Higher Gauge Theory

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More details at:
http://math.ucr.edu/home/baez/highergauge/
Gauge Theory

Ordinary gauge theory describes how point particles transform as they move along paths:

\[
\begin{array}{c}
\bullet \\
\text{\(g\)} \\
\bullet
\end{array}
\]

It is natural to assign a group element to each path:

Why?
Composition of paths corresponds to multiplication:

\[ g \quad \quad \quad \quad g' \]

Reversing the direction of a path corresponds to taking inverses:

\[ g^{-1} \]

The associative law makes parallel transport along a triple composite unambiguous:

\[ g \quad g' \quad g'' \]

So: *the topology dictates the algebra!*

The electromagnetic field is described using the group $U(1)$. Other forces are described using other groups.
Higher Gauge Theory

Higher gauge theory describes not just how point particles but also how 1-dimensional objects transform as we move them. This leads to the concept of a 2-group.

A 2-group has objects:

\[ g \]

and also morphisms:

\[ g' \]

\[ f \]
We can multiply objects:

\[ g \quad g' \]

multiply morphisms:

\[ g_1 \quad g' \quad g_2 \quad g' \]

and also compose morphisms:

\[ g \quad g' \quad f \quad f' \quad g'' \]

Various laws should hold... all obvious from the pictures!
Each operation has a unit and inverses. Each operation is associative, so these are well-defined:

Finally, the **interchange law**, holds, meaning this is well-defined:

That’s all a 2-group is.
Crossed Modules

A 2-group $\mathcal{G}$ is determined by the quadruple $(G, H, t, \alpha)$ consisting of:

- the group $G$ consisting of all objects of $\mathcal{G}$:

  $\bullet$  

  ![Diagram of G](image)

- the group $H$ consisting of all morphisms of $\mathcal{G}$ with source 1:

  $\bullet$

  ![Diagram of H](image)
• the homomorphism $t: H \to G$ sending each element of $H$ to its target:

\[
\begin{array}{c}
\bullet \\
\downarrow h \\
\bullet \\
\end{array}
\quad 1 \\
\begin{array}{c}
\bullet \\
t(h) \\
\end{array}
\]

• the action $\alpha$ of $G$ on $H$ defined by:

\[
\alpha(g)(h) = \begin{array}{c}
\bullet \\
\downarrow 1_g \\
g \\
g \\
\downarrow 1 \\
t(h) \\
\bullet \\
\downarrow 1_g^{-1} \\
g^{-1} \\
g^{-1}
\end{array}
\]

For any 2-group $\mathcal{G}$, the quadruple $(G, H, t, \alpha)$ satisfies two equations making it a **crossed module**. Conversely, any crossed module gives a 2-group.
Examples of 2-Groups

- Any group $G$ gives a 2-group with $H = 1$. So:

  ordinary gauge theory $\subseteq$ higher gauge theory

In ordinary gauge theory, the gauge field is a connection: locally a $g$-valued 1-form $A$. We cleverly integrate this along paths to get elements of $G$:

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) [circle, draw] {};
\node (b) at (1,0) [circle, draw] {};
\draw [->] (a) to node [auto] {$g$} (b);
\end{tikzpicture}
\end{center}
Any abelian 2-group $H$ gives a 2-group with $G = 1$.

If we take $H = U(1)$, we get the 2-group for **2-form electromagnetism**. Here the gauge field is locally a 2-form $B$. The action is

$$\int_M \text{tr}(G \wedge *G)$$

where $G = dB$. Extremizing the action, we get

$$*d *G = 0$$

which looks just like the vacuum Maxwell equation!

We can integrate $B$ over surfaces to get elements of $H$:

$$\bullet \quad \begin{array}{c} 1 \\ h \\ 1 \end{array} \quad \bullet$$
• Any representation $\alpha$ of a group $G$ on a vector space $H$ gives a 2-group with trivial $t: H \to G$.

If $H = \mathfrak{g}$ we get the **tangent 2-group** of $G$. This is the 2-group for **BF theory** in 4 dimensions. Here the fields are a $\mathfrak{g}$-valued 1-form $A$ and an $\mathfrak{h}$-valued 2-form $B$. The action is

$$ \int_M \text{tr}(B \wedge F) $$

where $M$ is a 4-manifold. Extremizing the action, we get equations of motion

$$ d_A B = 0, \quad F = 0. $$

The second implies that we get well-behaved parallel transport over surfaces!
**Theorem.** If $M$ is a manifold and $(G, H, t, \alpha)$ is a Lie crossed module, then smooth maps sending paths and surfaces in $M$ to objects and morphisms in the corresponding 2-group:

\[
\begin{array}{c}
\xymatrix{
  x \ar[r]^\gamma & y \ar[r]_{\text{hol}(\gamma)} & \\
  \Sigma \ar[ur]_{\eta} & \ar[u] & \ar[u]_{\text{hol}(\Sigma)}
} ,
\end{array}
\]

compatible with composition and multiplication, are in 1-1 correspondence with pairs consisting of

- a $\mathfrak{g}$-valued 1-form $A$ on $M$
- an $\mathfrak{h}$-valued 1-form $B$ on $M$

satisfying the **fake flatness** condition:

\[ F + dt(B) = 0. \]
bullet Any group $G$ gives a 2-group where $H = G$, $t : H \to G$ is the identity, and the action $\alpha$ of $G$ on $H$ is given by conjugation.

This is the 2-group for **BF theory with cosmological term** in 4 dimensions. Here the fields are a $\mathfrak{g}$-valued 1-form $A$ and a $\mathfrak{g}$-valued 2-form $B$. The action is

$$\int_M \text{tr}(B \wedge F + \frac{1}{2} B \wedge B)$$

Extremizing this, we get equations of motion

$$d_A B = 0, \quad F + B = 0.$$ 

Since $F + dt(B) = 0$, we again get well-behaved parallel transport over surfaces!
Our last example is related to the String group.

Suppose $G$ is a compact, simply-connected, simple Lie group — for example $SU(n)$ or $Spin(n)$. Then

$$\pi_3(G) = \mathbb{Z}$$

and the topological group obtained by killing the third homotopy group of $G$ is called $\hat{G}$.

When $G = Spin(n)$, $\hat{G}$ is called String$(n)$:

$$\text{String}(n) \rightarrow Spin(n) \rightarrow SO(n) \rightarrow O(n).$$

To define spinors on $M$, we need to pick a spin structure. To define spinors on the free loop space $LM$, we need to pick a ‘string structure’. So, getting our hands on String$(n)$ is important — but tricky!
For any $k \in \mathbb{Z}$ there is a 2-group called $\mathcal{P}_k G$. We will use this to construct $\hat{G}$.

An object of $\mathcal{P}_k G$ is a smooth path $f : [0, 2\pi] \to G$ starting at the identity. A morphism from $f_1$ to $f_2$ is an equivalence class of pairs $(D, \lambda)$ consisting of a disk $D$ going from $f_1$ to $f_2$ together with $\lambda \in U(1)$:

What’s the equivalence relation?
Any two such pairs \((D_1, \lambda_1)\) and \((D_2, \lambda_2)\) have a 3-ball \(B\) whose boundary is \(D_1 \cup D_2\). The pairs are equivalent when
\[
\exp \left( 2\pi i k \int_B \nu \right) = \frac{\lambda_2}{\lambda_1}
\]
where \(\nu\) is the left-invariant closed 3-form on \(G\) with
\[
\nu(x, y, z) = \langle [x, y], z \rangle
\]
and \(\langle \cdot, \cdot \rangle\) is the Killing form, normalized so that \([\nu]\) generates \(H^3(G, \mathbb{Z})\).

**Theorem.** The morphisms in \(\mathcal{P}_k G\) starting at the constant path form the level-\(k\) central extension of the loop group \(\Omega G\):
\[
1 \longrightarrow U(1) \longrightarrow \Omega_k G \longrightarrow \Omega G \longrightarrow 1
\]
So, the 2-group $P_kG$ corresponds to the crossed module $(PG, \widehat{\Omega_kG}, t, \alpha)$ where:

- $PG$ consists of paths in $G$ starting at the identity.
- $\widehat{\Omega_kG}$ is the level-$k$ central extension of the loop group $\Omega G$.
- $t : \widehat{\Omega_kG} \to PG$ is given by:

\[
1 \xrightarrow{} U(1) \xrightarrow{} \widehat{\Omega_kG} \xrightarrow{} \Omega G \xrightarrow{} 1 \xrightarrow{t} PG
\]

- $\alpha$ is ‘conjugation’ of elements of $\widehat{\Omega_kG}$ by paths in $PG$. One must prove this is well-defined!
The **nerve** of a topological 2-group \( G \) is a simplicial topological group. When we take its **geometric realization** we get a topological group \( |G| \).

**Theorem.** When \( k = \pm 1 \),

\[
|\mathcal{P}_kG| \simeq \hat{G}.
\]

So, when \( G = \text{Spin}(n) \), \( |\mathcal{P}_kG| \) is the string group!
**QUESTION:** Which higher gauge theory uses the 2-group $\mathcal{P}_k G$ as its ‘gauge 2-group’?

**POSSIBLE ANSWER:** Chern–Simons theory in 3 dimensions! This is normally viewed as an ordinary gauge theory, but we may be able to see it as a higher gauge theory with this gauge 2-group.

For more detail, see the work of Urs Schreiber online at the $n$-Category Café.
The $M$-theory 3-Group?

String theory involves 1-dimensional objects — strings! Higher gauge theory with 2-groups describes the parallel transport of 1-dimensional objects. So, we should not be surprised to find some 2-groups (like $\mathcal{P}_k G$) that are related to string theory.

$M$-theory involves 2-dimensional objects — 2-branes! Higher gauge theory with 3-groups should describe the parallel transport of 2-dimensional objects. So, we should not be surprised to find some 3-groups that are related to $M$-theory.
QUESTION: Which 3-groups – or 3-supergroups – show up in $M$-theory?

POSSIBLE ANSWER: $M$-theory is the mysterious quantized version of 11d supergravity. 11d supergravity involves these fields:

- a 1-form valued in the 11d Poincaré Lie superalgebra
- a 3-form

So, maybe it is a higher gauge theory whose 3-supergroup has ‘Lie 3-superalgebra’ with:

- the 11d Poincaré Lie superalgebra as objects
- $\{0\}$ as morphisms
- $\mathbb{R}$ as 2-morphisms
In fact the concept of Lie 3-superalgebra is understood — and a nontrivial one like this exists!

For this and other reasons, it seems 11d supergravity is a higher (super)gauge theory. But, much more work needs to be done to understand this. The Lie 3-supergroup for $M$-theory seems to involve extra ingredients — like the exceptional group $E_8$.

For more detail see the work of Castellani, D’Auria and Fré, Aschieri and Jurčo, and Urs Schreiber.