coordinate of every point in A is less than some number y_0, the y coordinate of every point in α is greater than y_0, and similarly for A' and α' for some number y_0'. Without loss of generality we assume that $y_0 \leq y_0'$. Let J be an equivalence isotopy between α and α' such that $J(x,y,z,t,s) = (x,y,z,t)$ for all $y < y_0$, and note that J is also an equivalence isotopy between $A \times I_3 \times I_4$ and itself. Let H be an equivalence isotopy between A and A' such that $H(x,y,s) = (x,y)$ for all $y > y_0'$, and let $K(x,y,z,t,s) = (H(x,y,s),z,t)$. Note that K is an equivalence isotopy between $A \times I_3 \times I_4$ and $A' \times I_3 \times I_4$, and also an equivalence isotopy between α' and itself. Following J by K we get an equivalence isotopy between $(A \times I_3 \times I_4) \cup \alpha$ and $(A' \times I_3 \times I_4) \cup \alpha'$. Thus $A \otimes \alpha$ is independent of the choice of representatives used to define it. $\square$

2.2.3 The tensorator

For any 1-morphisms $f\colon A \to A'$ and $g\colon B \to B'$ we define the tensorator

$$\bigotimes_{f,g}\colon (A \otimes g)(f \otimes B') \Rightarrow (f \otimes B)(A' \otimes g)$$

as follows. Take a representative of the 1-morphism $(A \otimes g)(f \otimes B')$ that consists of straight vertical lines outside small regions containing representatives of f and g. Let $H\colon I_1 \times I_2 \times I_3 \times [0,1] \to I_1 \times I_2 \times I_3$ be an isotopy that slides the region containing f up, and the region containing g down, until f is above g, as in Fig. 12. We may choose H to be independent of s near $s = 0, 1$, and also choose it so that

$$S = \{(H(x,y,z,s),s)\colon\ (x,y,z) \in (A \otimes g)(f \otimes B')\}$$

has generic projections $p(S)$ and $\pi(p(S))$. Then S represents a 2-morphism, which we define to be $\bigotimes_{f,g}$.

For a fixed choice of representatives of f and g, any two isotopies of the above type will determine the same 2-morphism $\bigotimes_{f,g}$, since they act only by shifting the heights of f and g. Also, any two representatives of f (or g) that satisfy the above conditions will be isotopic by an equivalence isotopy that is constant outside the regions specified for f and g, and this isotopy can be extended in a natural way to an equivalence isotopy between the two corresponding 2-tangles. Hence $\bigotimes_{f,g}$ is well defined. Since H is an isotopy, $\bigotimes_{f,g}$ is a 2-isomorphism, with $\bigotimes_{f,g}^{-1}$ being the 2-morphism represented by

$$\{(H(x,y,z,s),1-s)\colon\ (x,y,z) \in (A \otimes g)(f \otimes B')\}.$$

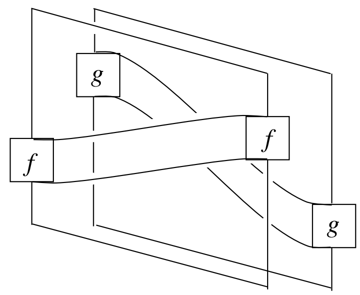


Figure 12: The tensorator $\bigotimes_{f,g}$

2.2.4 Verifying the conditions

We conclude by checking that the structures defined above make $\mathcal{T}$ into a monoidal 2-category.

Lemma 8. *$\mathcal{T}$ is a monoidal 2-category.*

Proof - By Lemma 6, $\mathcal{T}$ is a 2-category. To prove that the structures defined above make $\mathcal{T}$ into a monoidal 2-category, we check that it satisfies the following conditions listed in Lemma 4 of HDA1.

(i) For any object A, the maps $A \otimes -\colon \mathcal{T} \to \mathcal{T}$ and $- \otimes A\colon \mathcal{T} \to \mathcal{T}$ are 2-functors. We only consider the case of tensoring on the left by A, as the tensoring on the right is similar. First we check that tensoring by A preserves identities. Choosing appropriate representatives of A and B, the 1-morphism $A \otimes 1_B$ is represented by $(A \times I_3) \cup (B \times I_3)$. Since this equals $(A \cup B) \times I_3$, which represents $1_{A \otimes B}$, we have $A \otimes 1_B = 1_{A \otimes B}$. One can similarly check that $A \otimes 1_f = 1_{A \otimes f}$ for any 1-morphism f.

Next we should check that tensoring with A preserves all three forms of composition:

$$A \otimes fg = (A \otimes f)(A \otimes g),$$

$$A \otimes (\alpha\beta) = (A \otimes \alpha)(A \otimes \beta),$$

$$A \otimes (\alpha \cdot \beta) = (A \otimes \alpha) \cdot (A \otimes \beta)$$

Since the arguments for all three cases are similar, we consider only the third. Choose representatives of α and β that satisfy the conditions in the definition of vertical composition (the target tangle of α equals the source tangle of β, and α, β have a product structure in the t direction for $t \in [1/4, 1]$ and $t \in [0, 3/4]$, respectively) and lie to the right of a representative of A. Then the 2-tangle

$$(A \times I_3 \times I_4) \cup (\alpha \cap I_1 \times I_2 \times I_3 \times [0, 1/2]) \cup (\beta \cap I_1 \times I_2 \times I_3 \times [1/2, 1])$$

representing $A \otimes (\alpha \cdot \beta)$ and the 2-tangle

$$((A \times I_3 \times I_4 \cup \alpha) \cap I_1 \times I_2 \times I_3 \times [0, 1/2]) \cup ((A \times I_3 \times I_4 \cup \beta) \cap I_1 \times I_2 \times I_3 \times [1/2, 1]))$$

representing $(A \otimes \alpha) \cdot (A \otimes \beta)$ are equal, as desired.

(ii) For x any object, morphism or 2-morphism, $x \otimes I = I \otimes x = x$. Since I is represented by the empty set, its product with the intervals I_3 and I_4 is also empty, hence $x \otimes I = I \otimes x = x$ for any object, morphism or 2-morphism x.

(iii) For x any object, morphism or 2-morphism, and for any objects A and B, we have $A \otimes (B \otimes x) = (A \otimes B) \otimes x$, $A \otimes (x \otimes B) = (A \otimes x) \otimes B$ and $x \otimes (A \otimes B) = (x \otimes A) \otimes B$. This follows from the property that equivalence isotopies of objects, 1-morphisms and 2-morphisms all allow shifts in the y direction.

(iv) For any 1-morphisms $f\colon A \to A'$, $g\colon B \to B'$ and $h\colon C \to C'$, we have $\bigotimes_{A \otimes g,h} = A \otimes \bigotimes_{g,h}$, $\bigotimes_{f \otimes B,h} = \bigotimes_{f,B \otimes h}$ and $\bigotimes_{f,g \otimes C} = \bigotimes_{f,g} \otimes C$. The first follows from the fact that we may choose a representative of $\bigotimes_{A \otimes g,h}$ for which the tangle representing 1_A is straight, and the component containing it is flat and disjoint from the surface containing the representative of g. Arguments for the other cases are similar.

(v) For any objects A and B we have $1_A \otimes B = A \otimes 1_B = 1_{A \otimes B}$, and for any 1-morphisms $f\colon A \to A'$, $g\colon B \to B'$, we have $\bigotimes_{1_A,g} = 1_{A \otimes g}$ and $\bigotimes_{f,1_B} = 1_{f \otimes B}$. The properties for objects are clear from the definitions. The properties for 1-morphisms follow from the definition of the tensorator: sliding a straight vertical segment up or down has no effect, so the only effect of the isotopy defining $\bigotimes_{1_A,g}$ or $\bigotimes_{f,1_B}$ is to slide g or f up or down. This sliding is an equivalence isotopy, so the 2-tangle representing $\bigotimes_{1_A,g}$ or $\bigotimes_{f,1_B}$ is equivalent to one with a product structure in the t direction. It thus represents an identity 2-morphism.

(vi) For any 1-morphisms $f\colon A \to A'$, $g, h\colon B \to B'$ and any 2-morphism $\beta\colon g \Rightarrow h$,

$$((A \otimes \beta)(1_f \otimes B')) \cdot \bigotimes_{f,h} = \bigotimes_{f,g} \cdot ((f \otimes B)(A' \otimes \beta)).$$

(vii) For any 1-morphisms $f, g\colon A \to A'$, $h\colon B \to B'$ and any 2-morphism $\alpha\colon f \Rightarrow g$,

$$((A \otimes h)(\alpha \otimes B')) \cdot \bigotimes_{g,h} = \bigotimes_{f,h} \cdot ((\alpha \otimes B)(A' \otimes h)).$$

The arguments for (vi) and (vii) are similar, so we consider only the latter. A picture of the proof is shown in Fig. 13 below. We slide a piece of the surface that represents α in the standard representative of $((A \otimes h)(\alpha \otimes B')) \cdot \bigotimes_{g,h}$ along the path specified by $\bigotimes_{g,h}$, resulting in a surface in which α follows $\bigotimes_{f,h}$, and which represents $\bigotimes_{f,h} \cdot ((\alpha \otimes B)(A' \otimes h))$. Clearly, this sliding can be accomplished by an equivalence isotopy.

(viii) For any 1-morphisms $f\colon A \to A'$, $g\colon B \to B'$ and $g'\colon B' \to B''$,

$$\bigotimes_{f,gg'} = ((A \otimes g)\bigotimes_{f,g'}) \cdot (\bigotimes_{f,g}(A' \otimes g')),$$

and for any 1-morphisms $f\colon A \to A'$, $f'\colon A' \to A''$ and $g\colon B \to B'$,

$$\bigotimes_{ff',g} = (\bigotimes_{f,g}(f' \otimes B')) \cdot ((f \otimes B) \bigotimes_{f',g}).$$

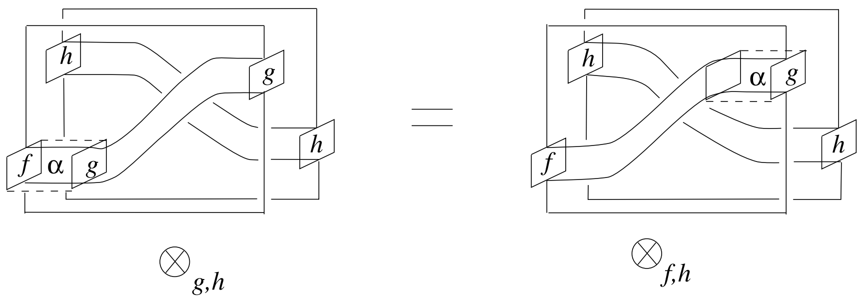


Figure 13: $((A \otimes h)(\alpha \otimes B')) \cdot \bigotimes_{g,h} = \bigotimes_{f,h} \cdot ((\alpha \otimes B)(A' \otimes h))$

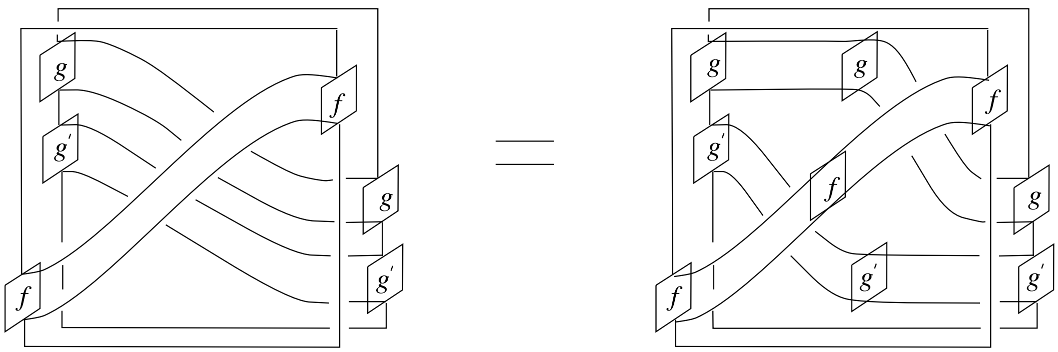


Figure 14: $\bigotimes_{f,gg'} = ((A \otimes g) \bigotimes_{f,g'}) \cdot (\bigotimes_{f,g}(A' \otimes g'))$

We only the check the first of these conditions, as the second is similar. Pictures of $((A \otimes g) \bigotimes_{f,g'}) \cdot (\bigotimes_{f,g}(A' \otimes g'))$ and $\bigotimes_{f,gg'}$ are shown in Fig. 14, for which the representatives of f, g and g' are straight outside small regions. Clearly there is an equivalence isotopy between these surfaces. □

2.3 $\mathcal{T}$ is a Braided Monoidal 2-Category

The definition of 'braided monoidal 2-category' has a somewhat complex history. The first definition was given by Kapranov and Voevodsky [20]. This definition was modified somewhat in HDA1. These modifications are necessary for the proper treatment of 2-tangles, and especially for an unambiguous statement of the Zamolodchikov tetrahedron equation, as had been noted by Breen [8]. Day and Street [16] later gave a more terse formulation of the definition in HDA1. Then Crans [15] noted an error in the proof of Theorem 18 of HDA1, and explained how to fix it by adding some conditions concerning the unit object to the definition of a braided monoidal 2-category. In what follows, by a 'braided monoidal 2-category' we mean a semistrict braided monoidal 2-category as defined by Crans.

In the following sections we first introduce some extra structures on the monoidal 2-category $\mathcal{T}$, and then prove they make it into a braided monoidal 2-category. These structures are:

1. For any objects A, B an equivalence $R_{A,B}\colon A \otimes B \to B \otimes A$, called the *braiding* of A and B.

2. For any 1-morphism $f\colon A \to A'$ and object B, a 2-isomorphism

$$R_{f,B}\colon (f \otimes B)R_{A',B} \Rightarrow R_{A,B}(B \otimes f)$$

   called the *braiding* of f and B, and for any object A and 1-morphism $g\colon B \to B'$, a 2-isomorphism

$$R_{A,g}\colon (A \otimes g)R_{A,B'} \Rightarrow R_{A,B}(g \otimes A)$$

   called the *braiding* of A and g.

3. For any objects A, B, and C, 2-isomorphisms

$$\tilde{R}_{(A|B,C)}\colon (R_{A,B} \otimes C)(B \otimes R_{A,C}) \Rightarrow R_{A,B\otimes C}$$

   and

$$\tilde{R}_{(A,B|C)}\colon (A \otimes R_{B,C})(R_{A,C} \otimes B) \Rightarrow R_{A\otimes B,C},$$

   called *braiding coherence 2-morphisms.*

2.3.1 Braiding for objects

Given objects $A, B \in \mathcal{T}$, we define the braiding $R_{A,B}\colon A \otimes B \to B \otimes A$ to be the 1-morphism represented by a tangle consisting of only positive crossings, such that each strand beginning at a point p of A crosses all strands beginning at points of B before any strand starting at a point of A to the left of p crosses any strands of B, as in Fig. 15. We define the 1-morphism $R^*_{A,B}$ by the reflection in z of a tangle representing $R_{A,B}$. Clearly, there are surfaces (which can be defined in terms of repeated Reidemeister II moves) that represent 2-isomorphisms between $R_{A,B}R^*_{A,B}$ and $1_{A\otimes B}$, and between $R^*_{A,B}R_{A,B}$ and $1_{B\otimes A}$. Thus $R_{A,B}$ is an equivalence.

2.3.2 Braiding for an object and a 1-morphism

Given a 1-morphism $f\colon A \to A'$ and an object B in $\mathcal{T}$, we define the 2-morphism

$$R_{f,B}\colon (f \otimes B)R_{A',B} \Rightarrow R_{A,B}(B \otimes f)$$

as follows. Consider a representative of the source $(f \otimes B)R_{A',B}$ for which f lies behind (has greater x values than) the strands beginning at B. Then there is an isotopy that moves f past the strands of B, as in Fig. 15. If $H(x, y, z, t)$ is such an isotopy, the generic 2-tangle representing $R_{f,B}$ is given by

$$\{(H(x, y, z, t), t)\colon\ (x, y, z) \in (f \otimes B)R_{A',B}\}.$$

Since this 2-tangle is traced out by an isotopy in this manner, $R_{A,f}$ has an inverse represented by the 2-tangle traced out by the reverse isotopy:

$$\{(H(x,y,z,1-t),t)\colon (x,y,z) \in (f \otimes B)R_{A',B}\}.$$

Given an object A and 1-morphism $g\colon B \to B'$, we define the 2-isomorphism

$$R_{A,g}\colon (A \otimes g)R_{A,B'} \Rightarrow R_{A,B}(g \otimes A)$$

in a similar way.

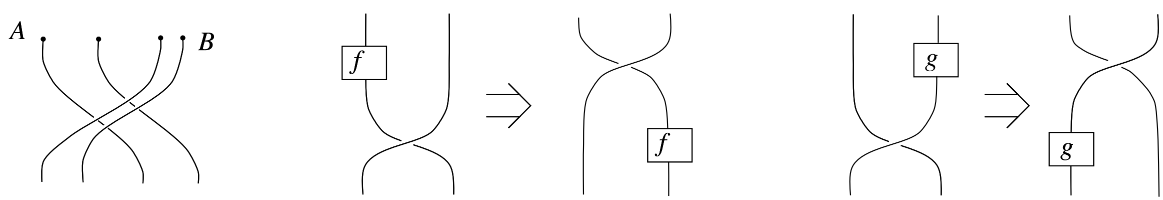


Figure 15: $R_{A,B}$, $R_{f,B}$ and $R_{A,g}$

2.3.3 Braiding coherence 2-morphisms

Given objects $A, B, C \in \mathcal{T}$, we define the 2-isomorphism

$$\tilde{R}_{(A|B,C)}\colon (R_{A,B} \otimes C)(B \otimes R_{A,C}) \Rightarrow R_{A,B\otimes C}$$

to be represented by a surface built from an isotopy that changes only the order of distant crossings (crossings with no strands in common) in a representative of $(R_{A,B} \otimes C)(B \otimes R_{A,C})$ to give a representative of $R_{A,B\otimes C}$. Since this 2-morphism is defined in terms of an isotopy, we can construct an inverse for it using the isotopy with t reversed, so it is a 2-isomorphism. Note that if Z is the object represented by a single point, $\tilde{R}_{(Z|B,C)}$ is an identity 2-morphism. We illustrate a nontrivial case of $\tilde{R}_{(A|B,C)}$ in Fig. 16.

Given objects $A, B, C \in \mathcal{T}$, we define $\tilde{R}_{(A,B|C)}$ to be the identity. This makes sense because its source and target are equal: $(A \otimes R_{B,C})(R_{A,C} \otimes B) = R_{A\otimes B,C}$.

2.3.4 Verifying the conditions

We conclude by checking that the structures defined above make $\mathcal{T}$ into a braided monoidal 2-category.

Lemma 9. *$\mathcal{T}$ is a braided monoidal 2-category.*

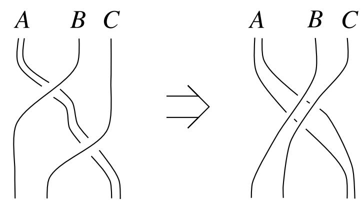


Figure 16: $\tilde{R}_{(A|B,C)}$

Proof - By Lemma 8, $\mathcal{T}$ is a monoidal 2-category. To prove that the structures defined above make $\mathcal{T}$ into a braided monoidal 2-category, we verify the conditions listed in Lemma 7 of HDA1, together with the conditions added by Crans. As in HDA1, we list some of these conditions using the 'hieroglyphic' notation of Kapranov and Voevodsky.

($\rightarrow \otimes \rightarrow$) *For any 1-morphisms $f\colon A \to A'$ and $g\colon B \to B'$, we have*

$$((f \otimes B)R_{A',g}) \cdot (R_{f,B}(g \otimes A')) \cdot (R_{A,B} \bigotimes_{g,f})$$

$$= (\bigotimes_{f,g}^{-1} R_{A',B'}) \cdot ((A \otimes g)R_{f,B'}) \cdot (R_{A,g}(B' \otimes f)).$$

This equation corresponds to the equivalence of the 2-tangles shown in Fig. 17. Since f lies behind g (i.e., the value of the x coordinate on points in the tangle representing f is greater than on points of that representing g), it is clear that the 2-tangles shown are equivalent.

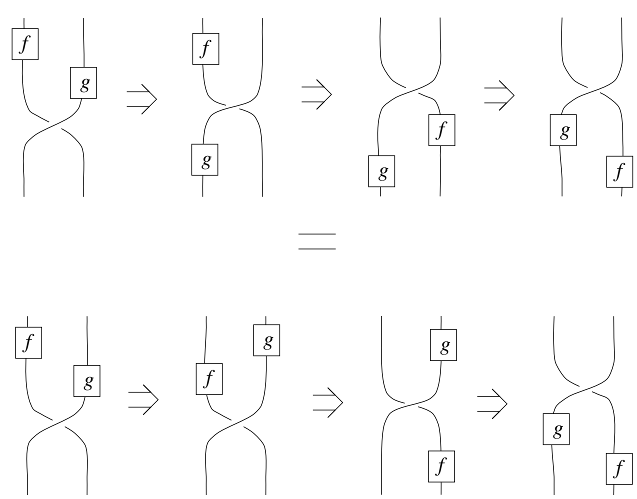


Figure 17: ($\rightarrow \otimes \rightarrow$)

($\bullet\otimes \Downarrow$) *For any 1-morphisms $f, f'\colon A \to A'$, 2-morphism $\alpha\colon f \Rightarrow f'$, and object B, we have*

$$R_{f,B} \cdot (R_{A,B}(B \otimes \alpha)) = ((\alpha \otimes B)R_{A',B}) \cdot R_{f',B}.$$

This equation corresponds to the equivalence of the 2-tangles shown in Fig. 18.

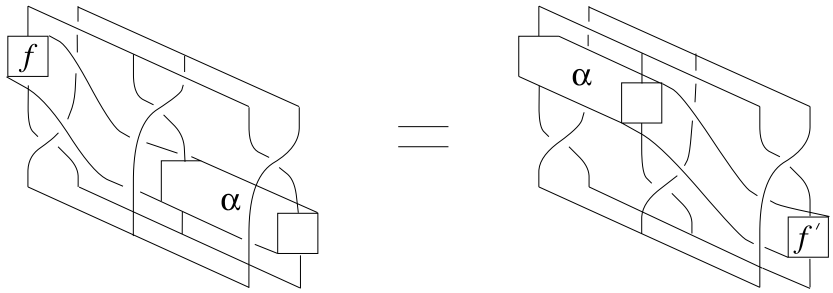


Figure 18: $(\bullet\otimes \Downarrow)$

$(\Downarrow \otimes\bullet)$ This is similar to $(\bullet\otimes \Downarrow)$, and follows from an analogous argument.

$(\to\to \otimes\bullet)$ *For any pair of 1-morphisms $f\colon A \to A'$, $f'\colon A' \to A''$ and object B, we have*

$$((f \otimes B)R_{f',B}) \cdot (R_{f,B}(B \otimes f')) = R_{ff',B}.$$

This equation corresponds to the equivalence of the 2-tangles illustrated in Fig. 19. Since the tangles representing f and f' lie behind those representing the braidings $R_{A,B}$, $R_{A',B}$ and $R_{A'',B}$, it is clear that there is an isotopy between the 2-tangles shown.

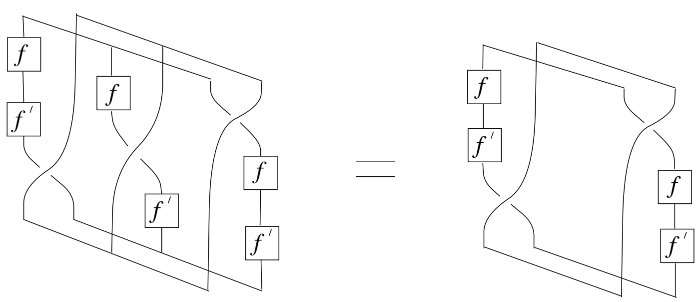


Figure 19: $(\to\to \otimes\bullet)$

$(\bullet\otimes \to\to)$ This is similar to $(\to\to \otimes\bullet)$.

$(\bullet \otimes (\bullet\otimes \to))$ *For any objects A, B, C and 1-morphism $f\colon C \to C'$, we have*

$$((AB \otimes f)\tilde{R}_{(A|B,C')}) \cdot R_{A,B\otimes f} =$$

$$((\bigotimes\nolimits_{R_{A,B},f}(B \otimes R_{A,C'})) \cdot ((R_{A,B} \otimes C)(B \otimes R_{A,f})) \cdot (\tilde{R}_{(A|B,C)}(B \otimes f \otimes A)).$$

This equation corresponds to the equivalence of the 2-tangles shown in Fig. 20.

$((\bullet \otimes \bullet)\otimes \to)$ *For any objects A, B, C and 1-morphism $f\colon C \to C'$, we have*

$$(A\otimes B\otimes f)\tilde{R}_{(A,B|C')} \cdot R_{A\otimes B,f} = ((A\otimes R_{B,f})(R_{A,C'}\otimes B)) \cdot ((A\otimes R_{B,C})(R_{A,f}\otimes B)) \cdot \tilde{R}_{(A,B|C)}.$$

This is similar to $(\bullet \otimes (\bullet\otimes \rightarrow))$, but simpler, because $R_{(A,B|C)}$ and $R_{(A,B|C')}$ are identity 2-morphisms.

$(\rightarrow \otimes(\bullet \otimes \bullet))$, $((\rightarrow \otimes\bullet) \otimes \bullet)$, $((\bullet\otimes \rightarrow) \otimes \bullet)$, $(\bullet \otimes (\rightarrow \otimes\bullet))$ These conditions are similar to the relations $(\bullet \otimes (\bullet\otimes \rightarrow))$ and $((\bullet \otimes \bullet)\otimes \rightarrow)$, and and are proved analogously.

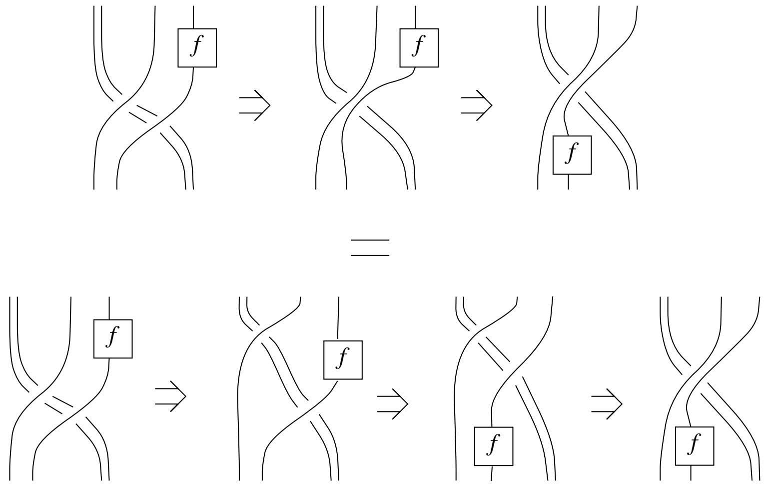


Figure 20: $(\bullet \otimes (\bullet\otimes \rightarrow))$

$((\bullet\otimes\bullet\otimes\bullet)\otimes\bullet)$ This condition holds trivially, since all of the 2-morphisms involved are of the form $\tilde{R}_{(\cdot,\cdot|\cdot)}$, which are identity morphisms.

$(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$ *For any objects A, B, C, D, we have*

$$((R_{A,B}\otimes C\otimes D)(B\otimes \tilde{R}_{(A|C,D)})\cdot \tilde{R}_{(A|B,C\otimes D)} = ((\tilde{R}_{(A|B,C)}\otimes D)(B\otimes C\otimes R_{A,D}))\cdot \tilde{R}_{(A|B\otimes C,D)}.$$

This equation corresponds to the equivalence of 2-tangles shown in Fig. 21.

$((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$ *For any objects A, B, C, D, we have*

$$((A\otimes R_{B,C}\otimes D)\bigotimes_{R_{A,C},R_{B,D}}(C\otimes R_{A,D}\otimes B))\cdot((\tilde{R}_{(A,B|C)}\otimes D)(C\otimes \tilde{R}_{(A,B|D)}))\cdot \tilde{R}_{(A\otimes B|C,D)}$$

$$= ((A \otimes \tilde{R}_{(B|C,D)})(\tilde{R}_{(A|C,D)} \otimes B)) \cdot \tilde{R}_{(A,B|C\otimes D)}.$$

Using the fact that 2-morphisms of the form $\tilde{R}_{(\cdot,\cdot|\cdot)}$ are identities, this equation simplifies to

$$((A \otimes R_{B,C} \otimes D)\bigotimes_{R_{A,C},R_{B,D}}(C \otimes R_{A,D} \otimes B)) \cdot \tilde{R}_{(A\otimes B|C,D)}$$

$$= (A \otimes \tilde{R}_{(B|C,D)})(\tilde{R}_{(A|C,D)} \otimes B).$$

When the objects A, B are represented by a single point, the right side of this equation is an identity 2-morphism because $\tilde{R}_{(B|C,D)}$ and $\tilde{R}_{(A|C,D)}$ are identities. The left side

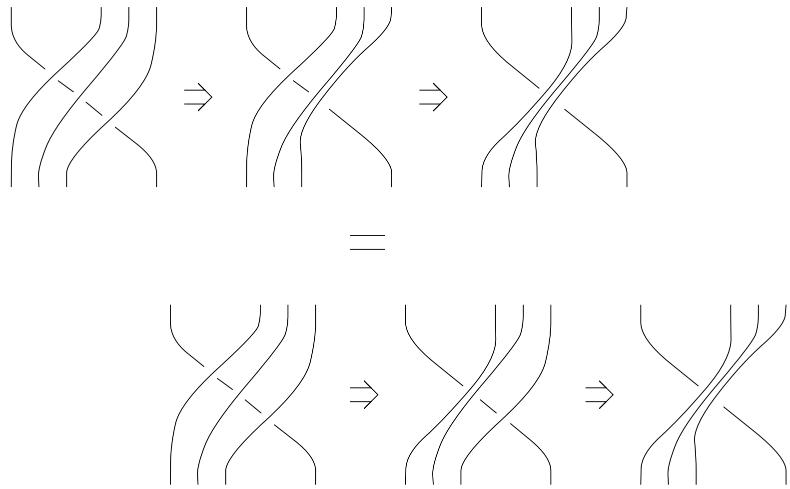


Figure 21: $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$

is also an identity 2-morphism, because $\tilde{R}_{(A\otimes B|C,D)}$ is the inverse of the other factor. Thus the equation is true in this case. The result for other objects A, B follows similarly, and indeed follows inductively from the case we just considered.

$S^{+}_{A,B,C} = S^{-}_{A,B,C}$. *For any objects* A, B, C, *we have*

$$(\tilde{R}^{-1}_{(A|B,C)}(R_{B,C} \otimes A)) \cdot R^{-1}_{A,R_{B,C}} \cdot ((A \otimes R_{B,C})\tilde{R}_{(A|B,C)}) =$$

$$((R_{A,B} \otimes C)\tilde{R}_{(A,B|C)}) \cdot R_{R_{A,B},C} \cdot (\tilde{R}^{-1}_{(A,B|C)}(C \otimes R_{A,B})).$$

This condition says that the isotopies corresponding to two different factorizations of the Reidemeister III move give rise to equivalent 2-tangles. Using the fact that 2-morphisms of the form $\tilde{R}_{(\cdot,\cdot|\cdot)}$ are identities, it corresponds to the equivalence of the 2-tangles shown in Fig. 22.

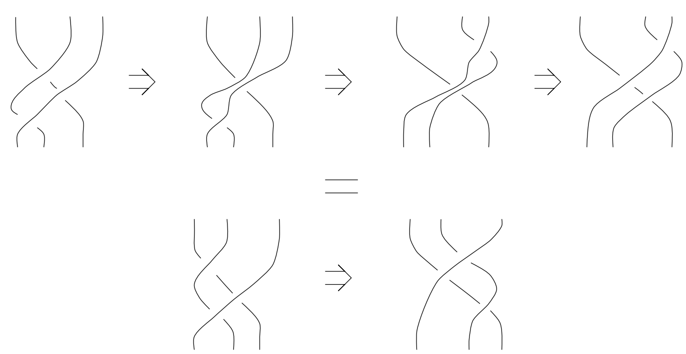


Figure 22: $S^{+}_{A,B,C} = S^{-}_{A,B,C}$

Lastly, we need to check the extra conditions introduced by Crans. These say that $R_{\cdot,\cdot}$, $\tilde{R}_{(\cdot|\cdot,\cdot)}$, and $\tilde{R}_{(\cdot,\cdot|\cdot)}$ are the identity whenever one of the arguments is the unit object I. These follow from the fact that I is represented by the empty set. $\square$

2.4 $\mathcal{T}$ is a Monoidal 2-Category with Duals

Now we introduce still more structure on $\mathcal{T}$ and show that this makes $\mathcal{T}$ into a 'monoidal 2-category with duals'. This is a categorification of the concept of 'monoidal category with duals', which was discussed in HDA0 and HDA2. Before giving a precise definition of a monoidal 2-category with duals, let us sketch the key points, using $\mathcal{T}$ as an example.

There are three levels of duality in $\mathcal{T}$. First, we can form the dual of an object in $\mathcal{T}$ by reflecting a set representing it in the y direction. Second, we can form the dual of a 1-morphism in $\mathcal{T}$ by reflecting a generic tangle representing it in the z direction. Third, we can form the dual of a 2-morphism in $\mathcal{T}$ by reflecting a generic 2-tangle representing it in the t direction.

In addition, the dual of any object A comes equipped with 'unit' and 'counit' 1-morphisms:

$$i_A\colon I \to A \otimes A^*, \qquad e_A\colon A^* \otimes A \to I,$$

familiar from other contexts, such as monoidal categories with duals. If Z is the object corresponding to a single point in the unit square, the unit i_Z corresponds to a tangle with one strand and a single maximum, while the counit e_Z corresponds to a tangle with one strand and a single minimum, as shown in Fig. 23. In singularity theory, the singularity occurring at such a maximum or minimum is called a 'fold'.

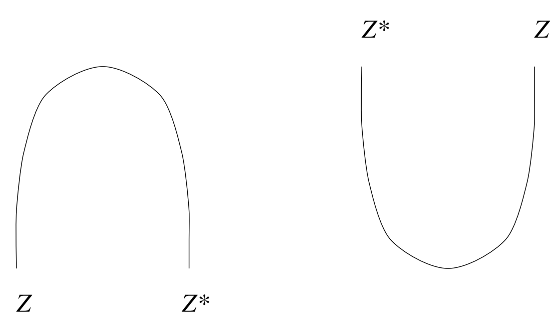
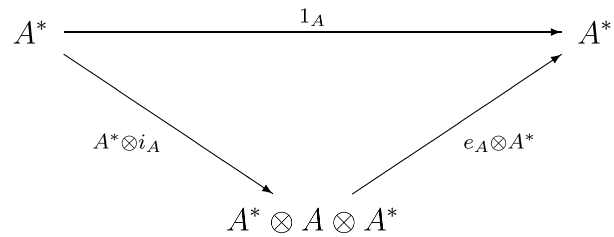


Figure 23: Folds corresponding to the unit i_Z and counit e_Z

In a monoidal category with duals, the unit and counit satisfy 'triangle identities' saying the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ & {\scriptstyle i_A \otimes A} \searrow \quad \nearrow {\scriptstyle A \otimes e_A} & \\ & A \otimes A^* \otimes A & \end{array}$$



In a monoidal 2-category with duals these diagrams no longer commute on the nose. Instead, they commute up to specified 2-isomorphisms. Using duality it turns out to be sufficient to consider only the first, so we demand the existence of 2-isomorphism

$$T_A\colon (i_A \otimes A)(A \otimes e_A) \Rightarrow 1_A$$

called the 'triangulator'. When Z is the object corresponding to a single point, T_Z corresponds to the 2-tangle shown in Fig. 24. This 2-tangle describes the process of cancellation of two folds, a maximum and a minimum. The singularity occurring at the moment of cancellation is known as a 'cusp'. In this context, Carter, Rieger and Saito call it a 'cusp on a fold line'.

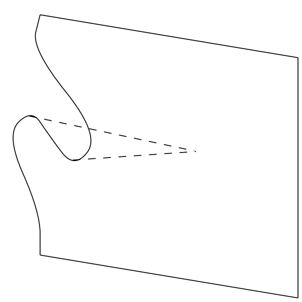


Figure 24: Cusp corresponding to the triangulator T_Z

As usual, when we categorify and replace equations by specified isomorphisms, the isomorphisms should satisfy new equations of their own, called coherence laws. Fig. 25 depicts the most interesting coherence law satisfied by the triangulator in the case $A = Z$. The left side of the equation is a 2-tangle with two cusps, while the right side has no cusps. The equation arises from an isotopy between these two 2-tangles corresponding to a cancellation of cusps. The singularity occurring when the cusps cancel is called a 'swallowtail'.

There is a fascinating recursive pattern here. The fold e_Z describes the process of cancellation of the object Z and its dual Z^*. The cusp T_Z describes the process of cancellation of two folds. Similarly, the swallowtail coherence law for T_Z comes from the process of cancellation of two cusps. It will be interesting to see if this pattern continues in higher-dimensional situations.

There is one more piece of structure in a monoidal 2-category with duals. Namely, the dual of a 1-morphism $f\colon A \to B$ comes equipped with 'unit' and 'counit' 2-

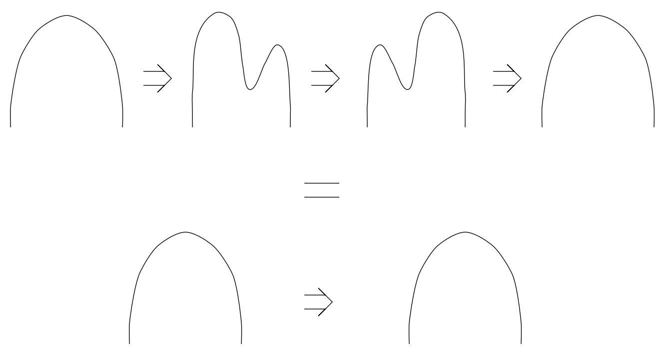


Figure 25: Swallowtail coherence law for the triangulator T_Z

morphisms:

$$i_f\colon 1_A \Rightarrow ff^*, \qquad e_f\colon f^*f \Rightarrow 1_B.$$

These units and counits satisfy the triangle identities strictly, as equations. When f is the braiding $R_{Z,Z}$, the unit i_f and counit e_f correspond to two forms of the Reidemeister II move, as shown in Fig. 26. When f is the unit i_Z, the unit i_f and counit e_f correspond to surfaces with a minimum and a saddle point, respectively, as shown in Fig. 27. The minimum is also called 'the birth of a circle'. Finally, when f is the counit e_Z, the counit e_f and unit i_f correspond to surfaces with a maximum and a different sort of saddle point, as shown in Fig. 28. The maximum is also called 'the death of a circle'. The triangle identities satisfied by i_{i_Z}, e_{i_Z}, i_{e_Z} and e_{e_Z} correspond to cancellation of critical points as in Morse theory, with the t coordinate serving as the Morse function.

There is no unit or counit associated to the dual of a 2-morphism, since these would have to be 3-morphisms. In general, we expect duality to be 'truncated' like this in a monoidal n-category with duals: there should be duals of objects, 1-morphisms, and so on up to n-morphisms, with units and counits for these duals except at the n-morphism level.

Finally, there are various coherence laws that need to be satisfied. Apart from the swallowtail, these are of four forms. First, there are formulas for the duals of the 1-morphisms i_A, e_A and the 2-morphisms i_f, e_f, and T_A. Second, there are compatibility relations between duality and the various forms of composition: tensoring, composition of 1-morphisms, and vertical and horizontal composition of 2-morphisms. Third, the triangulator of the unit object is an identity 2-morphism. Fourth, there is a compatibility relation between duality for 1-morphisms and 2-morphisms, which is item 12 in the definition below.

The precise definition is as follows:

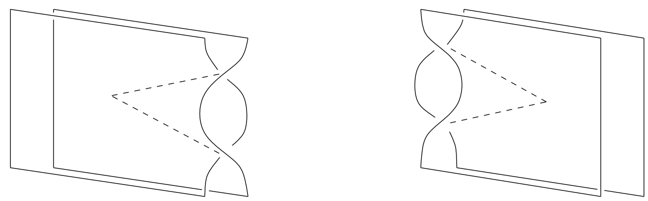


Figure 26: 2-Tangles corresponding to $i_{R_{Z,Z}}$ and $e_{R_{Z,Z}}$

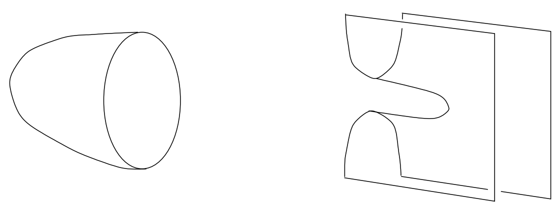


Figure 27: 2-Tangles corresponding to i_{i_Z} and e_{i_Z}

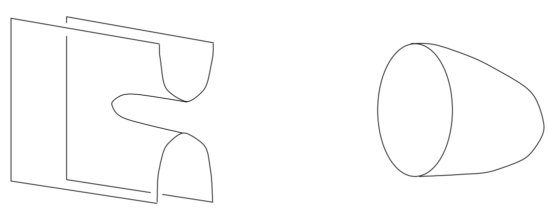


Figure 28: 2-Tangles corresponding to i_{e_Z} and e_{e_Z}

Definition 10. *A* monoidal 2-category with duals *is a monoidal 2-category equipped with the following structures:*

1. For every 2-morphism $\alpha\colon f \Rightarrow g$ there is a 2-morphism $\alpha^*\colon g \Rightarrow f$ called the *dual* of α.

2. For every morphism $f\colon A \to B$ there is a morphism $f^*\colon B \to A$ called the *dual* of f, and 2-morphisms $i_f\colon 1_A \Rightarrow ff^*$ and $e_f\colon f^*f \Rightarrow 1_B$, called the *unit* and *counit* of f, respectively.

3. For any object A, there is a object A^* called the *dual* of A, 1-morphisms $i_A\colon I \to A \otimes A^*$ and $e_A\colon A^* \otimes A \to I$ called the *unit* and *counit* of A, respectively, and a 2-morphism $T_A\colon (i_A \otimes A)(A \otimes e_A) \Rightarrow 1_A$ called the *triangulator* of A.

We say that a 2-morphism α is unitary *if it is invertible and $\alpha^{-1} = \alpha^*$. Given a 2-morphism $\alpha\colon f \Rightarrow g$, we define the* adjoint $\alpha^\dagger\colon g^* \Rightarrow f^*$ *by*

$$\alpha^\dagger = (g^* i_f) \cdot (g^* \alpha f^*) \cdot (e_g f^*).$$

In addition, the structures above are required to satisfy the following conditions:

1. $X^{**} = X$ for any object, morphism or 2-morphism X.

2. $1_X^* = 1_X$ for any object or morphism X.

3. For all objects A, B, 1-morphisms f, g, and 2-morphisms α, β for which both sides of the following equations are well-defined, we have

$$(\alpha \cdot \beta)^* = \beta^* \cdot \alpha^*,$$

$$(\alpha\beta)^* = \alpha^*\beta^*,$$

$$(fg)^* = g^* f^*,$$

$$(A \otimes \alpha)^* = A \otimes \alpha^*, \qquad (\alpha \otimes A)^* = \alpha^* \otimes A,$$

$$(A \otimes f)^* = A \otimes f^*, \qquad (f \otimes A)^* = f^* \otimes A,$$

   and

$$(A \otimes B)^* = B^* \otimes A^*.$$

4. For all 1-morphisms f and g, the 2-morphism $\bigotimes_{f,g}$ is unitary.

5. For any object or 1-morphism X we have $i_{X^*} = e_X^*$ and $e_{X^*} = i_X^*$.

6. For any object A, the 2-morphism T_A is unitary.

7. If I is the unit object, $T_I = 1_{1_I}$.

8. For any objects A and B we have

$$i_{A\otimes B} = i_A(A \otimes i_B \otimes A^*),$$

$$e_{A\otimes B} = (B^* \otimes e_A \otimes B)e_B,$$

and

$$T_{A\otimes B} = [(i_A \otimes A \otimes B)(A \otimes \bigotimes^{-1}_{i_B,e_A} \otimes B)(A \otimes B \otimes e_B)] \cdot [(T_A \otimes B)(A \otimes T_B)].$$

9. For any object A and morphism f we have

$$i_{A\otimes f} = A \otimes i_f, \qquad i_{f\otimes A} = i_f \otimes A,$$

$$e_{A\otimes f} = A \otimes e_f, \qquad e_{f\otimes A} = e_f \otimes A.$$

10. For any 1-morphisms f and g, $i_{fg} = i_f \cdot (f i_g f^*)$ and $e_{fg} = (g^* e_f g) \cdot e_g$.

11. For any 1-morphism f, $i_f f \cdot f e_f = 1_f$ and $f^* i_f \cdot e_f f^* = 1_{f^*}$.

12. For any 2-morphism α, $\alpha^{\dagger *} = \alpha^{*\dagger}$.

13. For any object A we have

$$[i_A(A \otimes T^\dagger_{A^*})] \cdot [\bigotimes^{-1}_{i_A,i_A}(A \otimes e_A \otimes A^*)] \cdot [i_A(T_A \otimes A^*)] = 1_{i_A}.$$

In what follows we first introduce the structures on $\mathcal{T}$ that make it into a monoidal 2-category with duals, and then verify that they satisfy the conditions in the above definition. In fact, some of these conditions are redundant. All the equational laws involving counits but not units can be derived from those involving units but not counits by taking duals. For example, starting from the first equation in condition 8,

$$i_{A\otimes B} = i_A(A \otimes i_B \otimes A^*),$$

and taking duals, we obtain

$$e_{B^*\otimes A^*} = (A \otimes e_{B^*} \otimes A^*)e_{A^*},$$

which, since it holds for all objects A and B, implies the second equation in condition 8:

$$e_{A\otimes B} = (B^* \otimes e_A \otimes B)e_B.$$

Conversely, of course, the equational laws involving units but not counits can be derived from those involving counits but not units. Also, the two equations $i_{X^*} = e^*_X$ and $e_{X^*} = i^*_X$ imply each other.

2.4.1 Duality for objects

Given an object $A \in \mathcal{T}$, we define its dual A^* to equal A. Given a specific representative of an object in $A \in \mathcal{T}$, we use its image under the reflection $y \mapsto 1 - y$ as a standard representative of A^*.

To obtain a representative of the unit $i_A: I \to A \otimes A^*$, we first rotate a representative of A lying in $I_1 \times [0, 1/2] \times \{1\}$ through an angle of 180 degrees around the axis $\{y = 1/2, z = 1\}$; the submanifold of the cube traced out by this rotation is a disjoint union of semicircles with endpoints in the target plane, $\{z = 1\}$, with the right endpoints representing $A^* = A$, and the left endpoints representing A. We then straighten this submanifold using a small isotopy so that it has a product structure in the z direction near $z = 1$; if the isotopy is sufficiently close to the identity, we obtain a generic tangle, which we take as a representative of i_A. The 1-morphism i_A is independent of our choice of a representative for A, since any two such representatives are equivalent by an equivalence isotopy that is the identity except for $y < 1/2$, and such an equivalence isotopy can be extended to the tangle representing i_A. Moreover, i_A is independent of the isotopy used for straightening near $z = 1$, provided this isotopy is sufficiently close to the identity.

Similarly, we obtain a representative of the counit $e_A: A^* \otimes A \to I$ by rotating a representative of A lying in $I_1 \times [1/2, 1] \times \{0\}$ through 180 degrees around the axis $\{y = 1/2, z = 0\}$. The submanifold traced out by this rotation is a disjoint union of semicircles with endpoints in the source plane, $\{z = 0\}$, with the right endpoints representing A, and the left endpoints representing A^*. Straightening it near $z = 0$, we obtain a generic tangle which we take as a representative of e_A.

To define the triangulator T_A we proceed inductively, using the fact that any object $A \in \mathcal{T}$ is a tensor product of copies of Z, where Z is the object represented by the one-point set. We define T_I to be 1_I. We define T_Z by first choosing a representative of Z, and then choosing an isotopy between the standard representative of 1_Z and the standard representative of $(i_Z \otimes Z)(Z \otimes e_Z)$. We require that this isotopy have the property that the surface S traced out by this isotopy is a generic 2-tangle, and then let T_Z be the 2-morphism represented by S. One can check that T_Z is independent of the choices made. Since any other object $A \in \mathcal{T}$ is of the form $Z \otimes A'$ for some object A', we define T_A inductively by the relation in item 7 of Definition 10:

$$T_A = [(i_Z \otimes Z \otimes A')(Z \otimes \bigotimes_{i_{A'}, e_Z}^{-1} \otimes A')(Z \otimes A' \otimes e_{A'})] \cdot [(T_Z \otimes A')(Z \otimes T_{A'})].$$

2.4.2 Duality for 1-morphisms

Given a 1-morphism $f: A \to B$ in $\mathcal{T}$, we define the dual 1-morphism $f^*: B \to A$ as follows: given any representative of f, we take its image under the reflection $z \mapsto 1 - z$ as a representative for f^*. Any equivalence between representatives of f can be similarly reflected, so f^* is well defined.

To define i_f, we first rotate a representative of f lying in $I_1 \times I_2 \times [0, 1/2] \times \{1\}$ through an angle of 180 degrees around the plane $\{z = 1/2, t = 1\}$; the submanifold of the 4-cube traced out by this rotation intersects the target hyperplane $\{t = 1\}$ in a generic tangle representing ff^*. We then straighten this submanifold using a small isotopy to make it have a product structure in the t direction near $t = 1$; if the isotopy is sufficiently small, we obtain a generic 2-tangle, which we take as a representative of $i_f \colon 1_A \Rightarrow ff^*$. As in the previous section, one can show that this 2-morphism i_f is independent of the choices made.

We define e_f in a similar way by rotating an representative of f^* lying in $I_1 \times I_2 \times [1/2, 1] \times \{0\}$ around $\{z = 1/2, t = 0\}$, and then straightening the result. We obtain a generic 2-tangle which we take as a representative of $e_f \colon f^* f \Rightarrow 1_B$.

2.4.3 Duality for 2-morphisms

Given a 2-morphism $\alpha \colon f \Rightarrow g$ in $\mathcal{T}$, we define the dual 2-morphism $\alpha^* \colon g \Rightarrow f$ by taking the image of any representative of α under the reflection $t \mapsto 1 - t$. Any equivalence isotopy between representatives of α can be likewise reflected, so α^* is well defined.

2.4.4 Verifying the conditions

We conclude by checking that the structures defined above make $\mathcal{T}$ into a monoidal 2-category with duals.

Lemma 11. *$\mathcal{T}$ is a monoidal 2-category with duals.*

Proof - We check that $\mathcal{T}$ equipped with the structures given in the previous sections satisfies the conditions listed in Definition 10.

1. Duals of objects, 1-morphisms or 2-morphisms are defined by reflecting representatives. Since reflecting twice is an identity map, $X = X^{**}$ for any object, 1-morphism or 2-morphism X.

2. A standard representative of 1_A is a collection of vertical line segments. This is unaffected by the reflection $z \mapsto 1 - z$, so $1_A = 1_A^*$. Similarly, $1_f = 1_f^*$, since a standard representative for 1_f is a product of a representative of f with the unit interval in the t direction, and this 2-tangle is unaffected by the reflection $t \mapsto 1 - t$.

3. In a standard representative of the vertical composite $\alpha \cdot \beta$, β follows α in the t direction, so when we apply the reflection $t \mapsto 1 - t$, not only are the β and α components reflected, but also the reflected β now precedes the reflected α, hence $(\alpha \cdot \beta)^* = \beta^* \cdot \alpha^*$.

In a standard representative of the horizontal composite $\alpha\beta$, β is below α. This z ordering is not changed by the reflection $t \mapsto 1 - t$, so $(\alpha\beta)^* = \alpha^*\beta^*$.

In a standard representative of fg, g is below f. Since $(fg)^*$ is obtained by the reflection $z \mapsto 1 - z$, the order of the reflected f and g is reversed, so $(fg)^* = g^* f^*$.

In a standard representative of $A \otimes B$, B is to the right of A. Since $(A \otimes B)^*$ is obtained by the reflection $y \mapsto 1-y$, the order of the reflected A and B is reversed, so $(A \otimes B)^* = B^* \otimes A^*$.

The tensor product of an object A with a 1-morphism (resp. 2-morphism) X is represented by a disjoint union of X with 1_A (resp. 1_{1_A}) on the right or left. Since the reflections defining duals for 1-morphisms and 2-morphisms do not change the y coordinate, the order of the tensor product remains the same after reflection. Since $1_X^* = 1_X$, we have $(A \otimes X)^* = A \otimes X^*$ and $(X \otimes A)^* = X^* \otimes A$.

4. A standard representative of $\bigotimes_{f,g}$ is generated by an isotopy that moves f and g past each other in the z direction. A representative of $\bigotimes_{f,g}^{-1}$ is obtained using the same isotopy with t reversed. Reversing t in the isotopy amounts to reflecting the 2-tangle representing $\bigotimes_{f,g}$ in the t direction, so $\bigotimes_{f,g}^{-1} = \bigotimes_{f,g}^*$.

5. A standard representative of e_A is obtained by rotating a representative of A lying in the right half of $I_1 \times I_2 \times \{0\}$ about the axis $\{y = 1/2, z = 0\}$, and then straightening the resulting submanifold near $z = 0$. Thus a representative of e_A^* can be obtained by rotating a representative of A lying in the right half of $I_1 \times I_2 \times \{1\}$ about the axis $\{y = 1/2, z = 1\}$, and straightening the resulting submanifold near $z = 1$. However, the resulting tangle also serves as a representative of i_{A^*}, since we may also obtain this tangle by rotating a representative of A^* lying in the left half of $I_1 \times I_2 \times \{1\}$ about the axis $\{y = 1/2, z = 1\}$, and straightening the resulting submanifold near $z = 1$. Thus we have $i_{A^*} = e_A^*$. Similarly, one can show that a representative for i_{f^*} is also a representative for e_f^*, so these 2-morphisms are equal. The conditions $e_{A^*} = i_A^*$ and $e_{f^*} = i_f^*$ follow by taking duals.

6. A standard representative of T_Z is obtained from an isotopy between its source and target tangles, so the 2-morphism T_Z is unitary (it has an inverse that is given by reversing t in the isotopy). The fact that T_A is unitary in general follows from the fact that T_Z and $\bigotimes_{\cdot,\cdot}$ are unitary, together with the relation that defines T_A.

7. We have $T_I = 1_I$ by definition.

8. A standard representative of $i_{A \otimes B}$ is defined by rotating a representative of $A \otimes B$ lying in the left half of $I_1 \times I_2 \times \{1\}$ about the axis $\{y = 1/2, z = 1\}$, and straightening the resulting submanifold near $z = 1$. This produces a generic tangle consisting of nested semicircles, and the maxima of the semicircles with boundary $A \cup A^*$ are above the maxima of the semicircles with boundary $B \cup B^*$. If we take a small isotopy that straightens the strands for z in a small interval just above the maxima of the semicircles with boundary $B \cup B^*$, we get an equivalent generic tangle that represents $i_A \cdot (A \otimes i_B \otimes A^*)$. It follows that $i_{A \otimes B} = i_A \cdot (A \otimes i_B \otimes A^*)$. The condition $e_{A \otimes B} = (B^* \otimes e_A \otimes B) \cdot e_B$ follows from this by taking duals.

The relation

$$T_{A \otimes B} = [(i_A \otimes A \otimes B)(A \otimes \bigotimes_{i_B, e_A}^{-1} \otimes B)(A \otimes B \otimes e_B)] \cdot [(T_A \otimes B)(A \otimes T_B)]$$

is a consequence of the relation defining T_A, and in fact is identical to that relation for $A = Z$.

9. A standard representative of $i_{A\otimes f}$ is generated by rotating a representative of $A \otimes f$ lying in $I_1 \times I_2 \times [0, 1/2] \times \{1\}$ around the plane $\{z = 1/2, t = 1\}$. A standard representative of $A \otimes f$ consists of straight strands for the points of A to the left of a tangle representing f. When rotated, the strands associated with A become flat planes to the left of a 2-tangle representing i_f, so $i_{A\otimes f} = A \otimes i_f$. A similar argument shows that $i_{f\otimes A} = i_f \otimes A$. The analogous conditions for the counits, $e_{A\otimes f} = A \otimes e_f$ and $e_{f\otimes A} = e_f \otimes A$, follow by taking duals.

10. A representative of i_{fg} is generated by rotating a representative of fg lying in $I_1 \times I_2 \times [0, 1/2] \times \{1\}$ around the plane $\{z = 1/2, t = 1\}$. We may choose a representative of fg so that the tangle representing f lies in the region where $0 \leq z \leq 1/4$, while that representing g lies in the region where $1/4 \leq z \leq 1/2$. Then, after straightening the resulting surface near $t = 3/4$, the 2-tangle representing i_{fg} is the vertical composite of a 2-tangle in $I_1 \times I_2 \times I_3 \times [0, 3/4]$ representing $i_{f\otimes f^*}$ and a 2-tangle in $I_1 \times I_2 \times I_3 \times [3/4, 1]$ representing fi_gf^*. We thus have $i_{fg} = i_f \cdot (fi_gf^*)$. The condition $e_{fg} = (g^*e_fg) \cdot e_g$ follows by taking duals.

11. We specify a representative of $i_ff \cdot fe_f$ as in the left-hand side of Fig. 29. More precisely, we start by choosing a representative of f that consists of straight vertical lines outside the region where $z < 1/3 - \epsilon$ for some $\epsilon > 0$. By abuse of language let us call this representative simply f. Then we take $(f \cap I_1 \times I_2 \times [0, 1/3]) \times \{1/2\}$ and rotate it 180 degrees around $\{z = 1/3, t = 1/2\}$, so that it sweeps out a surface lying in the region with $t \leq 1/2$. We straighten this to obtain a surface S that has a product structure in the t direction near $t = 1/2$. Next, we take $S \cap (I_1 \times I_2 \times [1/3, 2/3] \times \{1/2\})$ and rotate it 180 degrees around $\{z = 2/3, t = 1/2\}$ so that it sweeps out a surface lying in the region with $t \geq 1/2$. We straighten this surface to obtain a surface S' with a product structure in the t direction near $t = 1/2$, for which the union $S \cup S'$ is a smooth submanifold of the 4-cube. Finally, we continue $S \cup S'$ to the boundary of the 4-cube by the product structure in t, obtaining a generic 2-tangle that represents $i_ff \cdot fe_f$. This 2-tangle is isotopic to a generic 2-tangle representing 1_f, as shown in the right-hand side of Fig. 29. Hence we conclude that the triangle identity $i_ff \cdot fe_f = 1_f$ holds. A similar argument shows that $fe_f \cdot i_ff = 1_f$.

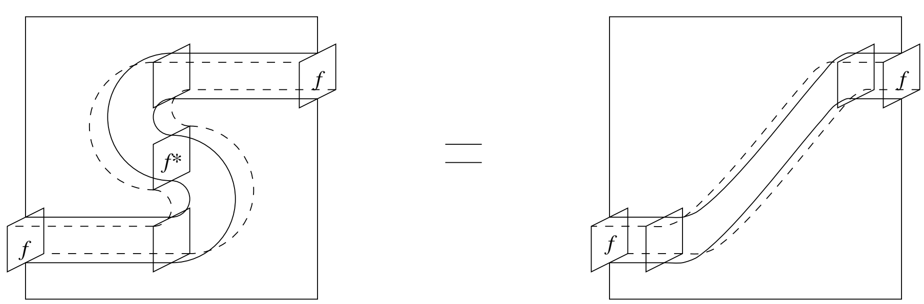


Figure 29: The triangle identity for 1-morphisms

12. We can find a representative of $\alpha^{*\dagger}$ as in the left-hand side of Fig. 30. More precisely, we can find a representative with a product structure in the t direction outside the representatives of α^* in $I_1 \times I_2 \times [1/3, 2/3] \times [1/3, 2/3]$ i_g in $I_1 \times I_2 \times [1/3, 1] \times [0, 1/3]$, and e_f in $I_1 \times I_2 \times [0, 2/3] \times [2/3, 1]$. We may also assume that the surfaces representing i_g and e_f are formed by rotating the appropriate tangles around $\{z = 2/3, t = 1/3\}$ and $\{z = 1/3, t = 2/3\}$, and then straightening to obtain surface with a product structure in the t direction near $t = 1/3$ and $t = 2/3$. As shown in Fig. 30, this representative of $\alpha^{*\dagger}$ is equivalent to a representative of $f^* i_f \cdot e_f f^* \cdot \alpha^*_r$, where α^*_r is obtained by applying the transformation $(x, y, z, t) \mapsto (x, y, -z, -t)$ to a representative of α^*. Since $f e_f \cdot i_f f = 1_f$, $i_{f^*} = e^*_f$ and $e_{f^*} = i^*_f$, we have $f^* i_f \cdot e_f f^* \cdot \alpha^*_r = \alpha^*_r$. Similarly, we can find an equivalence isotopy from a representative of $\alpha^{\dagger *}$ to α^*_r. Thus we conclude $\alpha^{\dagger *} = \alpha^{*\dagger}$.

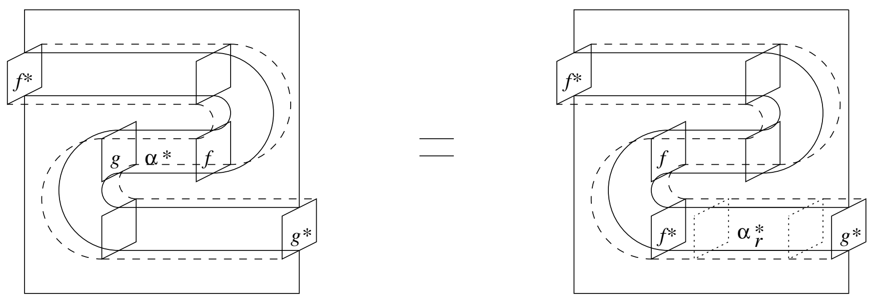


Figure 30: Representatives of $\alpha^{*\dagger}$

13. The swallowtail coherence law

$$[i_A(A \otimes T^\dagger_{A^*})] \cdot [\bigotimes^{-1}_{i_A, i_A} (A \otimes e_A \otimes A^*)] \cdot [i_A(T_A \otimes A^*)] = 1_{i_A}$$

clearly holds when A is the object Z represented by a one-element set; see Fig. 25. For A represented by an n-element set, we can inductively straighten sheets beginning either at the outside or inside, and get the result

$$[i_A(A \otimes T^\dagger_{A^*})] \cdot [\bigotimes^{-1}_{i_A, i_A} (A \otimes e_A \otimes A^*)] \cdot [i_A(T_A \otimes A^*)] = 1_{i_A}$$

for any object A. □

2.5 $\mathcal{T}$ is a Braided Monoidal 2-Category with Duals

In general, when an n-category with duals is equipped with extra structure, it is natural to demand that the structural n-isomorphisms be unitary. For example, in our definition of a monoidal 2-category with duals, we demanded that the tensorator and triangulator be unitary. We incorporate this principle in our definition of a 'braided monoidal 2-category with duals' by requiring that the braiding coherence

2-morphisms and the braiding for an object and a 1-morphism 2-isomorphisms be unitary. On the other hand, since the braiding for a pair of objects is a 1-morphism, we require only that it be unitary *up to specified unitary 2-morphisms.* In other words, given a pair of objects A, B, we do not demand that $R_{A,B}R^*_{A,B} = 1_{A\otimes B}$ and $R^*_{A,B}R_{A,B} = 1_{B\otimes A}$. Instead, we demand only that the 2-morphisms

$$i_{R_{A,B}}\colon 1_{A\otimes B} \Rightarrow R_{A,B}R^*_{A,B}$$

and

$$e_{R_{A,B}}\colon R^*_{A,B}R_{A,B} \Rightarrow 1_{B\otimes A}$$

be unitary. For more on this point, see Section 2.3.1.

Definition 12. *A* braided monoidal 2-category with duals *is a monoidal 2-category with duals that is also a braided monoidal 2-category for which the braiding is unitary in the sense that:*

1. *For any objects A, B, the 2-morphisms $i_{R_{A,B}}$ and $e_{R_{A,B}}$ are unitary.*
2. *For any object A and morphism f, the 2-morphisms $R_{A,f}$ and $R_{f,A}$ are unitary.*
3. *For any objects A, B, C, the 2-morphisms $\tilde{R}_{(A,B|C)}$ and $\tilde{R}_{(A|B,C)}$ are unitary.*

In HDA2 we showed that in a braided monoidal category with duals every object A has an automorphism $b_A\colon A \to A$ called the 'balancing', given by

$$b_A = (e^*_A \otimes A)(A^* \otimes R_{A,A})(e_A \otimes A).$$

In the study of tangles, the balancing is closely related to the subtle issue of framings. For example, the category of framed oriented tangles is the free braided monoidal category with duals on an object Z corresponding to a single positively oriented point in the unit square. In this category the balancing b_Z corresponds to a strand with a single twist in its framing. The category of unframed oriented tangles has an extra relation saying that $b_Z = 1$. In the language of knot theory, this relation is called the Reidemeister I move.

Similar ideas apply to braided monoidal 2-categories with duals. We may define the balancing by the same formula as in a braided monoidal category with duals. We expect that the 2-category of framed oriented 2-tangles is the free braided monoidal 2-category on the object Z corresponding to a single positively oriented point in the unit square. However, the 2-category of unframed oriented 2-tangles should be obtained, not by setting b_Z equal to the identity, but instead by adjoining a unitary 2-morphism $V_Z\colon b_Z \Rightarrow 1_Z$ satisfying a new coherence law of its own. This 2-morphism corresponds to the *process* of performing the Reidemeister I move. In other words, it describes the process of undoing a twist in the framing.

Actually, the connection to the work of Carter, Rieger and Saito will be clearer if we work not with the balancing b_Z but with the closely related morphism $i_{Z^*}R_{Z^*,Z}\colon 1 \to$

$Z \otimes Z^*$. The process of inserting a right-handed twist in the framing then corresponds to a unitary 2-morphism

$$W_Z \colon i_Z \Rightarrow i_{Z^*} R_{Z^*,Z}$$

which we call the 'writhing', as shown in Fig. 31. For any object A in a braided monoidal 2-category with duals, one can construct a unitary 2-morphism $V_A \colon b_A \Rightarrow 1_A$ given a unitary 2-morphism $W_A \colon i_A \Rightarrow i_{A^*} R_{A^*,A}$, and conversely.

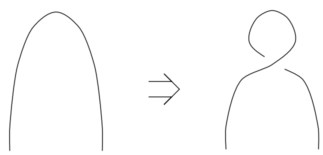


Figure 31: The writhing W_Z

Since we are studying unoriented unframed 2-tangles, where the generating object Z satisfies $Z = Z^*$, we shall only give the coherence law for the writhing in the case of a self-dual object.

Definition 13. *A* self-dual object *in a braided monoidal 2-category with duals is an object A with $A^* = A$.*

To state the coherence law for the writhing, it is convenient to introduce the following 2-morphisms for any pair of objects X, Y in a braided monoidal 2-category with duals:

$$H_{X,Y} \colon (i_X \otimes Y)(X \otimes R_{X^*,Y}) \Rightarrow (Y \otimes i_X)(R^*_{X,Y} \otimes X^*)$$

and

$$\bar{H}_{X,Y} \colon (i_X \otimes Y)(X \otimes R^*_{Y,X^*}) \Rightarrow (Y \otimes i_X)(R_{Y,X} \otimes X^*),$$

defined as follows:

$$\begin{aligned} H_{X,Y} \quad = \quad & [(i_X \otimes Y)(X \otimes R_{X^*,Y})(i_{R_{X,Y}} \otimes X^*)] \cdot \\ & [(i_X \otimes Y)\tilde{R}_{(X,X^*|Y)}(R^*_{X,Y} \otimes X^*)] \cdot \\ & [R_{i_X,Y}(R^*_{X,Y} \otimes X^*)] \end{aligned}$$

$$\begin{aligned} \bar{H}_{X,Y} \quad = \quad & [(i_X \otimes Y)(X \otimes R^*_{Y,X*})(i_{R^*_{Y,X}} \otimes X^*)] \cdot \\ & [(i_X \otimes Y)\tilde{R}^{\dagger *}_{(Y|X,X^*)}(R_{Y,X} \otimes X^*)] \cdot \\ & [R^{\dagger}_{Y,i^*_X}(R_{Y,X} \otimes X^*)] \end{aligned}$$

When both X and Y are the object $Z \in \mathcal{T}$, Carter, Rieger and Saito [9] call these 2-morphisms 'a double point arc crossing a fold line'. They can be visualized as in Fig. 32.

Figure 32: $H_{X,Y}$ and $\bar{H}_{X,Y}$

Definition 14. *A self-dual object A in a braided monoidal 2-category with duals is* unframed *if it is equipped with a unitary 2-morphism*

$$W_A\colon i_A \Rightarrow i_A R_{A,A}$$

satisfying the equation

$$T_A^\dagger \cdot ((A \otimes W_A)(e_A \otimes A)) \cdot ((A \otimes i_A)\bar{H}^{\dagger *}_{A,A}) =$$

$$T_A^{-1} \cdot ((i_A \otimes A)(A \otimes i_{R_{A,A}} e_A)) \cdot ((i_A \otimes A)(A \otimes R_{A,A} W_A^\dagger)) \cdot (H_{A,A}(A \otimes e_A))$$

A unitary 2-morphism satisfying this equation is called a writhing *for A.*

Fig. 33 depicts the coherence law satisfied by the writhing in the case where A is the object $Z \in \mathcal{T}$ corresponding to a single point in the unit square.

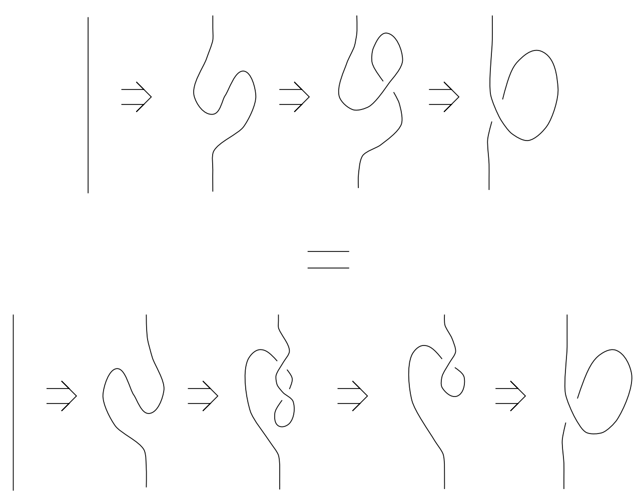


Figure 33: The coherence law for the writhing

Lemma 15. *$\mathcal{T}$ is a braided monoidal 2-category with duals, and the object $Z \in \mathcal{T}$ corresponding to a single point is an unframed self-dual object.*

Proof - For any objects $A, B \in \mathcal{T}$, one can straighten out a representative of $i_{R_{A,B}} \cdot i^*_{R_{A,B}}$ using an equivalence isotopy to obtain the standard representative of $1_{1_{A\otimes B}}$. One can similarly straighten out representatives of $i^*_{R_{A,B}} \cdot i_{R_{A,B}}$, $e_{R_{A,B}} \cdot e^*_{R_{A,B}}$, and $e^*_{R_{A,B}} \cdot e_{R_{A,B}}$, showing that $i_{R_{A,B}}$ and $e_{R_{A,B}}$ are unitary.

For any objects $A, B, C \in \mathcal{T}$ and any 1-morphism f in $\mathcal{T}$, the 2-morphisms $R_{A,f}$, $R_{f,A}$, $\tilde{R}_{(A,B|C)}$ and $\tilde{R}_{(A|B,C)}$ are represented by surfaces traced out by an isotopy. It follows that their inverses are defined by the isotopy with time reversed, so these 2-morphisms are unitary.

Finally, we show that the object Z represented by a single object is an unframed self-dual object. We have $Z = Z^*$ by definition, and we define the writhing W_Z to be the 2-morphism represented by the generic 2-tangle traced out by the Reidemeister I move. Since this move is an isotopy, W_Z is unitary. Finally, the coherence law

$$T^\dagger_Z \cdot ((Z \otimes W_Z)(e_Z \otimes Z)) \cdot ((Z \otimes i_Z)\bar{H}^{\dagger *}_{Z,Z}) =$$

$$T^{-1}_Z \cdot ((i_Z \otimes Z)(Z \otimes i_{R_{Z,Z}} e_Z)) \cdot ((i_Z \otimes Z)(Z \otimes R_{Z,Z} W^\dagger_Z)) \cdot (H_{Z,Z}(Z \otimes e_Z))$$

corresponds the equation shown in Fig. 33. Carter, Rieger and Saito [9] call this 'a branch point passing through a cusp' and show that it holds in $\mathcal{T}$. □

The coherence law for the writhing is somewhat mysterious from an algebraic point of view, but we can offer a partial explanation for it as follows. Suppose that A is a self-dual object in a braided monoidal 2-category with duals, and that A is equipped with a 2-morphism

$$W_A \colon i_A \Rightarrow i_A R_{A,A}.$$

Then we can form a unitary 2-morphism from i_A to $i_A R^*_{A,A}$ in two different ways, and the coherence law for the writhing says that these are equal. When A is the object $Z \in \mathcal{T}$, both these 2-morphisms insert a left-handed twist in the framing.

The first way to form a 2-morphism from i_A to $i_A R^*_{A,A}$ uses all three levels of duality. Suppose we have a 2-morphism $\alpha \colon f \Rightarrow g$ between 1-morphisms $f, g \colon X \to Y$. Then each level of duality gives us a different way to 'reverse' α. As already discussed, duality at the 2-morphism level gives us a 2-morphism $\alpha^* \colon g \Rightarrow f$, while duality at the 1-morphism level gives us the 2-morphism $\alpha^\dagger \colon g^* \Rightarrow f^*$ defined by $\alpha^\dagger = (g^* i_f) \cdot (g^* \alpha f^*) \cdot (e_g f^*)$. In addition, duality at the object level lets us define 1-morphisms

$$f^\dagger, g^\dagger \colon Y^* \to X^*$$

and a 2-morphism

$$\hat{\alpha} \colon g^\dagger \Rightarrow f^\dagger$$

as follows:

$$f^\dagger = (Y^* \otimes i_X)(Y^* \otimes f \otimes X^*)(e_Y \otimes X^*),$$

$$g^\dagger = (Y^* \otimes i_X)(Y^* \otimes g \otimes X^*)(e_Y \otimes X^*),$$

$$\hat{\alpha} = (Y^* \otimes i_X)(Y^* \otimes \alpha \otimes X^*)(e_Y \otimes X^*).$$

A more detailed analysis of the relationships between these operations can be found in the recent work of Mackaay [22]; this is one place where there may be room for improvement in our definition of 'monoidal 2-category with duals'.

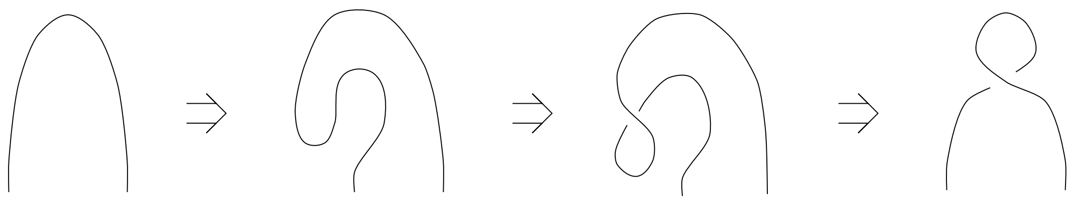


Figure 34: The 2-morphism $\bar{W}_Z$

Using these operations, we can form the 2-morphism

$$\bar{W}_A : i_A \Rightarrow i_A R^*_{A,A}$$

as the vertical composite of the three 2-morphisms depicted in Fig. 34 in the case $A = Z$. The first and third 2-morphisms in this composite are really just 'padding'. The meat of the sandwich is the second 2-morphism, which is formed by reversing W_A in all three ways listed above. More precisely, we have

$$\bar{W}_A = \gamma \cdot \widehat{W^{\dagger *}_A} \cdot \delta$$

where

$$\gamma = i_A(T^\dagger_A \otimes A)$$

and

$$\begin{aligned} \delta \;=\; & [i_A(H^*_{A,A} \otimes A)(e_A \otimes A \otimes A)] \cdot [i_A(i_A \otimes A \otimes A)(\bar{H}^{\dagger *}_{A,A} \otimes A)] \cdot \\ & [\bigotimes_{i_A R^*_{A,A}, i_A} (A \otimes e_A \otimes A)] \cdot [i_A R^*_{A,A}(A \otimes T^{\dagger *}_A)]. \end{aligned}$$

The second way to form a 2-morphism from i_A to $i_A R^*_{A,A}$ is to form the vertical composite of

$$i_A i_{R_{A,A}} : i_A \Rightarrow i_A R_{A,A} R^*_{A,A}$$

and

$$W^*_A R^*_{A,A} : i_A R_{A,A} R^*_{A,A} \Rightarrow i_A R^*_{A,A}.$$

In the case $A = Z$, this amounts to doing a Reidemeister II move and then removing a right-handed twist in the framing, as in Fig. 35.

Lemma 16. *Let A be an self-dual object in a braided monoidal 2-category with duals. Then a unitary 2-morphism $W_A : i_A \Rightarrow i_A R_{A,A}$ is a writhing for A if and only if it satisfies the equation*

$$\bar{W}_A = (i_A i_{R_{A,A}}) \cdot (W^*_A R^*_{A,A}).$$

Proof - We omit the proof, since it is a straightforward although lengthy calculation. □

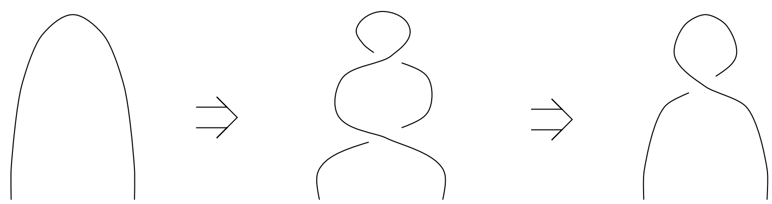


Figure 35: The 2-morphism $(i_Z i_{R_{Z,Z}}) \cdot (W^*_Z R^*_{Z,Z})$

3 A Combinatorial Description of 2-Tangles

In this section we describe a 2-category $\mathcal{C}$ that is isomorphic to the 2-category $\mathcal{T}$ of unframed unoriented 2-tangles. Our description is purely combinatorial, in the sense that we list the objects of $\mathcal{C}$ and describe its 1-morphisms and 2-morphisms using the method of generators and relations. Our list of generators and relations is based on the work of Carter, Rieger and Saito, so we can use their results to show that $\mathcal{C}$ and $\mathcal{T}$ are isomorphic [9]. This isomorphism makes $\mathcal{C}$ into a braided monoidal 2-category with duals.

Our presentation of the 2-category $\mathcal{C}$ makes use of a 'bar' operation on morphisms and 2-morphisms that corresponds in $\mathcal{T}$ to taking the image of a representative tangle or 2-tangle under the reflection $x \mapsto 1 - x$. Thus given a 1-morphism $f\colon A \to B$ we have $\bar{f}\colon A \to B$, and given a 2-morphism $\alpha\colon f \Rightarrow g$ we have $\bar{\alpha}\colon \bar{f} \Rightarrow \bar{g}$. Some examples have already been mentioned in Section 2.5.

In a deeper treament, this bar operation would be built into the definition of 'braided monoidal 2-category with duals'. Since a braided monoidal 2-category is a special sort of 4-category, a braided monoidal 2-category with duals should really have four duality operations. In the study of 2-tangles these operations correspond to reflection along the x, y, z, and t axes. The last three of these correspond to duality for objects, morphisms, and 2-morphisms. The first one remains obscure in our approach, but it gives rise to the bar operation.

3.1 The 2-category $\mathcal{C}$

In what follows we list the objects of $\mathcal{C}$ and give presentations for its 1-morphisms and 2-morphisms. For a more systematic treatment of 2-category presentations, see the work of Street [26].

3.1.1 Objects

Objects in $\mathcal{C}$ correspond to natural numbers, and are denoted by A_n, where $n \in \mathbb{N}$. These correspond to finite sets of points in the unit square, where n is the number of points.

3.1.2 1-Morphisms

Each 1-morphism in $\mathcal{C}$ represents a planar diagram of a generic tangle. Using the notation of Carter, Rieger, and Saito, we describe these 1-morphisms using the technique of generators and relations. The generating 1-morphisms are

$$X_{i,j}\colon A_{i+j+2} \to A_{i+j+2},$$

$$\bar{X}_{i,j}\colon A_{i+j+2} \to A_{i+j+2},$$

$$\cup_{i,j}\colon A_{i+j+2} \to A_{i+j},$$

$$\cap_{i,j}\colon A_{i+j} \to A_{i+j+2},$$

and the identity 1-morphisms

$$1_n\colon A_n \to A_n.$$

Here X denotes a right-handed crossing, $\bar{X}$ denotes a left-handed crossing, $\cup$ denotes a minimum, $\cap$ denotes a maximum, and the subscripts $i, j \geq 0$ denote the number of vertical strands to the left and right respectively of this crossing, maximum or minimum. The identity 1-morphism 1_n corresponds to a tangle consisting of nothing but n vertical strands.

Every 1-morphism in $\mathcal{C}$ is a composite of these generators, with the only relations being associativity and the left and right unit laws. Note that these relations do not include equivalence under Reidemeister moves, nor changing the height order of crossings and local extrema in the corresponding tangles.

We define duals of 1-morphisms in $\mathcal{C}$ as follows. For any morphism f, there is a 1-morphism f^* defined recursively by the following equations: $X^*_{i,j} = \bar{X}_{i,j}$, $\cap^*_{i,j} = \cup_{i,j}$, $1^*_n = 1_n$, $f^{**} = f$, and $(fg)^* = g^* f^*$.

3.2 2-Morphisms

Our description of the 2-morphisms in $\mathcal{C}$ is compatible with that given by Carter, Rieger and Saito, with the notation changed to better fit the 2-category structure: we describe surfaces explicitly in terms of 2-morphisms, rather than the sources and targets of these 2-morphisms. Again, we use the technique of generators and relations. Having already specified the 1-morphisms of $\mathcal{C}$, we now give a list of 2-morphism generators going between these 1-morphisms, together with a list of relations satisfied by these generators. Every 2-morphism in $\mathcal{C}$ is formed from these generating 2-morphisms by taking vertical and horizontal composites. Two such composites define the same 2-morphism in $\mathcal{C}$ if and only if one can get from one to the other using the relations in our list, together with the equational laws in the definition of a 2-category.

The generating 2-morphisms correspond to what Carter, Rieger and Saito call the 'full set of elementary string interactions', together with identity 2-morphisms for all the 1-morphisms in $\mathcal{C}$. We list these generating 2-morphisms below. The subscripts denote the number of strands to the left and right that are not affected.

1. The birth of a circle:

   $I_{m,n}\colon 1_{m+n} \Rightarrow \cap_{m,n}\cup_{m,n}$.

2. A saddle point:

   $E_{m,n}\colon \cup_{m,n}\cap_{m,n} \Rightarrow 1_{m+n+2}$.

3. The Reidemeister I move:

   $W_{m,n}\colon \cap_{m,n} \Rightarrow \cap_{m,n}X_{m,n}$,

4. The Reidemeister II move:

   $II_{m,n}\colon 1_{m+n+2} \Rightarrow X_{m,n}\bar{X}_{m,n}$.

5. Three forms of the Reidemeister III move:

   $S_{0;m,n}\colon X_{m,n+1}X_{m+1,n}X_{m,n+1} \Rightarrow X_{m+1,n}X_{m,n+1}X_{m+1,n}$,
   $S_{1;m,n}\colon X_{m,n+1}X_{m+1,n}\bar{X}_{m,n+1} \Rightarrow \bar{X}_{m+1,n}X_{m,n+1}X_{m+1,n}$,
   $S_{2;m,n}\colon X_{m,n+1}\bar{X}_{m+1,n}\bar{X}_{m,n+1} \Rightarrow \bar{X}_{m+1,n}\bar{X}_{m,n+1}X_{m+1,n}$.

6. A double point arc crossing a fold line:

   $H_{m,n}\colon \cap_{m,n+1}X_{m+1,n} \Rightarrow \cap_{m+1,n}\bar{X}_{m,n+1}$,

7. A cusp on a fold line:

   $T_{m,n}\colon \cap_{m,n+1}\cup_{m+1,n} \Rightarrow 1_{m+n+1}$,

8. Shifting relative heights of distant crossings and local extrema:

   $N_{Y_{m,n},Z_{i,j}}\colon Z_{i',j}Y_{m,n'} \Rightarrow Y_{m,n}Z_{i,j}$

   where Y and Z stand for $X, \bar{X}, \cup$ or $\cap$, where $i \geq m+2$ if $Y \neq \cup$, and $i \geq m$ if $Y = \cup$, and i, j, m, n, i', n' are chosen so that the composite 1-morphisms $Z_{i',j}Y_{m,n'}$ and $Y_{m,n}Z_{i,j}$ are well defined.

9. Identity 2-morphisms $1_f\colon f \Rightarrow f$ for any 1-morphism f.

10. For each of the above generating 2-morphisms $\alpha\colon f \Rightarrow g$, a generating 2-morphism $\bar{\alpha}\colon \bar{f} \Rightarrow \bar{g}$, where $\bar{f}$ is obtained from f by replacing each occurence of $X_{i,j}$ by $\bar{X}_{i,j}$ and vice versa in a product of generating 1-morphisms representing f.

11. For each of the above generating 2-morphisms $\alpha\colon f \Rightarrow g$, a generating 2-morphism $\alpha^\dagger\colon g^* \Rightarrow f^*$.

12. For each of the above generating 2-morphisms $\alpha\colon f \Rightarrow g$, a generating 2-morphism $\alpha^*\colon g \Rightarrow f$.

Next we list the relations that these generating 2-morphisms satisfy. With the help of the relations at the end of our list, relations 1-30 below are equivalent to the correspondingly numbered 'movie moves' in Theorem 3.5.5 of Carter, Rieger and Saito [9]. (We omit their 31st movie move, since it follows from the definition of a 2-category.) These authors explain how their movie moves arise from singularity theory, and illustrate many of them with beautiful figures. To help the reader find these figures in their paper, we include Carter, Rieger and Saito's names for the relations below when possible.

1. An elliptic confluence of branch points: $W_{m,n}W^*_{m,n} = 1$.

2. A hyperbolic confluence of branch points: $W^*_{m,n}W_{m,n} = 1$.

3. An elliptic confluence of double points: $II_{m,n}II^*_{m,n} = 1$.

4. A hyperbolic confluence of double points: $II^*_{m,n}II_{m,n} = 1$.

5. Cancelling triple points: $S_{i;m,n}S^*_{i;m,n} = 1$ and $S^*_{i;m,n}S_{i;m,n} = 1$ for $i = 0, 1, 2$.

6. A quadruple point in the isotopy (the Zamolodchikov tetrahedron equation):
$(Z_{i;m,n+1}X_{m+2,n}X_{m+1,n+1}X_{m,n+2}) \cdot (C_{m+1,n+1}B_{m,n+2}Z_{A;m+1,n}X_{m,n+2})$
$\cdot(C_{m+1,n+1}N^*_{B_{m,n+2},X_{m+2,n}}X_{m+1,n+1}A_{m+2,n}X_{m,n+2})$
$\cdot(C_{m+1,n+1}X_{m+2,n}B_{m,n+2}X_{m+1,n+1}N_{X_{m,n+2},A_{m+2,n}})$
$\cdot(C_{m+1,n+1}X_{m+2,n}Z_{B;m,n+1}A_{m+2,n}) \cdot (Z_{C;m+1,n}X_{m,n+2}B_{m+1,n+1}A_{m+2,n})$
$\cdot(X_{m+2,n}X_{m+1,n+1}N_{X_{m,n+2},C_{m+2,n}}B_{m+1,n+1}A_{m+2,n})$
$= (A_{m,n+2}B_{m+1,n+1}N^*_{C_{m,n+2}X_{m+2,n}}X_{m+1,n+1}X_{m,n+2})$
$\cdot(A_{m,n+2}B_{m+1,n+1}X_{m+2,n}Z_{C;m,n+1}) \cdot (A_{m,n+2}Z_{B;m+1,n}X_{m,n+2}C_{m+1,n+1})$
$\cdot(N^*_{A_{m,n+2},X_{m+2,n}}X_{m+1,n+1}B_{m+2,n}X_{m,n+2}C_{m+1,n+1})$
$\cdot(X_{m+2,n}A_{m,n+2}X_{m+1,n+1}N_{X_{m,n+2},B_{m+2,n}}C_{m+1,n+1})$
$\cdot(X_{m+2,n}Z_{A;m,n+1}B_{m+2,n}C_{m+1,n+1}) \cdot (X_{m+2,n}X_{m+1,n+1}X_{m,n+2}Z_{i;m+1,n})$
where $Z = S$ or $\bar{S}$, $i = 0, 1, 2$, A, B, C equal either X or $\bar{X}$ in such a way that source$(Z_{i;m,n+1}) = A_{m,n+2}B_{m+1,n+1}C_{m,n+2}$, and $Z_{Y;j,k}$ equals $S_{0,j,k}$ if $Y = X$ or $\bar{S}_{2,j,k}$ if $Y = \bar{X}$.

7. A branch point moving through a triple point:
$(\cap_{m+1,n}II_{m,n+1}\bar{X}_{m+1,n}X_{m,n+1}) \cdot (\bar{H}^*_{m,n}\bar{X}_{m,n+1}\bar{X}_{m+1,n}X_{m,n+1})$
$\cdot(\cap_{m,n+1}\bar{X}_{m+1,n}\bar{S}_{1;m,n}) \cdot (\cap_{m,n+1}\bar{II}^*_{m+1,n}\bar{X}_{m,n+1}\bar{X}_{m+1,n}) \cdot (\bar{W}^*_{m,n+1}\bar{X}_{m+1,n})$
$= (\bar{W}^*_{m+1,n}X_{m,n+1}) \cdot \bar{H}^*_{m,n}$
and

$(\cap_{m+1,n} II_{m,n+1} X_{m+1,n} X_{m,n+1}) \cdot (\bar{H}^*_{m,n} \bar{X}_{m,n+1} X_{m+1,n} X_{m,n+1})$
$\cdot(\cap_{m,n+1} \bar{X}_{m+1,n} \bar{S}_{2;m,n}) \cdot (\cap_{m,n+1} \bar{II}^*_{m+1,n} X_{m,n+1} \bar{X}_{m+1,n}) \cdot (W^*_{m,n+1} \bar{X}_{m+1,n})$
$= (W^*_{m+1,n} X_{m,n+1}) \cdot \bar{H}^*_{m,n}$

8. An elliptic confluence of cusps: $T^*_{m,n} T_{m,n} = 1$.
9. A hyperbolic confluence of cusps: $T_{m,n} T^*_{m,n} = 1$.
10. A swallowtail on the fold lines (the swallowtail coherence law):
    $(\cap_{m,n} T^\dagger_{m+1,n}) \cdot (N^*_{\cap_{m,n}, \cap_{m+2,n}} \cup_{m+1,n+1}) \cdot (\cap_{m,n} T_{m,n+1}) = 1_{\cap_{m,n}}$
11. Removing redundant double points crossing the fold lines: $H_{m,n} H^*_{m,n}$ and $H^*_{m,n} H_{m,n}$ are identity 2-morphisms.
12. A branch point passes through a cusp:
    $(\cap_{m+1,n} H^\dagger_{m,n}) \cdot (\bar{W}^*_{m+1,n} \cup_{m,n+1}) \cdot T^{\dagger *}_{m,n} = (\bar{H}^*_{m,n} \cup_{m+1,n}) \cdot (\cap_{m,n+1} W^\dagger_{m+1,n}) \cdot T_{m,n}$
    and
    $(\cap_{m,n+1} \bar{H}^{\dagger *}_{m,n}) \cdot (\bar{W}^*_{m,n+1} \cup_{m+1,n}) \cdot T_{m,n} = (H_{m,n} \cup_{m,n+1}) \cdot (\cap_{m+1,n} W^\dagger_{m,n+1}) \cdot T^{\dagger *}_{m,n}$
13. A double arc passes over a fold line near a cusp:
    $(\cap_{m,n+2} H^\dagger_{m+1,n}) \cdot (N^*_{\cap_{m,n+2}, \bar{X}_{m+2,n}} \cup_{m+1,n+1}) \cdot (\bar{X}_{m,n} T_{m,n+1})$
    $= (H_{m,n+1} \cup_{m+2,n}) \cdot (\cap_{m+1,n+1} N^*_{\bar{X}_{m,n+2}, \cup_{m+2,n}}) \cdot (T_{m+1,n} \bar{X}_{m,n})$
14. A triple point near a fold line:
    $(\bar{H}_{m+1,n} A_{i;m,n+2} B_{i;m+1,n+1}) \cdot (\cap_{m+2,n} Z_{i;m,n+1}) \cdot (N_{B_{i;m,n}, \cap_{m+2,n}} A_{i;m+1,n+1} X_{m,n+2})$
    $\cdot(B_{i;m,n} J_{A;m+1,n+1} X_{m,n+2}) \cdot (B_{i;m,n} \cap_{m+1,n+1} N_{X_{m,n+2}, \bar{A}_{i;m+2,n}})$
    $= (\cap_{m+1,n+1} N_{A_{i;m,n+2}, \bar{X}_{m+2,n}} B_{i;m+1,n+1}) \cdot (J_{A;m,n+1} \bar{X}_{m+2,n} B_{i;m+1,n+1})$
    $\cdot(\cap_{m,n+2} \tilde{Z}_{i;m+1,n}) \cdot (N^*_{\cap_{m,n+2}, B_{i;m+2,n}} \bar{X}_{m+1,n+1} \bar{A}_{i;m+2,n}) \cdot (B_{i;m,n} \bar{H}_{m,n+1} \bar{A}_{i;m+2,n})$
    where $i = 0, 1, 2$, and $A_{0;j,k} = X_{j,k}$, $B_{0;j,k} = X_{j,k}$, $Z_{0;j,k} = S^*_{0;j,k}$, $\tilde{Z}_{0;j,k} = \bar{S}_{1;j,k}$; $A_{1;j,k} = X_{j,k}$, $B_{1;j,k} = \bar{X}_{j,k}$, $Z_{1;j,k} = \bar{S}^*_{2;j,k}$, $\tilde{Z}_{1;j,k} = \bar{S}_{0;j,k}$; and $A_{2;j,k} = \bar{X}_{j,k}$, $B_{2;j,k} = \bar{X}_{j,k}$, $Z_{2;j,k} = \bar{S}^*_{1;j,k}$, $\tilde{Z}_{2;j,k} = S_{2;j,k}$. Also, we set $J_{X;j,k} = \bar{H}^*_{j,k}$ and $J_{\bar{X};j,k} = H^*_{j,k}$.
15. $N_{Y_{m,n}, Z_{i,j}}$ is unitary for $Y, Z \in \{X, \bar{X}, \cap, \cup\}$, where m, n, i, j are chosen so that $N_{Y_{m,n}, Z_{i,j}}$ is defined.
16. $(N^*_{Y_{m,n}, Y'_{i,j}} Y''_{k,l}) \cdot (Y'_{i',j} N^*_{Y_{m,n'}, Y''_{k,l}}) \cdot (N^*_{Y'_{i',j}, Y''_{k',l}} Y_{m,n''})$
    $= (Y_{m,n} N^*_{Y'_{i,j}, Y''_{k,l}}) \cdot (N^*_{Y_{m,n}, Y''_{k'',l}} Y'_{i,j'}) \cdot (Y''_{k''',l} N^*_{Y_{m,n'''}, Y'_{i,j'}})$
    where, here and in the relations below, $Y, Y', Y'' \in \{X, \bar{X}, \cap, \cup\}$ and the subscripts are chosen so that all the above 2-morphisms and composites are defined.

17. $(N^*_{Y_{m,n},A_{j,k+1}} B_{j+1,k} C_{j,k+1}) \cdot (A_{j',k+1} N^*_{Y_{m,n},B_{j+1,k}} C_{j,k+1})$
$\cdot (A_{j',k+1} B_{j'+1,k} N^*_{Y_{m,n},C_{j,k+1}}) \cdot (Z_{A,B,C;j',k} Y_{m,n})$
$= (Y_{m,n} Z_{A,B,C;j,k}) \cdot (N^*_{Y_{m,n},C_{j+1,k}} B_{j,k+1} A_{j+1,k})$
$\cdot (C_{j'+1,k} N^*_{Y_{m,n},B_{j,k+1}} A_{j+1,k}) \cdot (C_{j'+1,k} B_{j',k+1} N^*_{Y_{m,n},A_{j+1,k}})$
where $n > k+2$, $A, B, C \in \{X, \bar{X}\}$ satisfy $A_{j,k+1} B_{j+1,k} C_{j,k+1} = \text{source}(Z_{A,B,C;j,k})$ for $Z_{A,B,C;j,k} = S_{i;j,k}$ or $\bar{S}_{i;j,k}$, and j' equals j or $j \pm 2$ depending on Y. Also similar relations where $m > j+2$ and $N^*_{Y_{m,n},\chi_{j,k}}$ is replaced by $N_{\chi_{j,k},Y_{m,n}}$ for $\chi = A, B, C$.

18. $(N^*_{Y_{m,n},\cap_{j,k+1}} \cup_{j+1,k}) \cdot (\cap_{j',k+1} N^*_{Y_{m,n+2},\cup_{j+1,k}}) \cdot (T_{j',k} Y_{m,n}) = Y_{m,n} T_{j,k}$
where $n > k$ and j' equals j or $j \pm 2$ depending on Y. Also:
$(N_{\cap_{j,k'+1},Y_{m+2,n}} \cup_{j+1,k}) \cdot (\cap_{j,k'+1} N_{\cup_{j+1,k'},Y_{m,n}}) \cdot (T_{j,k'} Y_{m,n}) = Y_{m,n} T_{j,k}$
where $m > j$ and k' equals k or $k \pm 2$ depending on Y.

19. $(N_{Y_{m,n},\cap_{j,k}}) \cdot (Y_{m,n} W_{j,k}) = (W_{j',k} Y_{m,n+2}) \cdot (\cap_{j',k} N_{Y_{m,n+2},X_{j,k}}) \cdot (N_{Y_{m,n},\cap_{j,k}} X_{j,k})$
where $n \geq k$ and j' equals j or $j \pm 2$ depending on Y. Also:
$(N^*_{\cap_{j,k},Y_{m+2,n}}) \cdot (Y_{m,n} W_{j,k'}) = (W_{j,k} Y_{m+2,n}) \cdot (\cap_{j,k} N^*_{X_{j,k},Y_{m+2,n}}) \cdot (N^*_{\cap_{j,k},Y_{m+2,n}} X_{j,k'})$
where $m \geq j$ and k' equals k or $k \pm 2$ depending on Y.

20. $(N^*_{Y_{m,n},\cap_{j+1,k}} \bar{X}_{j,k+1}) \cdot (\cap_{j'+1,k} N^*_{Y_{m,n+2},\bar{X}_{j,k+1}}) \cdot (H^*_{j',k} Y_{m,n+2})$
$= (Y_{m,n} H^*_{j,k}) \cdot (N^*_{Y_{m,n},\cap_{j,k+1}} X_{j+1,k}) \cdot (\cap_{j',k+1} N^*_{Y_{m,n+2},X_{j+1,k}})$
where $n > k$, and j' equals j or $j \pm 2$ depending on Y. Also a similar relation where $m > j+2$ and $N^*_{Y,\chi}$ is replaced by $N_{\chi,Y}$ for $\chi = \cap, X, \bar{X}$.

21. A double point arc becomes tangent to the plane of projection:
$(\cap_{m+1,n+1} N^*_{X_{m,n+2},\bar{X}_{m+2,n}} \cup_{m+1,n+1}) \cdot (\cap_{m+1,n+1} \bar{X}_{m+2,n} H^\dagger_{m,n+1})$
$\cdot (\bar{H}_{m+1,n} \bar{X}_{m+1,n+1} \cup_{m,n+2}) \cdot (\cap_{m+2,n} II^*_{m+1,n+1} \cup_{m,n+2}) \cdot (N_{\cup_{m,n},\cap_{m,n}})$
$= (\bar{H}^*_{m,n+1} \bar{X}_{m+2,n} \cup_{m+1,n+1}) \cdot (\cap_{m,n+2} \bar{X}_{m+1,n+1} H^{\dagger *}_{m+1,n})$
$\cdot (\cap_{m,n+2} \bar{II}^*_{m+1,n+1} \cup_{m+2,n}) \cdot (N^*_{\cap_{m,n+2},\cup_{m+2,n}})$

22. $(Y_{m,n} Z_{i,j}) \cdot (N^*_{Y_{m,n},\chi_{i,j}} \chi^*_{i,j}) = (Z_{i',j} Y_{m,n}) \cdot (\chi_{i',j} N_{Y_{m,n'},\chi^*_{i,j}})$
where $i \geq m$, $Z \in \{II, I, E^*\}$, $\text{target}(Z) = \chi\chi^*$, and the subscripts are chosen to be compatible with the restrictions for defining N and the compositions. Also a similar condition where $j \geq n$.

23. A cusp on the double point set: $(X_{m,n} \bar{II}_{m,n}) \cdot (II^*_{m,n} X_{m,n}) = 1_{X_{m,n}}$
Also, a cusp on the set of fold-lines: $(I_{m,n} \cap_{m,n}) \cdot (\cap_{m,n} E_{m,n}) = 1_{\cap_{m,n}}$

24. A horizontal cusp:

$(\cap_{m+1,n}E^*_{m,n+1})\cdot(T^{\dagger *}_{m,n}\cap_{m,n+1}) = (T^*_{m,n}\cap_{m+1,n})\cdot(\cap_{m,n+1}E_{m+1,n})$

25. A triple point passing through a maximum on the double point set:

$(II_{m,n+1}A_{m+1,n}B_{m,n+1})\cdot(X_{m,n+1}\bar{S}_{i;m,n})$

$= (A_{m+1,n}B_{m,n+1}II_{m+1,n})\cdot(Z_{i;m,n}\bar{X}_{m+1,n})$

where $A, B \in \{X, \bar{X}\}$, i satisfies source$(\bar{S}_{i;m,n}) = \bar{X}_{m,n+1}A_{m+1,n}B_{m,n+1}$, and $Z_{i;m,n} = S^*_{j;m,n}$ or $\bar{S}^*_{j;m,n}$ satisfies source$(Z_{i;m,n}) = A_{m+1,n}B_{m,n+1}X_{m+1,n}$.

26. A maximum point of the double point set being pushed through a branch point:

$(\cap_{m,n}II_{m,n})\cdot(W^*_{m,n}\bar{X}_{m,n}) = (\bar{W}_{m,n})$

27. A branch point passes over a maximum point of the surface:

$I_{m,n}\cdot(W_{m,n}\cup_{m,n}) = I_{m,n}\cdot(\cap_{m,n}\bar{W}^{\dagger *}_{m,n})$

28. A double point arc passes over a fold line near a maximum point:

$I_{m,n+1}\cdot(\cap_{m,n+1}II_{m+1,n}\cup_{m,n+1})\cdot(H_{m,n}\bar{X}_{m+1,n}\cup_{m,n+1})$

$= I_{m+1,n}\cdot(\cap_{m+1,n}\bar{II}_{m,n+1}\cup_{m+1,n})\cdot(\cap_{m+1,n}\bar{X}_{m,n+1}H^{\dagger}_{m,n})$

29. A branch point passes over a saddle point of the surface:

$(\cup_{m,n}W_{m,n})\cdot(E_{m,n}X_{m,n}) = (\bar{W}^{\dagger *}_{m,n}\cap_{m,n})\cdot(X_{m,n}E_{m,n})$

30. A double point arc passes over a fold line near a saddle point:

$(H^{\dagger *}_{m,n}\cap_{m+1,n}\bar{X}_{m,n+1})\cdot(X_{m,n+1}E_{m+1,n}\bar{X}_{m,n+1})\cdot II^*_{m,n+1}$

$= (\bar{X}_{m+1,n}\cup_{m,n+1}H^*_{m,n})\cdot(\bar{X}_{m+1,n}E_{m,n+1}X_{m+1,n})\cdot(\bar{II}^*_{m+1,n})$

In addition to each of the above relations $\alpha = \beta$, we include the analogous relations:

- $\bar{\alpha} = \bar{\beta}$, where we impose the relations $\bar{I}_{m,n} = I_{m,n}$, $\bar{E}_{m,n} = E_{m,n}$, $\bar{T}_{m,n} = T_{m,n}$, $\bar{N}_{Y_{m,n},Z_{i,j}} = N_{\bar{Y}_{m,n},\bar{Z}_{i,j}}$, $\bar{1}_f = 1_{\bar{f}}$, and we define $\bar{\alpha}$ for 2-morphisms α other than the generating 2-morphisms listed in items 1 through 9 using the relations $\bar{\bar{\alpha}} = \alpha$, $\overline{\alpha\cdot\beta} = \bar{\alpha}\cdot\bar{\beta}$, and $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$, $\overline{\alpha^\dagger} = \bar{\alpha}^\dagger$, and $\overline{\alpha^*} = \bar{\alpha}^*$.

- $\alpha^\dagger = \beta^\dagger$, where we impose the relations $I^\dagger = I^*$, $E^\dagger = E^*$, $II^\dagger = II^*$, $S^\dagger_{0,m,n} = \bar{S}^*_{0,m,n}$, $S^\dagger_{1,m,n} = S^*_{2,m,n}$, $N^\dagger_{f,g} = N_{f^*,g^*}$ and $1^\dagger_f = 1_{f^*}$, and define $\alpha^\dagger$ for 2-morphisms other than the generating 2-morphisms listed in items 1 through 10 using the relations $\alpha^{\dagger\dagger} = \alpha$, $(\alpha\cdot\beta)^\dagger = \beta^\dagger\cdot\alpha^\dagger$, $(\alpha\beta)^\dagger = \beta^\dagger\alpha^\dagger$, $\bar{\alpha}^\dagger = \overline{\alpha^\dagger}$, and $\alpha^{*\dagger} = \alpha^{\dagger *}$.

- $\alpha^* = \beta^*$, where we impose the relation $1^*_f = 1_f$, and define α^* for 2-morphisms other than the generating 2-morphisms listed in items 1 through 11 using the relations $\alpha^{**} = \alpha$, $(\alpha\cdot\beta)^* = \beta^*\cdot\alpha^*$, and $(\alpha\beta)^* = \alpha^*\beta^*$, $\bar{\alpha}^* = \overline{\alpha^*}$, and $\alpha^{\dagger *} = \alpha^{*\dagger}$.

3.3 $\mathcal{T}$ and $\mathcal{C}$ are Isomorphic

We now show that $\mathcal{T}$ and $\mathcal{C}$ are isomorphic as 2-categories. Recall that two 2-categories $\mathcal{A}, \mathcal{B}$ are said to be 'isomorphic' if there exist 2-functors $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$ that are inverses in the strictest possible sense: $FG = 1_{\mathcal{A}}$ and $GF = 1_{\mathcal{B}}$.

Theorem 17. *$\mathcal{T}$ and $\mathcal{C}$ are isomorphic 2-categories.*

It suffices to construct a 2-functor $F: \mathcal{T} \to \mathcal{C}$ and show that it is bijective on objects, 1-morphisms and 2-morphisms. Each object $A \in \mathcal{T}$ is determined by the number of points in a representative of A. We define $F(A) = A_n$, where n is the number of points in a representative of A. Clearly, F is well defined and bijective on objects, since objects in both $\mathcal{T}$ and $\mathcal{C}$ are uniquely determined by natural numbers.

Each 1-morphism $f_{\mathcal{T}}$ in $\mathcal{T}$ is represented by a generic tangle. The planar diagram of this tangle has crossings and extrema at distinct heights, and thus determines a 1-morphism $f_{\mathcal{C}}$ in $\mathcal{C}$ given as a composite of the generating 1-morphisms. Let $F(f_{\mathcal{T}}) = f_{\mathcal{C}}$. Due to the level-preserving property of equivalence isotopies, two generic tangles representing the same 1-morphism in $\mathcal{T}$ cannot differ by Reidemeister moves or by changing the order of heights of crossings or extrema, so F is well defined on 1-morphisms. Using standard techniques one can construct an equivalence isotopy between any pair of tangles whose equivalence classes are mapped by F to the same 1-morphism in $\mathcal{C}$, so F is injective on 1-morphisms. Finally, every 1-morphism in $\mathcal{C}$ can be realized as the image of a 1-morphism in $\mathcal{T}$. We conclude that F is bijective on 1-morphisms. One can also check that F as defined on objects and 1-morphisms is in fact a functor from the underlying category of $\mathcal{T}$ to the underlying category of $\mathcal{C}$.

Let $\mathcal{D}$ be the 2-category with the same underlying category as $\mathcal{C}$, but with 2-morphisms freely generated as horizontal and vertical composites of the generating 2-morphisms listed in Section 3.2. Each 2-morphism $\alpha_{\mathcal{T}}$ in $\mathcal{T}$ is represented by some generic 2-tangle S. By Theorem 3.5.4 of Carter, Rieger and Saito [9], each singularity of the projection of this 2-tangle to the square $I_3 \times I_4$ corresponds to a generating 2-morphism for $\mathcal{C}$. Furthermore, the proof of this theorem yields a procedure for assigning to S a unique 2-morphism $\alpha_{\mathcal{D}}$ in $\mathcal{D}$.

Now suppose that S' is another generic 2-tangle representing the 2-morphism $\alpha_{\mathcal{T}}$, and let $\alpha'_{\mathcal{D}}$ be the corresponding 2-morphism in $\mathcal{D}$. Then there is an equivalence isotopy carrying S to S'. By Theorem 3.5.5 of Carter, Rieger and Saito, it follows that one can go from $\alpha_{\mathcal{D}}$ to $\alpha'_{\mathcal{D}}$ using the relations listed in Section 3.2, together with the equational laws in the definition of a 2-category. It follows that $\alpha_{\mathcal{D}}$ and $\alpha'_{\mathcal{D}}$ map to the same 2-morphism $\alpha_{\mathcal{C}}$ under the canonical 2-functor from $\mathcal{D}$ to $\mathcal{C}$. Thus we may define F on 2-morphisms by $F(\alpha_{\mathcal{T}}) = \alpha_{\mathcal{C}}$. Theorem 3.5.4 of Carter, Rieger and Saito implies that F is surjective on 2-morphisms, while their Theorem 3.5.5 implies that it is injective. One can easily check that $F: \mathcal{T} \to \mathcal{C}$ is a 2-functor. □

Since $\mathcal{T}$ is a braided monoidal 2-category with duals, we can use the isomorphism $F\colon \mathcal{T} \to \mathcal{C}$ to give $\mathcal{C}$ the structure of a braided monoidal 2-category with duals. We then have the following monoidal, braiding and duality structures on $\mathcal{C}$:

1. $I = A_0$.

2. $A_m \otimes A_n = A_{m+n}$.

3. $A_n \otimes Y_{i,j} = Y_{n+i,j}$ and $Y_{i,j} \otimes A_n = Y_{i,j+n}$ for $Y = X, \bar{X}, \cap$ or $\cup$. $A_n \otimes 1_m = 1_n \otimes A_m = 1_{n+m}$. The tensor products of objects with other 1-morphisms are determined by the relations

$$A \otimes (fg) = (A \otimes f)(A \otimes g)$$

and

$$(fg) \otimes A = (f \otimes A)(g \otimes A).$$

4. $\bigotimes_{Y_{i,j},Z_{m,n}} = N_{Y_{i,j+z},Z_{y+m,n}}$, $\bigotimes_{Y_{i,j},1_n} = 1_{Y_{i,j+n}}$ and $\bigotimes_{1_n,Y_{i,j}} = 1_{Y_{n+i,j}}$ for $Y, Z \in \{X, \bar{X}, \cup, \cap\}$, where source$(Z_{m,n}) = A_z$ and target$(Y_{i,j}) = A_y$. The tensorator is determined for other 1-morphisms by the relations

$$\bigotimes_{f,gg'} = ((A \otimes g)\bigotimes_{f,g'}) \cdot (\bigotimes_{f,g}(A' \otimes g'))$$

for $f\colon A \to A'$, $g\colon B \to B'$ and $g'\colon B' \to B''$, and

$$\bigotimes_{ff',g} = (\bigotimes_{f,g}(f' \otimes B')) \cdot ((f \otimes B)\bigotimes_{f',g})$$

for $f\colon A \to A'$, $f'\colon A' \to A''$ and $g\colon B \to B'$.

5. $A_n \otimes Y_{i,j} = Y_{n+i,j}$ and $Y_{i,j} \otimes A_n = Y_{i,j+n}$ for $Y = I, E, W, II, S_{i,}, H, T$ and the $^-$, $*$, or $\dagger$ of these 2-morphisms; $A_n \otimes N_{Y_{i,j},Z_{k,l}} = N_{Y_{n+i,j},Z_{n+k,l}}$ and $N_{Y_{i,j},Z_{k,l}} \otimes A_n = N_{Y_{i,j+n},Z_{k,l+n}}$; $A_n \otimes 1_f = 1_{A_n \otimes f}$ and $1_f \otimes A_n = 1_{f \otimes A_n}$. The tensor products of objects with other 2-morphisms are determined by the relations

$$A \otimes (\alpha\beta) = (A \otimes \alpha)(A \otimes \beta)$$

and

$$(\alpha\beta) \otimes A = (\alpha \otimes A)(\beta \otimes A).$$

6. $R_{A_0,A_n} = R_{A_n,A_0} = 1_n$, $R_{A_1,A_1} = X_{0,0}$, and R_{A_n,A_m} is determined for other n, m by the relations

$$R_{A_1,A_{m+n}} = (R_{A_1,A_m} \otimes A_n)(A_m \otimes R_{A_1,A_n})$$

and

$$R_{A_{m+n},A_i} = (A_m \otimes R_{A_n,A_i})(R_{A_m,A_i} \otimes A_n).$$

7. $\tilde{R}_{(A,B|C)} = 1_{(A\otimes R_{B,C})(R_{A,C}\otimes B)}$.

8. $\tilde{R}_{(A_1|B,C)} = 1_{(R_{A_1,B}\otimes C)(B\otimes R_{A_1,C})}$, while $\tilde{R}_{(A_n|B,C)}$ is determined for other n using condition $((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$ in the definition of braided monoidal 2-category.

9. $R_{A_1,X_{0,0}} = S^*_{0;0,0}$, $R_{X_{0,0},A_1} = S_{0;0,0}$, $R_{A_1,\bar{X}_{0,0}} = S^*_{1;0,0}$, $R_{\bar{X}_{0,0},A_1} = \bar{S}_{2;0,0}$,
$R_{A_1,\cap_{0,0}} = (\bar{H}^*_{0,0}X_{1,0}) \cdot (\cap_{0,1}\bar{I\!I}^*_{1,0})$, $R_{\cap_{0,0},A_1} = (H_{0,0}X_{0,1}) \cdot (\cap_{1,0}\bar{I\!I}^*_{0,1})$,
$R_{A_1,\cup_{0,0}} = (I\!I_{0,1}\cup_{1,0})(X_{0,1}\bar{H}^\dagger_{0,0})$, $R_{\cup_{0,0},A_1} = (I\!I_{1,0}\cup_{0,1})(X_{1,0}H^{\dagger *}_{0,0})$,
$R_{A_1,1_n} = 1_{R_{A_1,A_n}}$ and $R_{1_n,A_1} = 1_{R_{A_n,A_1}}$, while $R_{A,f}$ and $R_{f,A}$ for other 1-morphisms f and objects A are determined using conditions $((\bullet \otimes \bullet)\otimes \rightarrow)$, $(\rightarrow \otimes(\bullet \otimes \bullet))$, $((\rightarrow \otimes\bullet) \otimes \bullet)$, $((\bullet\otimes \rightarrow) \otimes \bullet)$, $(\bullet \otimes (\rightarrow \otimes\bullet))$, and $(\bullet \otimes (\bullet\otimes \rightarrow))$ in the definition of braided monoidal 2-category.

10. $A^*_n = A_n$, and the duals of 1-morphisms and 2-morphisms are given as in the definition of $\mathcal{C}$.

11. $i_{A_0} = 1_{A_0}$ and $i_{A_1} = \cap_{0,0}$, while the unit is determined for other objects by the relation
$$i_{A\otimes B} = i_A(A \otimes i_B \otimes A^*).$$

12. $e_{A_0} = 1_{A_0}$ and $e_{A_1} = \cup_{0,0}$, while the counit is determined for other objects by the relation
$$e_{A\otimes B} = (A^* \otimes e_B \otimes A)e_A.$$

13. $i_{\cap_{m,n}} = I_{m,n}$, $i_{\cup_{m,n}} = E^*_{m,n}$, $i_{X_{m,n}} = I\!I_{m,n}$, $i_{\bar{X}_{m,n}} = \bar{I\!I}_{m,n}$, $i_{1_n} = 1_{1_n}$. For other 1-morphisms, the unit is determined using the relation
$$i_{fg} = i_f \cdot (f i_g f^*).$$

14. $e_{\cap_{m,n}} = E_{m,n}$, $e_{\cup_{m,n}} = I^*_{m,n}$, $e_{X_{m,n}} = \bar{I\!I}^*_{m,n}$, $e_{\bar{X}_{m,n}} = I\!I^*_{m,n}$, $e_{1_n} = 1_{1_n}$. For other 1-morphisms, the unit is determined using the relation
$$e_{fg} = (g^* e_f g) \cdot e_g.$$

15. $T_{A_0} = 1_{1_0}$, $T_{A_1} = T_{0,0}$, and T_{A_n} is determined for other n using the relation
$$T_{A\otimes B} = ((i_A \otimes A \otimes B)(A \otimes \bigotimes^{-1}_{i_B,e_A} \otimes B)(A \otimes B \otimes e_B)) \cdot ((T_A \otimes B)(A \otimes T_B)).$$

Moreover, the definition of the † operation on 2-morphisms of $\mathcal{C}$ agrees with the general definition of 'adjoint' for 2-morphisms in a monoidal 2-category with duals. For example, for the generator $H_{m,n}$, this follows from relations 23 and 28 in Section 3.2.

In what follows we sometimes apply the notation developed for $\mathcal{C}$ to the 2-category $\mathcal{T}$, using the isomorphism F. Note in particular that $A_1 \in \mathcal{C}$ is an unframed self-dual object with writhing $W_{0,0}$.

4 The Universal Property of 2-Tangles

In this section, we use the isomorphism between $\mathcal{T}$ and $\mathcal{C}$ to characterize $\mathcal{T}$ in terms of a universal property. Namely, we show that $\mathcal{T}$ is the free braided monoidal 2-category with duals on the unframed self-dual object Z corresponding to a single point in the unit square. To do this, we first define what it means for a braided monoidal 2-category with duals to be 'generated' by an unframed self-dual object, and show that $\mathcal{T}$ is generated by Z in this sense. To describe the sense in which $\mathcal{T}$ is 'freely' generated by Z, we define what it means for a strict monoidal 2-functor to 'semistrictly preserve braiding and duals on an unframed self-dual generating object'. Then we show that for any braided monoidal 2-category with duals $\mathcal{A}$ containing an unframed self-dual object A, there exists a unique strict monoidal 2-functor $F\colon \mathcal{T} \to \mathcal{A}$ with this property that maps Z to A. This universal property characterizes $\mathcal{T}$ up to isomorphism.

Definition 18. *We say a braided monoidal 2-category is* generated *by an unframed self-dual object A if:*

1. Every object can be obtained by tensoring from:
    (a) I,
    (b) A.
2. Every 1-morphism can be obtained by composition from:
    (a) 1_A,
    (b) i_A,
    (c) $R_{A,A}$,
    (d) tensor products of the above 1-morphisms with arbitrary objects,
    (e) duals of the above 1-morphisms.
3. Every 2-morphism can be obtained by horizontal and vertical composition from:
    (a) 2-morphisms 1_f for arbitrary 1-morphisms f,
    (b) 2-morphisms $\bigotimes_{f,g}$ for arbitrary 1-morphisms f and g,
    (c) 2-morphisms $R_{A,f}$ and $R_{f,A}$ for the 1-morphisms f listed in a) - e) above,
    (d) 2-morphisms i_f for arbitrary 1-morphisms f,
    (e) T_A,
    (f) W_A,
    (g) tensor products of arbitrary objects with the above 2-morphisms,

(h) duals of the above 2-morphisms.

Theorem 19. *$\mathcal{T}$ is a braided monoidal 2-category generated by the unframed self-dual object Z.*

Proof - Using the isomorphism between $\mathcal{T}$ and $\mathcal{C}$, the definition of $\mathcal{C}$ immediately implies that every object in $\mathcal{T}$ is either I or a tensor product of copies of Z. It also implies that every 1-morphism is a composite of the 1-morphisms $1_Z, i_Z, R_{Z,Z}$, tensor products of these 1-morphisms with objects of $\mathcal{T}$, and duals thereof.

Similarly, we immediately see that any 2-morphism in $\mathcal{T}$ can be obtained by horizontal and vertical composition from the 2-morphism generators listed in Section 3.2. Thus it suffices to describe these generators as horizontal and vertical composites of the 2-morphisms listed in clauses 3a) - 3h) of Definition 18. We can simplify this task with the help of a few observations. First, the case of identity 2-morphisms is trivial. Second, for any 2-morphism generator α, the corresponding 2-morphisms α^* and $\alpha^\dagger$ as given in Section 3.2 are the same as the dual and adjoint in the 2-category sense, so we do not need to describe these variants of α. Similarly, we do not need to describe the variant $\bar{\alpha}$ when it equals α, and we do need to describe $\bar{N}_{Y,Z}$, since $\bar{N}_{Y,Z} = N_{\bar{Y},\bar{Z}}$. Finally, since $\alpha_{m,n} = A_m \otimes \alpha_{0,0} \otimes A_n$, it suffices to describe the following 2-morphisms:

1. $I_{0,0} = i_{i_Z}$
2. $E_{0,0} = e_{i_Z}$
3. $W_{0,0} = W_Z$, $\bar{W}_{0,0} = (i_Z i_{R_{Z,Z}}) \cdot (W_Z^* R_{Z,Z}^*)$
4. $II_{0,0} = i_{R_{Z,Z}}$, $\bar{II}_{0,0} = i_{R^*_{Z,Z}}$
5. $S_{0;0,0} = R_{R_{Z,Z},Z}$, $S_{1;0,0} = R^*_{Z,R^*_{Z,Z}}$, $S_{2;0,0} = R^\dagger_{Z,R^*_{Z,Z}}$

   $\bar{S}_{0;0,0} = R^\dagger_{Z,R_{Z,Z}}$, $\bar{S}_{1;0,0} = R^{\dagger *}_{R^*_{Z,Z},Z}$, $\bar{S}_{2;0,0} = R_{R^*_{Z,Z},Z}$
6. $H_{0,0} = ((i_Z \otimes Z)(Z \otimes R_{Z,Z})(i_{R_{Z,Z}} \otimes Z)) \cdot (R_{i_Z,Z}(R^*_{Z,Z} \otimes Z))$

   $\bar{H}_{0,0} = ((i_Z \otimes Z)(Z \otimes R^*_{Z,Z})(i_{R^*_{Z,Z}} \otimes Z)) \cdot (R^\dagger_{Z,i^*_Z}(R_{Z,Z} \otimes Z))$
7. $T_{0,0} = T_Z$
8. $N_{Y_{m,n},Z_{i,j}} = \bigotimes_{Y_{m,n'},Z_{0,j}}$ for some n'

All the above are composites of the 2-morphisms listed in clauses 3a) - 3h) of Definition 18, so the braided monoidal 2-category $\mathcal{T}$ is generated by the unframed self-dual object Z. $\square$

Now we describe the sense in which $\mathcal{T}$ is 'freely' generated by the unframed self-dual object Z. Naively, one might hope that for any braided monoidal 2-category $\mathcal{B}$ containing an unframed self-dual object B, there would exist a unique 2-functor $F: \mathcal{T} \to \mathcal{B}$ sending Z to B and strictly preserving all the structure in sight, at least on the generating object: the monoidal structure, braiding and duals. However, this is too much to ask. Uniqueness is no problem, but such a 2-functor might not exist, because $\tilde{R}_{(Z|Z,Z)}$ and $\tilde{R}_{(Z,Z|Z)}$ are identity 2-morphisms, while this might not be true of $\tilde{R}_{(B|B,B)}$ and $\tilde{R}_{(B,B|B)}$. One way to deal with this is to include conditions ensuring this in the definition of 'unframed self-dual object'. This is basically the approach we took in our earlier short paper [6].

While this approach is consistent, it seems unnecessarily restrictive to impose these extra conditions on the object B. Probably there is a strictification theorem saying that these conditions represent no essential loss of generality. However, such a theorem has not yet been proved. The generalized center construction of HDA1 shows that one can safely assume either that $\tilde{R}_{(\cdot|\cdot,\cdot)}$ or $\tilde{R}_{(\cdot,\cdot|\cdot)}$ is the identity. Unfortunately, we do not see how to use it to simultaneously set *both* these braiding coherence 2-morphisms equal to the identity, even for a single object.

The approach we take here is thus to weaken our insistence that $F: \mathcal{T} \to \mathcal{B}$ strictly preserve all the structure in sight. This allows for the possibility that $\tilde{R}_{(B|B,B)}$ and $\tilde{R}_{(B,B|B)}$ are not identity 2-morphisms. As a result, these 2-morphisms appear as 'padding' in some of the equations in Definition 21.

Definition 20. *For monoidal 2-categories $\mathcal{A}$ and $\mathcal{B}$, a* strict monoidal 2-functor *$F: \mathcal{A} \to \mathcal{B}$ is a 2-functor such that $F(I) = I$, $F(A \otimes X) = F(A) \otimes F(X)$ and $F(X \otimes A) = F(X) \otimes F(A)$ for any object A and object, 1-morphism or 2-morphism X, and $F(\bigotimes_{f,g}) = \bigotimes_{F(f),F(g)}$ for any 1-morphisms f and g.*

Definition 21. *Suppose that $\mathcal{A}, \mathcal{B}$ are braided monoidal 2-categories with duals and that $\mathcal{A}$ is generated by an unframed self-dual object A with $\tilde{R}_{(A|A,A)} = 1$, $\tilde{R}_{(A,A|A)} = 1$. We say a strict monoidal 2-functor $F: \mathcal{A} \to \mathcal{B}$ mapping A to an unframed self-dual object $B \in \mathcal{B}$* preserves braiding and duals semistrictly on the generator *if:*

1. $F(X^*) = F(X)^*$ for every object, morphism, or 2-morphism X,
2. $F(i_f) = i_{F(f)}$ for every morphism f,
3. $F(i_A) = i_B$,
4. $F(T_A) = T_B$,
5. $F(W_A) = W_B$,
6. $F(R_{A,A}) = R_{B,B}$,

7. $F(R_{A,f}) = (B \otimes F(f))\tilde{R}_{(B|B,B)}) \cdot R_{B,F(f)} \cdot (\tilde{R}^{-1}_{(B|B,B)}(F(f) \otimes B))$
   and
   $F(R_{f,A}) = ((F(f) \otimes B)\tilde{R}_{(B,B|B)}) \cdot R_{F(f),B} \cdot (\tilde{R}^{-1}_{(B,B|B)}(B \otimes F(f)))$
   for $f = R_{A,A}, R^*_{A,A}$.

8. $F(R_{A,i_A}) = ((B \otimes i_B)\tilde{R}_{(B|B,B)}) \cdot R_{B,i_B}$
   and
   $F(R_{i_A,A}) = ((i_B \otimes B)\tilde{R}_{(B,B|B)}) \cdot R_{i_B,B}$

9. $F(R_{A,e_A}) = R_{B,e_B} \cdot (\tilde{R}^{-1}_{(B|B,B)}(e_B \otimes B))$
   and
   $F(R_{e_A,A}) = R_{e_B,B} \cdot (\tilde{R}^{-1}_{(B,B|B)}(B \otimes e_B))$

Note that in the above definition we still assume that $\tilde{R}_{(A|A,A)} = 1$ and $\tilde{R}_{(A,A|A)} = 1$ for the unframed self-dual object A generating the 2-category $\mathcal{A}$. We could drop this assumption at the expense of still more padding in conditions 7 – 9, but we do not need this extra generality.

The following theorem is the main result of this paper:

Theorem 22. *For any braided monoidal 2-category with duals $\mathcal{B}$ and unframed self-dual object $B \in \mathcal{B}$, there exists a unique strict monoidal 2-functor $F: \mathcal{T} \to \mathcal{B}$ with $F(Z) = B$ that preserves braiding and duals semistrictly on the generator.*

Proof - Uniqueness follows straightforwardly from the fact that $\mathcal{T}$ is generated by Z, together with the fact that F is a strict monoidal 2-functor preserving the braiding and duals semistrictly on the generator. Together these suffice to determine F on any object, morphism, or 2-morphism of $\mathcal{T}$.

For existence we use the isomorphism $\mathcal{T} \cong \mathcal{C}$ to describe $\mathcal{T}$ using generators and relations, and show that all the relations are mapped by F to equations that actually hold in $\mathcal{B}$. For objects and 1-morphisms there are no nontrivial relations. For 2-morphisms we need to check that $F(\alpha) = F(\beta)$ for every equation $\alpha = \beta$ in the list of 30 relations given in Section 3.2. In addition we need to show that $F(\alpha^*) = F(\beta^*)$, $F(\alpha^\dagger) = F(\beta^\dagger)$, and $F(\bar{\alpha}) = F(\bar{\beta})$. The first two follow automatically from $F(\alpha) = F(\beta)$, since

$$F(\alpha^*) = F(\alpha)^* = F(\beta)^* = F(\beta^*)$$

and

$$F(\alpha^\dagger) = F(\alpha)^\dagger = F(\beta)^\dagger = F(\beta^\dagger),$$

using the fact that if $\alpha\colon f \Rightarrow g$,

$$\begin{array}{rcl} F(\alpha^\dagger) &=& F((g^* i_f)\cdot(g^*\alpha f^*)\cdot(e_g f^*)) \\ &=& (F(g)^* i_{F(f)})\cdot(F(g)^* F(\alpha) F(f)^*)\cdot(e_{F(g)} F(f)^*) \\ &=& F(\alpha)^\dagger. \end{array}$$

Thus we only need to check the 'barred' version $F(\bar\alpha) = F(\bar\beta)$. We skip this in cases where $\bar\alpha = \alpha$ and $\bar\beta = \beta$.

In what follows we sketch how to show $F(\alpha) = F(\beta)$ and $F(\bar\alpha) = F(\bar\beta)$ for each equation $\alpha = \beta$ in the list of relations in Section 3.2. We only use the definition of a braided monoidal 2-category, the definition of an unframed self-dual object, and the properties of F. The full arguments are lengthy and complicated, so we just indicate the main ideas.

1,2. These follow from the unitarity of the writhing W_B.

3,4. These follow from the unitarity of $i_{R_{B,B}}$.

5. These and their barred versions follow from clause 7 in Definition 21, together with the fact that $R_{R_{B,B},B}$, $R_{B,R_{B,B}}$, $\tilde R_{(B|B,B)}$, and $\tilde R_{(B,B|B)}$ are unitary. For example,

$$\begin{array}{rcl} F(S_{0;0,0}) &=& F(R_{R_{Z,Z},Z}) \\ &=& ((R_{B,B}\otimes B)\tilde R_{(B,B|B)})\cdot R_{R_{B,B},B}\cdot(\tilde R^{-1}_{(B,B|B)}(B\otimes R_{B,B}). \end{array}$$

Being a composite of unitary 2-morphisms, $F(S_{0;0,0})$ is unitary. Using the relations between tensoring and duality, it follows that $F(S_{0;n,m})$ is unitary for all n, m.

6. Let us write $\alpha = \beta$ for any of these relations holding in $\mathcal{T}$. These are all variants of the Zamolodchikov tetrahedron equation. Because of clause 7 in Definition 21, the corresponding equations $F(\alpha) = F(\beta)$ contain extra 'padding' built from the 2-isomorphisms $\tilde R_{(B|B,B)}$, $\tilde R_{(B,B|B)}$, and their inverses. However, one can cancel out this padding, reducing $F(\alpha) = F(\beta)$ to the corresponding version of the Zamolodchikov tetrahedron equation for the object B in $\mathcal{B}$. This, in turn, follows from the definition of a braided monoidal 2-category (see Kapranov and Voevodsky [20] and the comments in HDA1). The barred versions $F(\bar\alpha) = F(\bar\beta)$ follow from the fact that $\bar R_{f,A} = R^\dagger_{A,\bar f^*}$ and $\bar R_{A,f} = R^\dagger_{\bar f^*,A}$.

7. These follow from conditions $(\bullet\otimes\Downarrow)$ and $(\Downarrow\otimes\bullet)$ in the definition of braided monoidal 2-category, applied to the 2-morphisms W_B and $\bar W_B$. Again, the equation in $\mathcal{B}$ contains padding built from $\tilde R_{(B|B,B)}$, $\tilde R_{(B,B|B)}$, and their inverses, but this padding cancels. And again, the barred versions follow from the fact that $\bar R_{f,A} = R^\dagger_{A,\bar f}$ and $\bar R_{A,f} = R^\dagger_{\bar f,A}$.

8,9. These follow from the unitarity of the triangulator T_B.

10. This follows from the swallowtail coherence law.

11. This and its barred version follow from clause 6 in Theorem 19, together with the fact that $i_{R_{B,B}}$, $i_{R^*_{B,B}}$, $R_{i_B,B}$, R_{B,i_B}, $\tilde R_{(B,B|B)}$, and $\tilde R_{(B|B,B)}$ are unitary.

12. This follows from the coherence law satisfied by the writhing W_B.

13. This follows from condition ($\Downarrow \otimes \bullet$) in the definition of braided monoidal 2-category, applied to the 2-morphism T. The barred version works similarly.

14. Half of these and their barred versions follow from condition ($\to \otimes \to$) in the definition of braided monoidal 2-category, applied to the 1-morphisms $R_{B,B}$ or its dual and i_B or its dual. The rest follow from ($\Downarrow \otimes \bullet$) and ($\bullet \otimes \Downarrow$) applied to the 2-morphisms R_{Z,i_Z} and $R^{\dagger}_{e_Z,Z}$, together with extensive use of the monoidal 2-category axioms and equation 25 below. The equations to be proved contain padding built from $\tilde{R}_{(B|B,B)}$, $\tilde{R}_{(B,B|B)}$, and their inverses, but this padding cancels.

15. These follow from the unitarity of the tensorator $\bigotimes_{f,g}$ for generating 1-morphisms f, g.

16. This is essentially the Yang-Baxter equation for the tensorator, which follows from conditions (vii) and (viii) in the definition of a monoidal 2-category, using an argument analogous to the usual proof of the Yang-Baxter equation for the braiding in a braided monoidal category. (Indeed, the latter is a special case of the former, since a monoidal 2-category with one object is a braided monoidal category.)

17. These follow from conditions (vi) and (vii) in the definition of a monoidal 2-category, applied to generating 1-morphisms and the 2-morphisms $R_{R_{B,B},B}$, $R^*_{B,R^*_{B,B}}$, $R^{\dagger}_{B,R^*_{B,B}}$, $R^{\dagger}_{Z,R_{Z,Z}}$, $R^{\dagger *}_{R^*_{Z,Z},Z}$, and $R_{R^*_{Z,Z},Z}$. Again, the equations to be proved contain padding built from $\tilde{R}_{(B|B,B)}$ and $\tilde{R}_{(B,B|B)}$, which cancels.

18. These follow from conditions (vi) and (vii) in the definition of a monoidal 2-category, applied to generating 1-morphisms and the 2-morphism T_B.

19. These and their barred versions follow from conditions (vi) and (vii) in the definition of a monoidal 2-category, applied to generating 1-morphisms and the 2-morphisms W_B and $\bar{W}_B$.

20. These and their barred versions follow from conditions (vi) and (vii) in the definition of a monoidal 2-category, applied to generating 1-morphisms and the 2-morphisms $H_{B,B}$ and $\bar{H}_{B,B}$. Again, the equations to be proved contain padding built from $\tilde{R}_{(B|B,B)}$ and $\tilde{R}_{(B,B|B)}$, which cancels.

21. This and its barred version follow from condition ($\to \otimes \to$) in the definition of braided monoidal 2-category, applied to the 1-morphisms i_B and e_B, together with extensive use of the 2-category axioms, the triangle equations, and the formulas

$$R_{e_B,B} = (i_{R_{B\otimes B,B}}(e_B \otimes B)) \cdot (R_{B\otimes B,B} R^{\dagger *}_{i_B,B}),$$

$$R_{i_B,B} = (R^{\dagger *}_{e_B,B} R_{B\otimes B,B}) \cdot ((B \otimes i_B) e_{R_{B\otimes B,B}}),$$

together with similar formulas for R_{B,e_B} and R_{B,i_B}. Again, the equations to be proved contain padding built from $\tilde{R}_{(B|B,B)}$ and $\tilde{R}_{(B,B|B)}$, which cancels.

22. These and their barred versions follow from conditions (vi) and (vii) in the definition of a monoidal 2-category, applied to generating 1-morphisms and the 2-morphisms $i_{R_{B,B}}, i_{R^*_{B,B}}, i_{i_B}$, and i_{e_B}. Again, the equations to be proved contain padding built from $\tilde{R}_{(B|B,B)}$ and $\tilde{R}_{(B,B|B)}$, which cancels.

23. These and their barred versions follow from the triangle equation in the definition of a monoidal 2-category with duals, $(i_f f) \cdot (f e_f) = 1_f$, applied to the 1-morphisms $R_{B,B}, R^*_{B,B}$, and i_B.

24. This follows from the triangle equation $(f^* i_f) \cdot (e_f f^*) = 1_{f^*}$ applied to the 1-morphism e_B, making extensive use of the monoidal 2-category axioms.

25. These and their barred versions follow from conditions $(\bullet\otimes \Downarrow)$ and $(\Downarrow \otimes\bullet)$ in the definition of braided monoidal 2-category, applied to the 2-morphisms $i_{R_{B,B}}$ and $i_{R^*_{B,B}}$. For some the proof is fairly straightforward; for the rest one must cleverly exploit the monoidal 2-category axioms, the invertibility of $i_{R_{B,B}}$ and $i_{R^*_{B,B}}$, and the definition of adjoint 2-morphisms. In all cases, the equations to be proved contain padding built from $\tilde{R}_{(B|B,B)}$ and $\tilde{R}_{(B,B|B)}$, which cancels.

26. This and its barred version follow from the coherence law for the writhing W_B, with the help of Lemma 16.

27. This and its barred version follows from the equation $\alpha^{\dagger *} = \alpha^{*\dagger}$ in the definition of a monoidal 2-category with duals applied to the 2-morphism W_B, together with the coherence law for the writhing W_B, the triangle equations, and extensive use of the 2-category axioms.

28. This follows from condition $(\Downarrow \otimes\bullet)$ in the definition of braided monoidal 2-category applied to the 2-morphism i_{i_B}, together with extensive use of the 2-category axioms and the formulas for $R_{e_B,B}$ and $R_{i_B,B}$ used in the proof of 21. Again, the equation to be proved contains padding built from $\tilde{R}_{(B|B,B)}$ and $\tilde{R}_{(B,B|B)}$, which cancels. The barred version works similarly.

29. This equation is actually redundant, since it follows from 23 and 27 using only the 2-category axioms.

30. This and its barred version follow from conditions $(\bullet\otimes \Downarrow)$ and $(\Downarrow \otimes\bullet)$ in the definition of braided monoidal 2-category, applied to the 2-morphism e_{i_B}. The equations to be proved contain padding built from $\tilde{R}_{(B|B,B)}$ and $\tilde{R}_{(B,B|B)}$, which cancels. □

5 Conclusions

We have shown that the 2-category of 2-tangles in 4 dimensions is the free braided monoidal 2-category with duals on an unframed self-dual object. Given any self-dual unframed object in a braided monoidal 2-category with duals, our proof of this result gives a concrete recipe for computing a 2-tangle invariant. Of course, for this to be useful, we need more examples of braided monoidal 2-categories with duals. Obtaining these will require further work in higher-dimensional algebra. There are a number of promising strategies.

First, there is plenty of evidence that a certain class of braided monoidal 2-categories with duals, the braided monoidal 2-groupoids, are essentially the same as homotopy 2-types of double loop spaces [3, 5]. These should give 2-tangle invariants with a 'purely homotopy-theoretic' flavor. In particular, for any compact

2-dimensional submanifold $\Sigma \subset \mathbb{R}^4$, these invariants should depend only on the homotopy type of $\mathbb{R}^4 - \Sigma$.

Second, one can construct braided monoidal 2-categories as the 'quantum doubles' of monoidal 2-categories [7, 15]. It seems plausible that applying this construction to a monoidal 2-category with duals will give a braided monoidal 2-category with duals. This reduces the question to obtaining monoidal 2-categories with duals. The 2-category of unitary representations of a 2-groupoid should be a monoidal 2-category with duals, just as the category of unitary representations of a groupoid is a monoidal category with duals [1]. Moreover, the 2-category of representations of any Hopf category is a monoidal 2-category [23], and when 'unitary' representations can be be defined, the 2-category of unitary representations should be a monoidal 2-category with duals. Some examples of Hopf categories and related structures have been studied by Neuchl [23] as well as Crane and Yetter [13, 14]. Also, Crane and Frenkel have sketched an interesting construction of a Hopf category from Kashiwara and Lusztig's canonical basis of a quantum group [12].

Third, just as one can construct braided monoidal categories from solutions of the Yang-Baxter equation, one can construct braided monoidal 2-categories from solutions of the Zamolodchikov tetrahedron equations [20]. Many such solutions are known [11], so one may hope that some give braided monoidal 2-categories with duals.

Finally, one expects 'braided monoidal 3-Hilbert spaces' to be interesting examples of braided monoidal 2-categories with duals [1]. However, to obtain these we will probably need to use some of the constructions sketched above.

Our result and its proof can probably be improved in various ways. First, we expect similar algebraic characterizations of the 2-category of framed and/or oriented 2-tangles in 4 dimensions, where we drop the conditions that the object be unframed and/or self-dual. Of course, our current definition of 'unframed' applies only to self-dual objects, so we need to more clearly separate these concepts. Moreover, our definition of 'braided monoidal 2-category with duals' may need some extra conditions to handle framed tangles. For example, the 2-morphism corresponding to the framed Reidemeister I move exists under the current definition of a braided monoidal 2-category with duals, without using the writhing, but we are unable to use this definition to show that this 2-morphism is unitary, even though topological considerations say it should be.

Second, one should be able to compress the definition of 'monoidal 2-category with duals' using more of the language of 2-category theory. Doing so will shed more light on the still mysterious general notion of 'n-category with duals'. It bodes well that the triangulator and its swallowtail coherence law have already been observed by Street in his study of adjunctions between 2-categories [27]. In general, we expect a close relation between the theory of n-categories with duals and the theory of adjunctions between n-categories. This has already been noted in work on 2-Hilbert spaces [1, 22], and the patterns found here should continue for higher n-Hilbert spaces.

Finally, and most importantly, there must be a way to state and prove the tangle hypothesis for all values of n and k that does not involve long lists of equations. Our treatment of the case $n = k = 2$ resembles moving a house across the country by taking it apart, sending the pieces by mail, and then rebuilding it on the other side. First Carter, Saito and Rieger deduced their list of movie moves using a classification of singularities. Then we showed that their movie moves are equivalent to a long form of the definition of 'braided monoidal 2-category with duals generated by an unframed self-dual object'. But this long definition can presumably expressed much more tersely using more sophisticated higher-dimensional algebra. There should thus be a more conceptual approach that proceeds at a higher level of abstraction. While already desirable for $n = k = 2$, the advantages of such an approach will be even greater for larger n and k.

Finding a more conceptual approach to the tangle hypothesis poses many interesting challenges in n-category theory. In particular, it will require a deeper understanding of the mysterious relationship between n-categories and singularity theory. We hope the present work provides some useful clues.

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Errata

Our previous paper [6] has the following errors:

1. The definition of an 'unframed object' A should include the requirement that $A^* = A$. All appearances of A^* in this definition can thus be replaced by A.

2. The pictures in Figure 1 and Figure 3 should be switched.

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