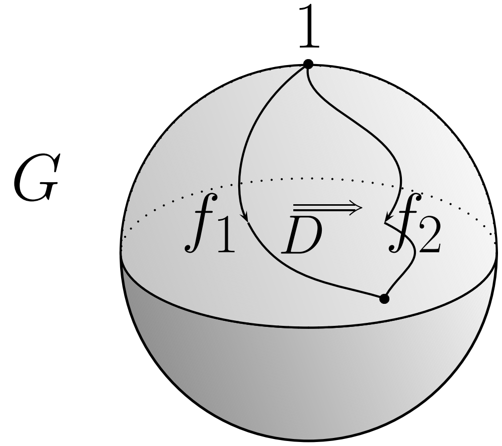
Higher Gauge Theory

John C. Baez

joint work with:
Toby Bartels, Alissa Crans, Aaron Lauda,
Urs Schreiber, Danny Stevenson.



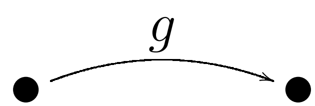
More details at:

http://math.ucr.edu/home/baez/highergauge/

Gauge Theory

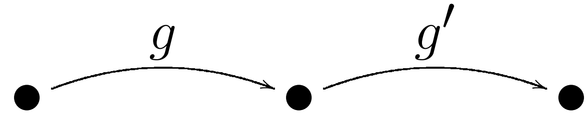
Ordinary gauge theory describes how point particles transform as they move along paths:

It is natural to assign a group element to each path:

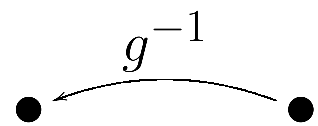


Why?

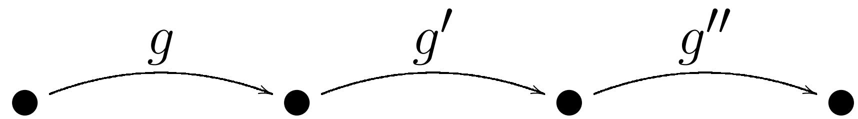
Composition of paths corresponds to multiplication:



Reversing the direction of a path corresponds to taking inverses:



The associative law makes parallel transport along a triple composite unambiguous:



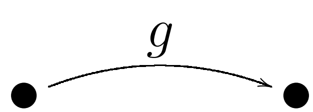
So: *the topology dictates the algebra!*

The electromagnetic field is described using the group U(1). Other forces are described using other groups.

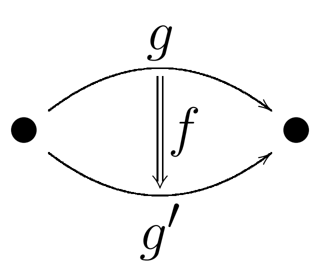
Higher Gauge Theory

Higher gauge theory describes not just how point particles but also how 1-dimensional objects transform as we move them. This leads to the concept of a **2-group**.

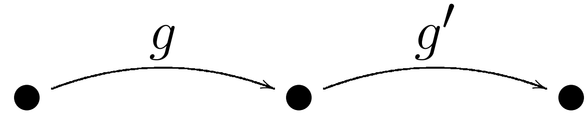
A 2-group has objects:



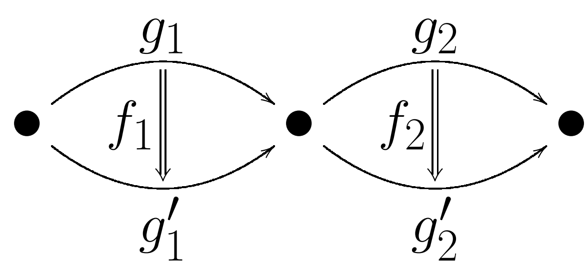
and also morphisms:



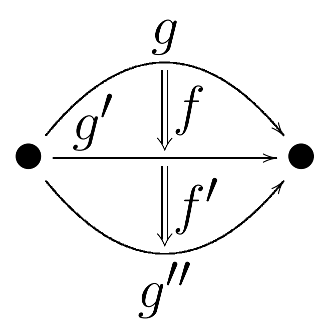
We can multiply objects:



multiply morphisms:

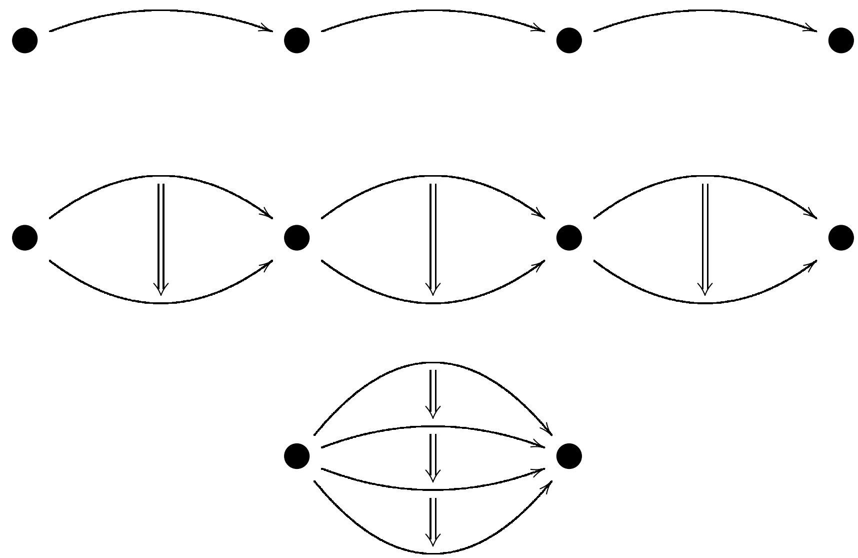


and also compose morphisms:

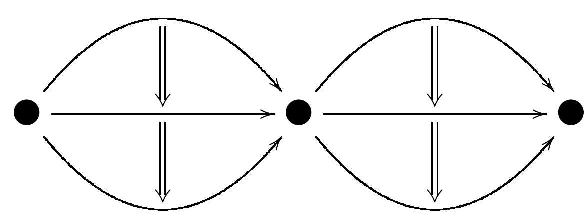


Various laws should hold... all obvious from the pictures!

Each operation has a unit and inverses. Each operation is associative, so these are well-defined:



Finally, the **interchange law**, holds, meaning this is well-defined:

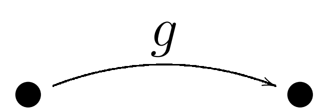


That's all a 2-group is.

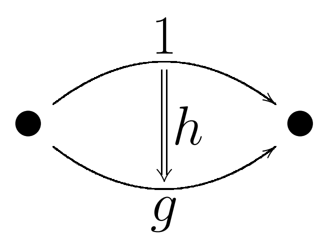
Crossed Modules

A 2-group $\mathcal{G}$ is determined by the quadruple (G, H, t, α) consisting of:

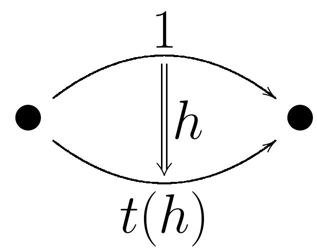
- the group G consisting of all objects of $\mathcal{G}$:



- the group H consisting of all morphisms of $\mathcal{G}$ with source 1:

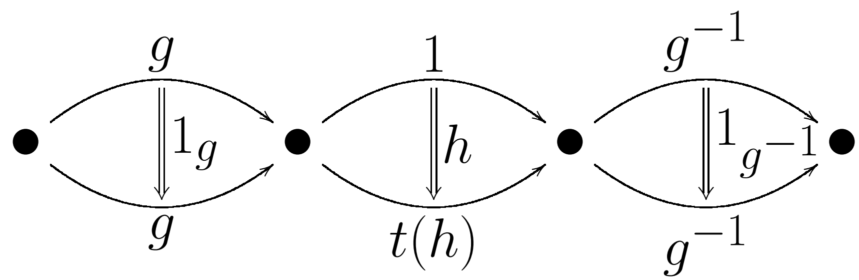


- the homomorphism $t\colon H \to G$ sending each element of H to its target:



- the action α of G on H defined by:

$\alpha(g)(h) =$



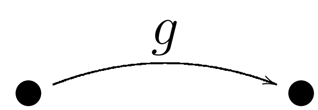
For any 2-group $\mathcal{G}$, the quadruple (G, H, t, α) satisfies two equations making it a **crossed module**. Conversely, any crossed module gives a 2-group.

Examples of 2-Groups

- Any group G gives a 2-group with $H = 1$. So:

$$\text{ordinary gauge theory} \subseteq \text{higher gauge theory}$$

In ordinary gauge theory, the gauge field is a connection: locally a $\mathfrak{g}$-valued 1-form A. We cleverly integrate this along paths to get elements of G:



• Any abelian 2-group H gives a 2-group with $G = 1$.

If we take $H = \mathrm{U}(1)$, we get the 2-group for **2-form electromagnetism**. Here the gauge field is locally a 2-form B. The action is

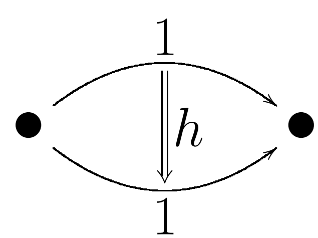
$$\int_M \mathrm{tr}(G \wedge *G)$$

where $G = dB$. Extremizing the action, we get

$$*d * G = 0$$

which looks just like the vacuum Maxwell equation!

We can integrate B over surfaces to get elements of H:



- Any representation α of a group G on a vector space H gives a 2-group with trivial $t\colon H \to G$.

If $H = \mathfrak{g}$ we get the **tangent 2-group** of G. This is the 2-group for ***BF* theory** in 4 dimensions. Here the fields are a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B. The action is

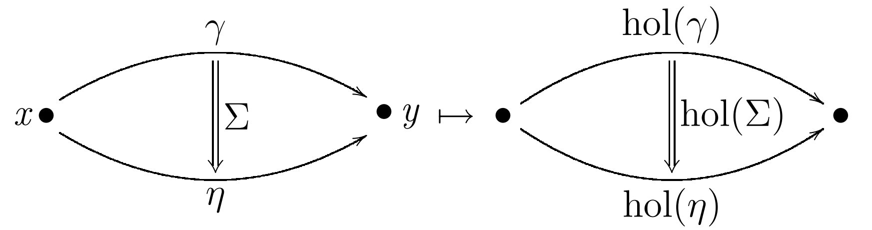
$$\int_M \mathrm{tr}(B \wedge F)$$

where M is a 4-manifold. Extremizing the action, we get equations of motion

$$d_A B = 0, \qquad F = 0.$$

The second implies that we get well-behaved parallel transport over surfaces!

Theorem. If M is a manifold and (G, H, t, α) is a Lie crossed module, then smooth maps sending paths and surfaces in M to objects and morphisms in the corresponding 2-group:



compatible with composition and multiplication, are in 1-1 correspondence with pairs consisting of

- a $\mathfrak{g}$-valued 1-form A on M
- an $\mathfrak{h}$-valued 1-form B on M

satisfying the **fake flatness** condition:

$$F + dt(B) = 0.$$

• Any group G gives a 2-group where $H = G$, $t\colon H \to G$ is the identity, and the action α of G on H is given by conjugation.

This is the 2-group for ***BF* theory with cosmological term** in 4 dimensions. Here the fields are a $\mathfrak{g}$-valued 1-form A and a $\mathfrak{g}$-valued 2-form B. The action is

$$\int_M \mathrm{tr}(B \wedge F + \frac{1}{2} B \wedge B)$$

Extremizing this, we get equations of motion

$$d_A B = 0, \qquad F + B = 0.$$

Since $F + dt(B) = 0$, we again get well-behaved parallel transport over surfaces!

• Our last example is related to the String group.

Suppose G is a compact, simply-connected, simple Lie group — for example $\mathrm{SU}(n)$ or $\mathrm{Spin}(n)$. Then

$$\pi_3(G) = \mathbb{Z}$$

and the topological group obtained by killing the third homotopy group of G is called $\widehat{G}$.

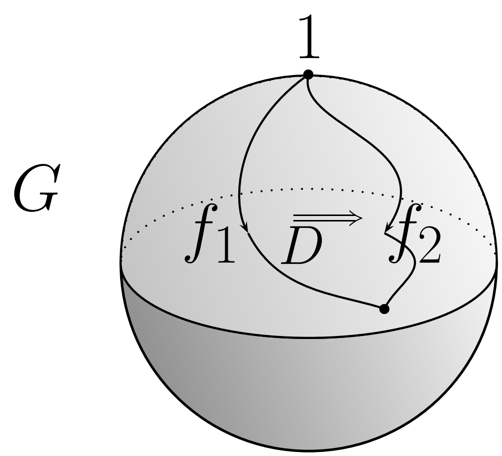
When $G = \mathrm{Spin}(n)$, $\widehat{G}$ is called $\mathrm{String}(n)$:

$$\mathrm{String}(n) \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to \mathrm{O}(n).$$

To define spinors on M, we need to pick a spin structure. To define spinors on the free loop space LM, we need to pick a 'string structure'. So, getting our hands on $\mathrm{String}(n)$ is important — but tricky!

For any $k \in \mathbb{Z}$ there is a 2-group called $\mathcal{P}_k G$. We will use this to construct $\widehat{G}$.

An object of $\mathcal{P}_k G$ is a smooth path $f \colon [0, 2\pi] \to G$ starting at the identity. A morphism from f_1 to f_2 is an equivalence class of pairs (D, λ) consisting of a disk D going from f_1 to f_2 together with $\lambda \in \mathrm{U}(1)$:



What's the equivalence relation?

Any two such pairs (D_1, λ_1) and (D_2, λ_2) have a 3-ball B whose boundary is $D_1 \cup D_2$. The pairs are equivalent when

$$\exp\left(2\pi i k \int_B \nu\right) = \lambda_2/\lambda_1$$

where ν is the left-invariant closed 3-form on G with

$$\nu(x, y, z) = \langle [x, y], z \rangle$$

and $\langle \cdot, \cdot \rangle$ is the Killing form, normalized so that $[\nu]$ generates $H^3(G, \mathbb{Z})$.

Theorem. The morphisms in $\mathcal{P}_k G$ starting at the constant path form the level-k central extension of the loop group ΩG:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \longrightarrow \Omega G \longrightarrow 1$$

So, the 2-group $\mathcal{P}_k G$ corresponds to the crossed module $(PG, \widehat{\Omega_k G}, t, \alpha)$ where:

- PG consists of paths in G starting at the identity.
- $\widehat{\Omega_k G}$ is the level-k central extension of the loop group ΩG.
- $t \colon \widehat{\Omega_k G} \to PG$ is given by:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \widehat{\Omega_k G} & \longrightarrow & \Omega G & \longrightarrow & 1 \\ & & & & & \overset{t}{\searrow} & \downarrow i & & \\ & & & & & & PG & & \end{array}$$

- α is 'conjugation' of elements of $\widehat{\Omega_k G}$ by paths in PG. One must prove this is well-defined!

The **nerve** of a topological 2-group G is a simplicial topological group. When we take its **geometric realization** we get a topological group $|G|$.

Theorem. When $k = \pm 1$,

$$|\mathcal{P}_k G| \simeq \widehat{G}.$$

So, when $G = \mathrm{Spin}(n)$, $|\mathcal{P}_k G|$ is the string group!

QUESTION: Which higher gauge theory uses the 2-group $\mathcal{P}_k G$ as its 'gauge 2-group'?

POSSIBLE ANSWER: Chern–Simons theory in 3 dimensions! This is normally viewed as an ordinary gauge theory, but we may be able to see it as a higher gauge theory with this gauge 2-group.

For more detail, see the work of Urs Schreiber online at the n-Category Café.

The M-theory 3-Group?

String theory involves 1-dimensional objects — strings! Higher gauge theory with 2-groups describes the parallel transport of 1-dimensional objects. So, we should not be surprised to find some 2-groups (like $\mathcal{P}_k G$) that are related to string theory.

M-theory involves 2-dimensional objects — 2-branes! Higher gauge theory with *3-groups* should describe the parallel transport of 2-dimensional objects. So, we should not be surprised to find some 3-groups that are related to M-theory.

QUESTION: Which 3-groups – or 3-*supergroups* – show up in M-theory?

POSSIBLE ANSWER: M-theory is the mysterious quantized version of 11d supergravity. 11d supergravity involves these fields:

- a 1-form valued in the 11d Poincaré Lie superalgebra
- a 3-form

So, maybe it is a higher gauge theory whose 3-supergroup has 'Lie 3-superalgebra' with:

- the 11d Poincaré Lie superalgebra as objects
- $\{0\}$ as morphisms
- $\mathbb{R}$ as 2-morphisms

In fact the concept of Lie 3-superalgebra is understood — and a nontrivial one like this exists!

For this and other reasons, it seems 11d supergravity is a higher (super)gauge theory. But, much more work needs to be done to understand this. The Lie 3-supergroup for M-theory seems to involve extra ingredients — like the exceptional group E_8.

For more detail see the work of Castellani, D'Auria and Fré, Aschieri and Jurčo, and Urs Schreiber.