Higher Linear Algebra

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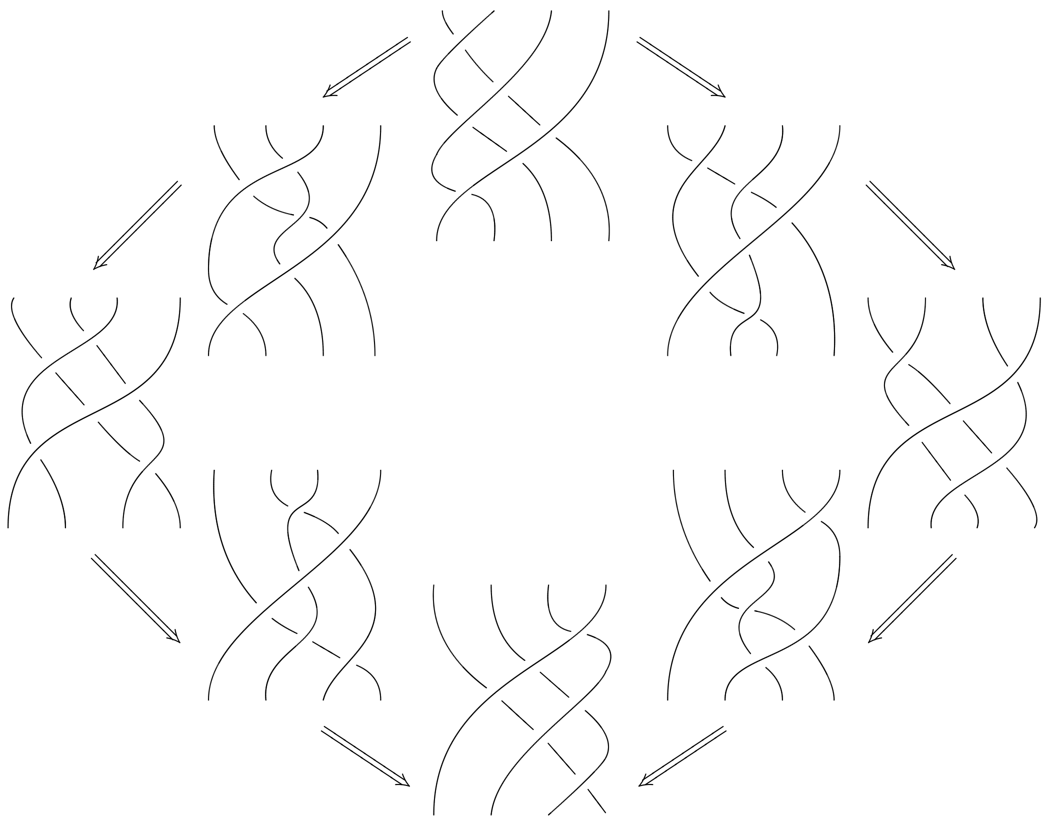

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Higher Dimensional Algebra VI: Lie 2-algebras
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Categorified vector spaces

- Kapranov and Voevodsky defined a finite-dimensional 2-vector space to be a category of the form Vect^n.

- Instead, we define a **2-vector space** to be a category in Vect.

  We have a 2-category, 2Vect whose:
  - objects are 2-vector spaces,
  - 1-morphisms are linear functors between these,
  - 2-morphisms are linear natural transformations between these

- Proposition: $2\mathrm{Vect} \simeq 2\mathrm{Term}$

A **2-vector space**, V, consists of:

- a **vector space** V_0 of objects,

- a **vector space** V_1 of morphisms,

together with:

- **linear** source and target maps

$$s, t \colon V_1 \to V_0,$$

- a **linear** identity-assigning map

$$i \colon V_0 \to V_1,$$

- a **linear** composition map

$$\circ \colon V_1 \times_{V_0} V_1 \to V_1$$

such that the following diagrams commute, expressing the usual category laws:

- laws specifying the source and target of identity morphisms:

$$\begin{tikzcd} V_0 \arrow[r, "i"] \arrow[dr, "1_{V_0}"'] & V_1 \arrow[d, "s"] \\ & V_0 \end{tikzcd} \qquad \begin{tikzcd} V_0 \arrow[r, "i"] \arrow[dr, "1_{V_0}"'] & V_1 \arrow[d, "t"] \\ & V_0 \end{tikzcd}$$

- laws specifying the source and target of composite morphisms:

$$\begin{tikzcd} V_1 \times_{V_0} V_1 \arrow[r, "\circ"] \arrow[d, "p_1"'] & V_1 \arrow[d, "s"] \\ V_1 \arrow[r, "s"] & V_0 \end{tikzcd} \qquad \begin{tikzcd} V_1 \times_{V_0} V_1 \arrow[r, "\circ"] \arrow[d, "p_2"'] & V_1 \arrow[d, "t"] \\ V_1 \arrow[r, "t"] & V_0 \end{tikzcd}$$

- the associative law for composition of morphisms:

$$\begin{tikzcd} V_1 \times_{V_0} V_1 \times_{V_0} V_1 \arrow[r, "\circ \times_{V_0} 1"] \arrow[d, "1 \times_{V_0} \circ"'] & V_1 \times_{V_0} V_1 \arrow[d, "\circ"] \\ V_1 \times_{V_0} V_1 \arrow[r, "\circ"] & V_1 \end{tikzcd}$$

- the left and right unit laws for composition of morphisms:

$$\begin{tikzcd} V_0 \times_{V_0} V_1 \arrow[r, "i \times 1"] \arrow[dr, "p_2"'] & V_1 \times_{V_0} V_1 \arrow[d, "\circ"] & V_1 \times_{V_0} V_0 \arrow[l, "1 \times i"'] \arrow[dl, "p_1"] \\ & V_1 & \end{tikzcd}$$

n-vector spaces

- An $(n{+}1)$**-vector space** is a strict n-category in Vect.

  We have a n-category, nVect whose:

  – objects are n-vector spaces,

  – 1-morphisms are strict linear functors between these,

  – 2-morphisms are strict linear natural transformations between these, …

  and so on!

- Proposition: $(n+1)\mathrm{Vect} \simeq (n+1)\mathrm{Term}$

- $\bigoplus$, $\bigotimes$ of n-vector spaces

Moral: Homological algebra is secretly categorified linear algebra!

Questions still remain:

- What about weak n-categories in Vect?
- What about weakening laws governing addition and scalar multiplication?

Categorified Linear Algebra

Associative algebras	A_∞-algebras
Commutative algebras	E_∞-algebras
Lie algebras	L_∞-algebras
Coalgebras	?
Hopf Algebras	?
$\vdots$	$\vdots$

Given a vector space V and an isomorphism

$$B\colon V \otimes V \to V \otimes V,$$

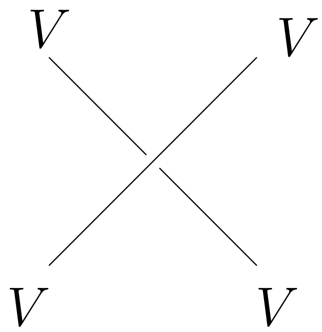
we say B is a **Yang–Baxter operator** if it satisfies the **Yang–Baxter equation**, which says that:

$$(B\otimes 1)(1\otimes B)(B\otimes 1) = (1\otimes B)(B\otimes 1)(1\otimes B),$$

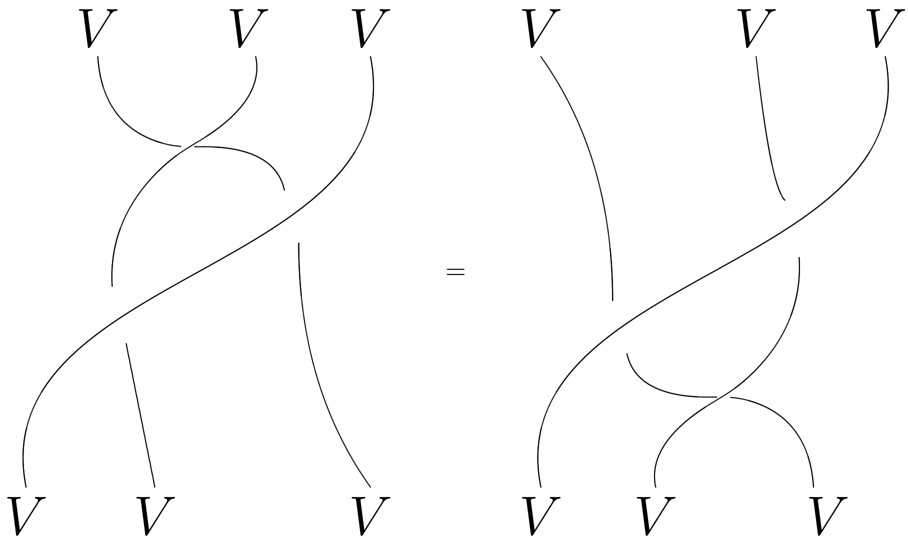
or in other words, that this diagram commutes:

$$\begin{tikzcd}
 & V \otimes V \otimes V \arrow[dl, "1\otimes B"'] \arrow[dr, "B\otimes 1"] & \\
V \otimes V \otimes V \arrow[d, "B\otimes 1"'] & & V \otimes V \otimes V \arrow[d, "1\otimes B"] \\
V \otimes V \otimes V \arrow[dr, "B\otimes 1"'] & & V \otimes V \otimes V \arrow[dl, "B\otimes 1"] \\
 & V \otimes V \otimes V &
\end{tikzcd}$$

If we draw $B \colon V \otimes V \to V \otimes V$ as a braiding:

$B =$ 

the Yang–Baxter equation says that:



Proposition: Let L be a vector space over k equipped with a skew-symmetric bilinear operation

$$[\cdot,\cdot]\colon L \times L \to L.$$

Let $L' = k \oplus L$ and define the isomorphism

$$B\colon L' \otimes L' \to L' \otimes L' \text{ by}$$

$$B((a,x)\otimes(b,y)) = (b,y)\otimes(a,x)+(1,0)\otimes(0,[x,y]).$$

Then B is a solution of the Yang–Baxter equation if and only if $[\cdot,\cdot]$ satisfies the Jacobi identity.

$\boxed{\textbf{Goal}}$

Develop a higher-dimensional analogue, obtained by categorifying everything in sight!

A **semistrict Lie 2-algebra** consists of:

- a 2-vector space L

equipped with:

- a skew-symmetric bilinear functor, the **bracket**,

$$[\cdot,\cdot]\colon L \times L \to L$$

- a completely antisymmetric trilinear natural isomorphism, the **Jacobiator**,

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$

that is required to satisfy:

- the **Jacobiator identity**:

$$J_{[w,x],y,z} \circ [J_{w,x,z}, y] \circ (J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}) =$$

$$[J_{w,x,y}, z] \circ (J_{[w,y],x,z} + J_{w,[x,y],z}) \circ [J_{w,y,z}, x] \circ [w, J_{x,y,z}]$$

for all $w, x, y, z \in L_0$.

$$
\begin{tikzcd}
 & {[[[w,x],y],z]} \arrow[dl, "{[J_{w,x,y,z}]}"'] \arrow[dr, "1"] & \\
{[[[w,y],x],z]+[[w,[x,y]],z]} \arrow[d, "{J_{[w,y],x,z}+J_{w,[x,y],z}}"'] & & {[[[w,x],y],z]} \arrow[d, "{J_{[w,x],y,z}}"] \\
\begin{array}{c}[[[w,y],z],x]+[[w,y],[x,z]] \\ +[w,[[x,y],z]]+[[w,z],[x,y]]\end{array} \arrow[d, "{[J_{w,y,z},x]}"'] & & {[[[w,x],z],y]+[[w,x],[y,z]]} \arrow[d, "{[J_{w,x,z},y]}"] \\
\begin{array}{c}[[[w,z],y],x]+[[w,[y,z]],x] \\ +[[w,y],[x,z]]+[w,[[x,y],z]]+[[w,z],[x,y]]\end{array} \arrow[dr, "{[w,J_{x,y,z}]}"'] & & \begin{array}{c}[[w,[x,z]],y] \\ +[[w,x],[y,z]]+[[[w,z],x],y]\end{array} \arrow[dl, "{\begin{array}{c}J_{w,[x,z],y} \\ +J_{[w,z],x,y}+J_{w,x,[y,z]}\end{array}}"] \\
 & \begin{array}{c}[[[w,z],y],x]+[[w,z],[x,y]]+[[w,y],[x,z]] \\ +[w,[[x,z],y]]+[[w,[y,z]],x]+[w,[x,[y,z]]]\end{array} &
\end{tikzcd}
$$

Zamolodchikov tetrahedron equation

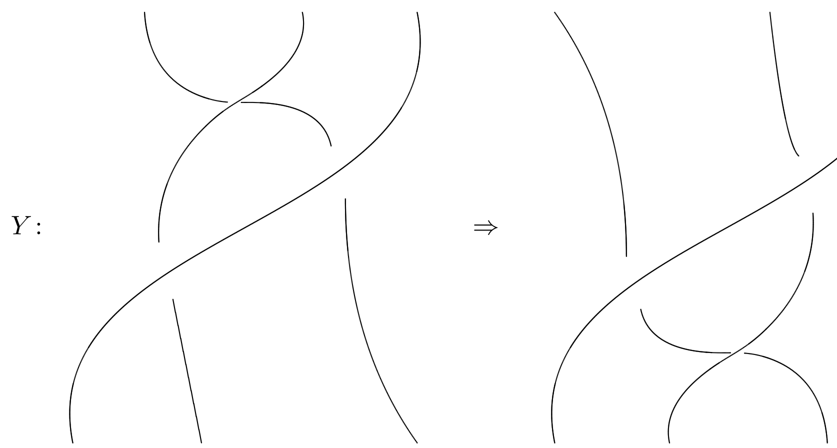
Given a 2-vector space V and an invertible linear functor $B\colon V\otimes V \to V\otimes V$, a linear natural isomorphism

$$Y\colon (B\otimes 1)(1\otimes B)(B\otimes 1) \Rightarrow (1\otimes B)(B\otimes 1)(1\otimes B)$$

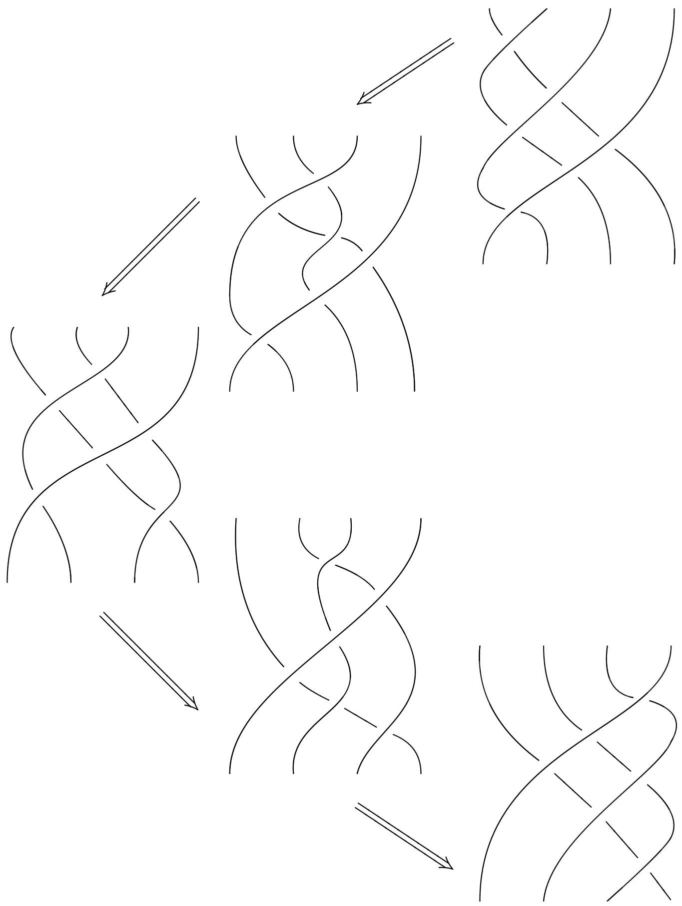
satisfies the **Zamolodchikov tetrahedron equation** if:

$$[Y\circ(1\otimes 1\otimes B)(1\otimes B\otimes 1)(B\otimes 1\otimes 1)][(1\otimes B\otimes 1)(B\otimes 1\otimes 1)\circ Y\circ(B\otimes 1\otimes 1)]$$
$$[(1\otimes B\otimes 1)(1\otimes 1\otimes B)\circ Y\circ(1\otimes 1\otimes B)][Y\circ(B\otimes 1\otimes 1)(1\otimes B\otimes 1)(1\otimes 1\otimes B)]$$
$$=$$
$$[(B\otimes 1\otimes 1)(1\otimes B\otimes 1)(1\otimes 1\otimes B)\circ Y][(B\otimes 1\otimes 1)\circ Y\circ(B\otimes 1\otimes 1)(1\otimes B\otimes 1)]$$
$$[(1\otimes 1\otimes B)\circ Y\circ(1\otimes 1\otimes B)(1\otimes B\otimes 1)][(1\otimes 1\otimes B)(1\otimes B\otimes 1)(B\otimes 1\otimes 1)\circ Y]$$

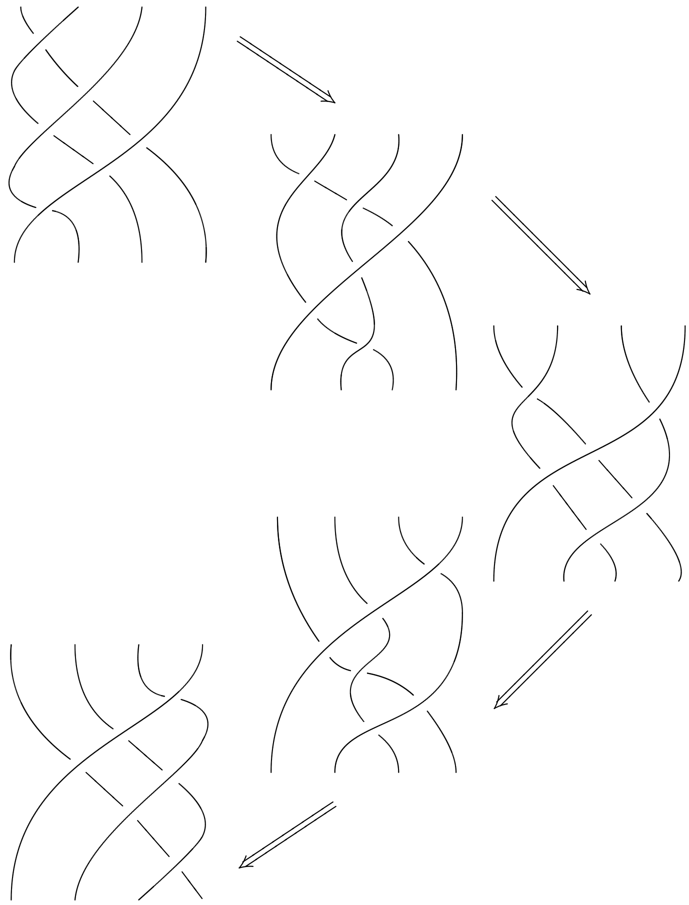
We should think of Y as the surface in 4-space traced out by the *process of performing* the third Reidemeister move:



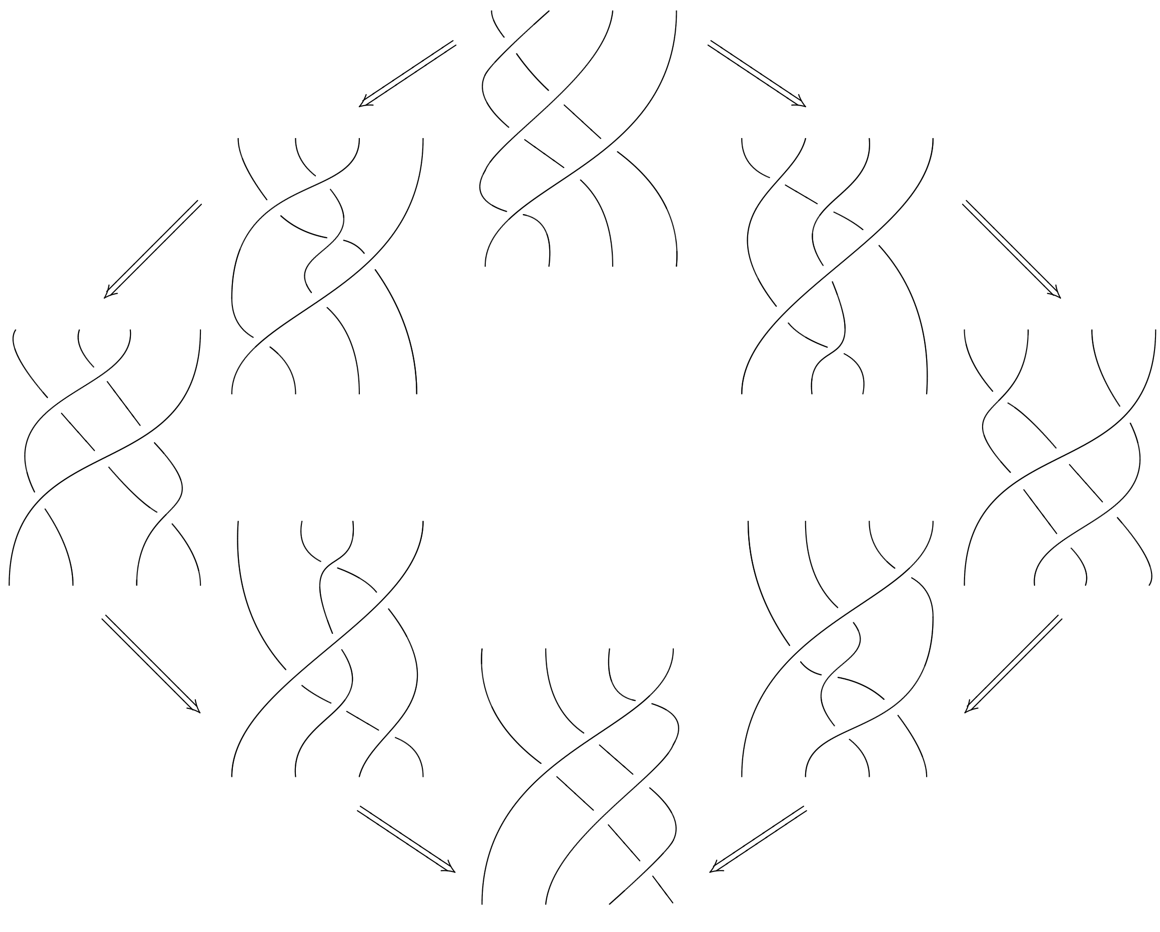
Left side of Zamolodchikov tetrahedron equation:



Right side of Zamolodchikov tetrahedron equation:



In short, the Zamolodchikov tetrahedron equation is a formalization of this commutative octagon:



Theorem: Let L be a 2-vector space, let $[\cdot,\cdot]\colon L\times L \to L$ be a skew-symmetric bilinear functor, and let J be a completely antisymmetric trilinear natural transformation with

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y].$$

Let $L' = K \oplus L$, where K is the categorified ground field.
Let $B\colon L'\otimes L' \to L'\otimes L'$ be defined as follows:

$$B((a,x)\otimes(b,y)) = (b,y)\otimes(a,x) + (1,0)\otimes(0,[x,y])$$

whenever (a,x) and (b,y) are both either objects or morphisms in L'. Finally, let

$$Y\colon (B\otimes 1)(1\otimes B)(B\otimes 1) \Rightarrow (1\otimes B)(B\otimes 1)(1\otimes B)$$

be defined as follows:

$$Y = \begin{tikzcd} L'\otimes L'\otimes L' \arrow[d, "p\otimes p\otimes p"] \\ \underset{(x,y,z)}{L\otimes L\otimes L} \arrow[d, bend right=60, "{[[x,y],z]}"'] \arrow[d, bend left=60, "{[x,[y,z]]+[[x,z],y]}"] \arrow[d, phantom, "\overset{J}{\Longrightarrow}"] \\ \underset{a}{L} \arrow[d, hook, "j"] \\ \underset{(1,0)\otimes(1,0)\otimes(0,a)}{L'\otimes L'\otimes L'} \end{tikzcd}$$

where a is either an object or morphism of L. Then Y is a solution of the Zamolodchikov tetrahedron equation if and only if J satisfies the Jacobiator identity.

Hierarchy of Higher Commutativity

Topology	**Algebra**
Crossing	Commutator
Crossing of crossings	Jacobi identity
Crossing of crossing of crossings	Jacobiator identity
⋮	⋮

Examples

Theorem: There is a one-to-one correspondence between equivalence classes of Lie 2-algebras (where equivalence is as objects of the 2-category Lie2Alg) and isomorphism classes of quadruples consisting of a Lie algebra $\mathfrak{g}$, a vector space V, a representation ρ of $\mathfrak{g}$ on V, and an element of $H^3(\mathfrak{g}, V)$.

Example: Given a finite-dimensional, simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, $H^3(\mathfrak{g}, \mathbb{C}) = \mathbb{C}$, so we can construct a Lie 2-algebra with nontrivial Jacobiator consisting of:

- $L_0 = \mathfrak{g}$
- $L_1 =$ identity morphisms
- $J_{x,y,z} = f(x, y, z)1_{[[x,y],z]}$ where $f\colon \mathfrak{g}^3 \to \mathbb{C}$ is a nontrivial 3-cocycle

In fact, we can take $J_{x,y,z} = \hbar\langle x, [y, z]\rangle 1$ for $\hbar \in \mathbb{C}$.

Lie 2-groups

A **coherent 2-group** is a weak monoidal category C in which every morphism is invertible and every object $x \in C$ is equipped with an adjoint equivalence $(x, \bar{x}, i_x, e_x)$.

A **Lie 2-group** is a coherent 2-group object in DiffCat.