The Homotopy Hypothesis

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Notes and references at:
http://math.ucr.edu/home/baez/homotopy/
The Big Idea

The theory of $\infty$-categories seeks to formalize our notions of *thing*, *process*, *metaprocess*, *meta-metaprocess* and so on:

At first glance, this has no more to do with *topology* than with any other subject.
But, a point in a topological space is a ‘thing’:

A path is a ‘process’:

A homotopy between paths is a ‘metaprocess’:

and so on.
So, any space $X$ should give an $\infty$-category! This amounts to using $X$ as a ‘blackboard’ on which to draw diagrams:

![Diagram](image)

This $\infty$-category should be an $\infty$-groupoid: the fundamental $\infty$-groupoid, $\Pi_\infty(X)$.

In its rawest form, the homotopy hypothesis asks:

*To what extent are spaces ‘the same’ as $\infty$-groupoids?*

Let’s warm up with ordinary groupoids....
The Fundamental Groupoid

From any space $X$ we can try to build a category whose objects are points of $X$ and whose morphisms are paths in $X$:

If we use homotopy classes of paths, this works and we get a groupoid: the fundamental groupoid, $\Pi_1(X)$.

There is a 2-functor

$$\Pi_1: \text{Top} \to \text{Gpd}$$

sending spaces, maps and homotopy classes of homotopies to groupoids, functors and natural transformations. So we can ask:

*To what extent are spaces secretly the same as groupoids?*
Eilenberg–Mac Lane Spaces

We can try to find an ‘inverse’ to $\Pi_1$, building a space from any groupoid $G$: the **Eilenberg–Mac Lane space** $|G|$. To do this we take a vertex for each object of $G$:

- $x$

an edge for each morphism of $G$:

- $f$

a triangle for each composable pair of morphisms:

\[ \begin{array}{c}
\bullet \\
|f| \\
\bullet \\
\circ \\
|g| \\
\bullet \\
\circ \\
|fg| \\
\bullet \\
\end{array} \]

a tetrahedron for each composable triple:

\[ \begin{array}{c}
\bullet \\
|f| \\
\bullet \\
\bullet \\
|g| \\
\bullet \\
\bullet \\
|fgh| \\
\bullet \\
\end{array} \]

and so on!
$|G|$ has $G$ as its fundamental groupoid, up to equivalence. $|G|$ is a **homotopy 1-type**: a CW complex whose homotopy groups above the 1st vanish for any basepoint. These facts characterize it up to homotopy equivalence.

Indeed, we have 2-functors going both ways:

$$\begin{array}{ccc}
\text{Top} & \xrightarrow{\Pi_1} & \text{Gpd} \\
|\cdot| & & \\
\end{array}$$

We have an equivalence

$$i : G \xrightarrow{\sim} \Pi_1(|G|)$$

for every groupoid $G$. We also have a map

$$e : |\Pi_1(X)| \longrightarrow X$$

for every space $X$. This is a homotopy equivalence if $X$ is a homotopy 1-type.
In fact, one can prove:

**Homotopy Hypothesis (dimension 1).** Let $1\text{Type}$ be the 2-category of homotopy 1-types, maps, and homotopy classes of homotopies between maps. Then

$$\Pi_1 : 1\text{Type} \to \text{Gpd}$$

is an equivalence of 2-categories.

Even better, Lack and Leinster have shown these 2-functors

$$\begin{array}{c}
\text{Top} \\
\downarrow \Pi_1 \\
\text{Gpd}
\end{array}$$

are adjoints (technically a ‘biadjunction’).
The Homotopy Hypothesis

Generalizing to (weak) \(n\)-groupoids:

**The Homotopy Hypothesis (dimension \(n\)).** There is an equivalence of \((n+1)\)-categories

\[
\Pi_n : n\text{Type} \to n\text{Gpd}
\]

where a **homotopy \(n\)-type** is a CW complex whose homotopy groups above the \(n\)th vanish for all basepoints, and \(n\text{Type}\) is the \((n+1)\)-category with:

- homotopy \(n\)-types as objects,
- continuous maps as 1-morphisms,
- homotopies as 2-morphisms,
- homotopies between homotopies as 3-morphisms,...

...homotopy classes of \((n+1)\)-fold homotopies as \((n+1)\)-morphisms.
The homotopy hypothesis for all finite $n$ should follow from:

**The Homotopy Hypothesis (dimension $\infty$).** There is an equivalence of $\infty$-categories

$$\Pi_\infty: \infty\text{Type} \to \infty\text{Gpd}$$

where $\infty\text{Type}$ is the $\infty$-category of *homotopy types*, with:

- CW complexes as objects,
- continuous maps as 1-morphisms,
- homotopies as 2-morphisms,
- homotopies between homotopies as 3-morphisms,
- homotopies between homotopies between homotopies as 4-morphisms,...
$(\infty, 1)$-Categories

Both $\infty$Type and $\infty$Gpd should be $(\infty, 1)$-categories: $\infty$-categories where all $j$-morphisms are weakly invertible for $j > 1$.

Any definition of $\infty$-category should give a definition of $(\infty, 1)$-category, as a special case. For example, Street’s simplicial $\infty$-categories have quasicategories as a special case.

There are also many ‘stand-alone’ approaches to $(\infty, 1)$-categories:

- simplicially enriched categories: categories enriched over SimpSet
- $A_\infty$-categories
- Segal categories

We can try to state and prove the homotopy hypothesis in any of these approaches. In some, it’s already been done!

But: no pain, no gain.
Any definition of $(\infty, 1)$-category should give a definition of $\infty$-groupoid, as a special case. For example, quasicategories have Kan complexes as a special case. A Kan complex is a simplicial set where every ‘horn’ has a ‘filler’:

\[\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1.25,1.25) -- (2.5,0) -- (0,0);
\node at (0,0) {$x$};
\node at (1.25,1.25) {$y$};
\node at (2.5,0) {$z$};
\node at (1.25,0.25) {$h$};
\draw (0,0) -- (0.5,0.5);
\draw (1.25,1.25) -- (1.75,0.75);
\draw (2.5,0) -- (2.0,0.5);
\node at (0.75,0.25) {$f$};
\node at (1.75,0.5) {$g$};
\node at (2.0,0.75) {$h$};
\end{tikzpicture}
\end{array}\]
If we take Kan complexes as our definition of $\infty$-groupoids, it is easy to define

$$\Pi_\infty : \infty \text{Type} \to \infty \text{Gpd}$$

as an ordinary functor between categories. People usually get this from the adjunction

$$\text{Top} \overset{\text{Sing}}{\leftarrow} \overset{|-|}{\longrightarrow} \text{SimpSet}$$

by noting the ‘singular simplicial set’ functor, $\text{Sing}$, maps all spaces to Kan complexes.

Similarly, ‘geometric realization’, $| - |$, maps all simplicial sets to CW complexes.

Top and SimpSet are ‘model categories’. Kan complexes are very nice objects in the model category SimpSet: they are ‘fibrant and cofibrant’. CW complexes are very nice in Top.

Every object in a model category is ‘weakly equivalent’ to a very nice one.
In any model category we have:

very nice objects,
morphisms,
homotopies between morphisms,
homotopies between homotopies between morphisms,....

So, for the $n$-category theorist,

\begin{center}
\textbf{Model categories are a trick for getting $(\infty, 1)$-categories.}
\end{center}

In particular: the model category Top gives the $(\infty, 1)$-category $\infty\text{Type}$. The model category SimpSet gives $\infty\text{Gpd}$.

One way to make this precise: any model category gives a simpli-
cially enriched category — Dwyer and Kan’s ‘simplicial localization’.

This can be defined using just the very nice objects, the morphisms, and the weak equivalences.
The adjunction

\[
\begin{array}{c}
\text{Top} \\ \text{SimpSet}
\end{array}
\begin{array}{c}
\text{Sing} \\ |-|
\end{array}
\]

is a ‘Quillen equivalence’ of model categories. For the \(n\)-category theorist,

*Quillen equivalences are a trick for getting equivalences between \((\infty, 1)\)-categories.*

In particular: the equivalence between \(\infty\text{Type}\) and \(\infty\text{Gpd}\).

One way to make this precise: Quillen equivalent model categories give ‘weakly equivalent’ simplicially enriched categories — as shown by Dwyer and Kan.
So, we can work simplicially and define

- $\infty$-groupoid := Kan complex
- $\infty$-$1$-category := simplicially enriched category
- equivalent $\infty$-$1$-categories := weakly equivalent simplicially enriched categories

Then Quillen, Dwyer and Kan showed:

**The Homotopy Hypothesis (simplicial version).** There is an equivalence of $\infty$-$1$-categories

$$ \Pi_\infty : \infty \text{Type} \to \infty \text{Gpd} $$

where $\infty \text{Type}$ arises from the model category Top by simplicial localization, and $\infty \text{Gpd}$ arises from the model category SimpSet.
So, why not just use simplicial methods...

...and forget about ‘globular’ $n$-categories?

Bad answer: \textit{because we always liked globular $n$-categories.}

Better answer: \textit{globular methods clarify the structure of $\infty$-categories, and thus $\infty$-groupoids, and thus homotopy types — given the homotopy hypothesis.}
Dimension 1

In any globular $n$-category, ‘cell colonies’ like this give us 1-morphisms:

\[ \bullet \xrightarrow{\text{composition}} \bullet \]

composition of 1-morphisms

For any pointed $n$-groupoid, this operation defines multiplication in the fundamental group, $\pi_1$.

$\pi_1$ classifies connected 1-groupoids up to equivalence.
In any $n$-category, these cell colonies give 2-morphisms:

- **Composition** of 2-morphisms

- **Whiskering** a 2-morphism by a 1-morphism

- The **associator**

For any pointed $n$-groupoid, these operations give a group $\pi_2$, an action of $\pi_1$ on $\pi_2$, and a cohomology class

$$[a] \in H^3(\pi_1, \pi_2) \quad \text{(the associator)}$$

Together with $\pi_1$, these classify connected 2-groupoids up to equivalence.
Dimension 3 and Beyond

Can we go on? These cell colonies give interesting 3-morphisms:

- **composition** of 3-morphisms

- **whiskering** a 3-morphism by a 1-morphism

- the **braiding** for 2-morphisms

- the **associator** for 2-morphisms

- **pseudonaturality** of the associator for 1-morphisms

- the **pentagonator** for 1-morphisms
How can we use these to *classify* connected 3-groupoids?

And how about $n$-groupoids for higher $n$?

Most homotopy theorists consider the combinatorics of homotopy types a complicated morass. *Maybe globular $n$-categories can help!*

Also: the homotopy hypothesis says that any sub-$\infty$-groupoid of an $\infty$-category corresponds to a homotopy type. So, we can use homotopy theory to study the *coherence laws* that hold — up to further coherence laws — in an $\infty$-category.

In short:

*The homotopy hypothesis may or may not help homotopy theory — but it’s already helped $n$-category theory, and will surely continue to do so!*