The Homotopy Hypothesis

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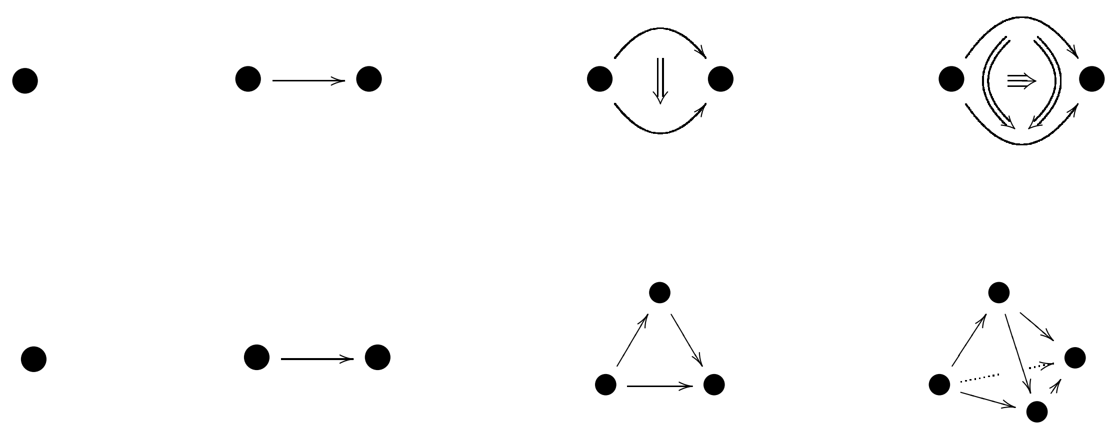

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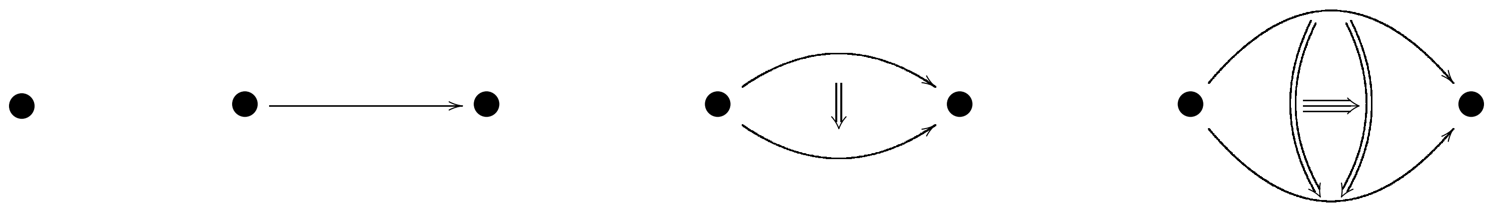
Notes and references at:

http://math.ucr.edu/home/baez/homotopy/

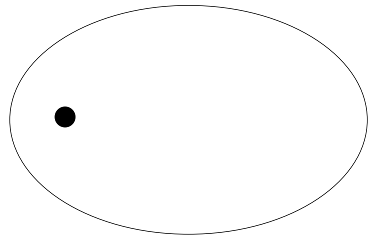
The Big Idea

The theory of ∞-categories seeks to formalize our notions of *thing*, *process*, *metaprocess*, *meta-metaprocess* and so on:

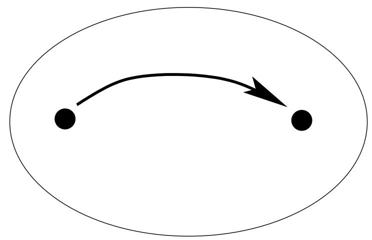


At first glance, this has no more to do with *topology* than with any other subject.

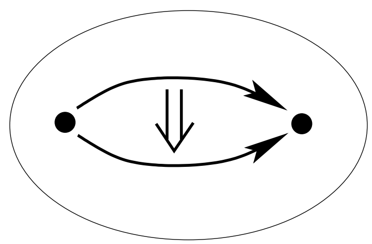
But, a point in a topological space is a ‘thing’:



A path is a ‘process’:

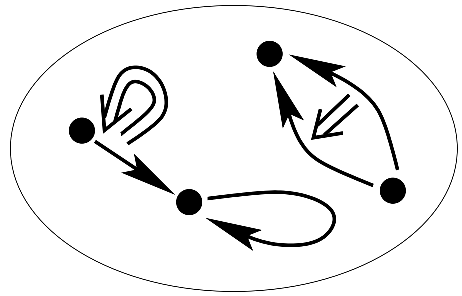


A homotopy between paths is a ‘metaprocess’:



and so on.

So, any space X should give an ∞-category! This amounts to using X as a 'blackboard' on which to draw diagrams:



This ∞-category should be an ∞-groupoid: the **fundamental ∞-groupoid**, $\Pi_\infty(X)$.

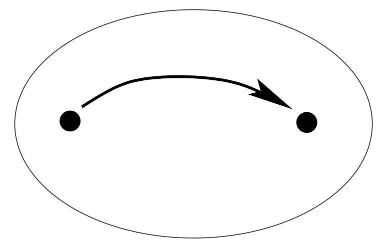
In its rawest form, the homotopy hypothesis asks:

To what extent are spaces 'the same' as ∞-groupoids?

Let's warm up with ordinary groupoids....

The Fundamental Groupoid

From any space X we can try to build a category whose objects are points of X and whose morphisms are paths in X:



If we use *homotopy classes of paths*, this works and we get a groupoid: the **fundamental groupoid**, $\Pi_1(X)$.

There is a 2-functor

$$\Pi_1 \colon \mathrm{Top} \to \mathrm{Gpd}$$

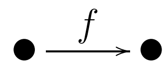
sending spaces, maps and homotopy classes of homotopies to groupoids, functors and natural transformations. So we can ask:

To what extent are spaces secretly the same as groupoids?

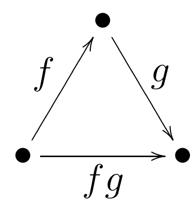
Eilenberg–Mac Lane Spaces

We can try to find an 'inverse' to Π_1, building a space from any groupoid G: the **Eilenberg–Mac Lane space** $|G|$. To do this we take a vertex for each object of G:

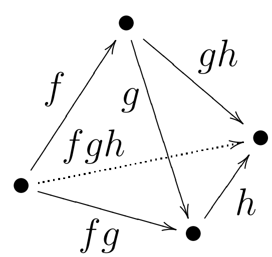
an edge for each morphism of G:



a triangle for each composable pair of morphisms:



a tetrahedron for each composable triple:



and so on!

$|G|$ has G as its fundamental groupoid, up to equivalence. $|G|$ is a **homotopy 1-type**: a CW complex whose homotopy groups above the 1st vanish for any basepoint. These facts characterize it up to homotopy equivalence.

Indeed, we have 2-functors going both ways:

$$\mathrm{Top} \underset{|-|}{\overset{\Pi_1}{\rightleftarrows}} \mathrm{Gpd}$$

We have an equivalence

$$i : G \xrightarrow{\sim} \Pi_1(|G|)$$

for every groupoid G. We also have a map

$$e : |\Pi_1(X)| \longrightarrow X$$

for every space X. This is a homotopy equivalence if X is a homotopy 1-type.

In fact, one can prove:

Homotopy Hypothesis (dimension 1). Let **1Type** be the 2-category of homotopy 1-types, maps, and homotopy classes of homotopies between maps. Then

$$\Pi_1 \colon \mathrm{1Type} \to \mathrm{Gpd}$$

is an equivalence of 2-categories.

Even better, Lack and Leinster have shown these 2-functors

$$\mathrm{Top} \underset{|-|}{\overset{\Pi_1}{\rightleftarrows}} \mathrm{Gpd}$$

are adjoints (technically a 'biadjunction').

The Homotopy Hypothesis

Generalizing to (weak) n-groupoids:

The Homotopy Hypothesis (dimension n). There is an equivalence of $(n+1)$-categories

$$\Pi_n \colon n\mathrm{Type} \to n\mathrm{Gpd}$$

where a **homotopy n-type** is a CW complex whose homotopy groups above the nth vanish for all basepoints, and **nType** is the $(n+1)$-category with:

homotopy n-types as objects,
continuous maps as 1-morphisms,
homotopies as 2-morphisms,
homotopies between homotopies as 3-morphisms,...

...homotopy classes of $(n{+}1)$-fold homotopies as $(n{+}1)$-morphisms.

The homotopy hypothesis for all finite n should follow from:

The Homotopy Hypothesis (dimension ∞). There is an equivalence of ∞-categories

$$\Pi_\infty \colon \infty\mathrm{Type} \to \infty\mathrm{Gpd}$$

where ∞**Type** is the ∞-category of **homotopy types**, with:

CW complexes as objects,
continuous maps as 1-morphisms,
homotopies as 2-morphisms,
homotopies between homotopies as 3-morphisms,
homotopies between homotopies between homotopies as 4-morphisms,....

$(\infty, 1)$-Categories

Both ∞Type and ∞Gpd should be **$(\infty, \mathbf{1})$-categories**: ∞-categories where all j-morphisms are weakly invertible for $j > 1$.

Any definition of ∞-category should give a definition of $(\infty, 1)$-category, as a special case. For example, Street's simplicial ∞-categories have quasicategories as a special case.

There are also many 'stand-alone' approaches to $(\infty, 1)$-categories:

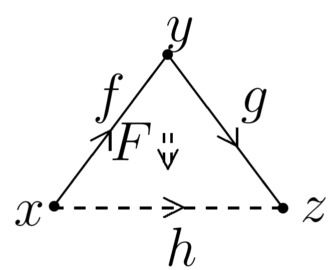
- simplicially enriched categories: categories enriched over SimpSet
- A_∞-categories
- Segal categories

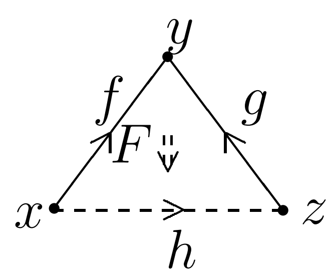
We can try to state and prove the homotopy hypothesis in any of these approaches. In some, it's already been done!

But: *no pain, no gain.*

∞-Groupoids

Any definition of $(\infty, 1)$-category should give a definition of ∞-groupoid, as a special case. For example, quasicategories have Kan complexes as a special case. A **Kan complex** is a simplicial set where every 'horn' has a 'filler':





If we take Kan complexes as our definition of ∞-groupoids, it is easy to define

$$\Pi_\infty \colon \infty\mathrm{Type} \to \infty\mathrm{Gpd}$$

as an ordinary *functor* between *categories*. People usually get this from the adjunction

$$\mathrm{Top} \underset{|-|}{\overset{\mathrm{Sing}}{\rightleftarrows}} \mathrm{SimpSet}$$

by noting the 'singular simplicial set' functor, Sing, maps all spaces to Kan complexes.

Similarly, 'geometric realization', $|-|$, maps all simplicial sets to CW complexes.

Top and SimpSet are 'model categories'. Kan complexes are **very nice** objects in the model category SimpSet: they are 'fibrant and cofibrant'. CW complexes are very nice in Top.

Every object in a model category is 'weakly equivalent' to a very nice one.

In any model category we have:

very nice objects,
morphisms,
homotopies between morphisms,
homotopies between homotopies between morphisms,....

So, for the n-category theorist,

Model categories are a trick for getting $(\infty, \mathbf{1})$-categories.

In particular: the model category Top gives the $(\infty, 1)$-category ∞Type. The model category SimpSet gives ∞Gpd.

One way to make this precise: any model category gives a simplicially enriched category — Dwyer and Kan's 'simplicial localization'.

This can be defined using just the very nice objects, the morphisms, and the weak equivalences.

The adjunction

$$\mathrm{Top} \underset{|-|}{\overset{\mathrm{Sing}}{\rightleftarrows}} \mathrm{SimpSet}$$

is a 'Quillen equivalence' of model categories. For the n-category theorist,

Quillen equivalences are a trick for getting equivalences between $(\infty, \mathbf{1})$-categories.

In particular: the equivalence between ∞Type and ∞Gpd.

One way to make this precise: Quillen equivalent model categories give 'weakly equivalent' simplicially enriched categories — as shown by Dwyer and Kan.

So, we can work simplicially and define

- ∞-groupoid := Kan complex
- $(\infty, 1)$-category := simplicially enriched category
- equivalent $(\infty, 1)$-categories := weakly equivalent simplicially enriched categories

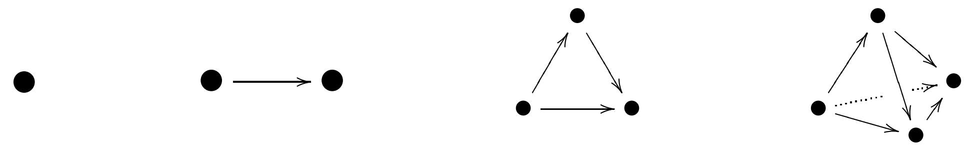
Then Quillen, Dwyer and Kan showed:

The Homotopy Hypothesis (simplicial version). There is an equivalence of $(\infty, 1)$-categories

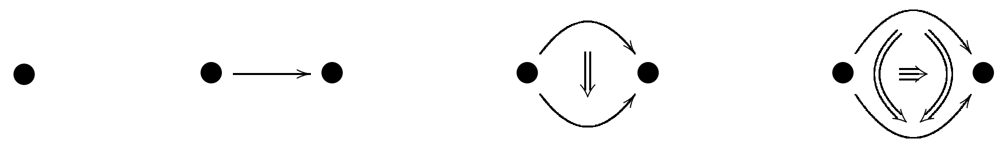
$$\Pi_\infty \colon \infty\mathrm{Type} \to \infty\mathrm{Gpd}$$

where ∞Type arises from the model category Top by simplicial localization, and ∞Gpd arises from the model category SimpSet.

So, why not just use simplicial methods...



...and forget about 'globular' n-categories?

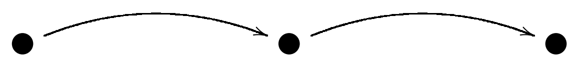


Bad answer: ***because we always liked globular n-categories.***

Better answer: ***globular methods clarify the structure of ∞-categories, and thus ∞-groupoids, and thus homotopy types — given the homotopy hypothesis.***

Dimension 1

In any globular n-category, 'cell colonies' like this give us 1-morphisms:



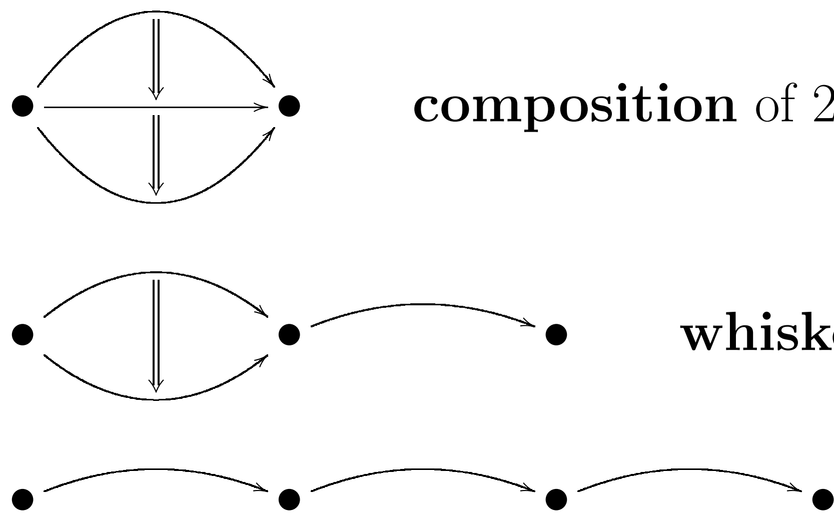
composition of 1-morphisms

For any pointed n-groupoid, this operation defines multiplication in the fundamental group, π_1.

π_1 *classifies* connected 1-groupoids up to equivalence.

Dimension 2

In any n-category, these cell colonies give 2-morphisms:



composition of 2-morphisms

whiskering a 2-morphism by a 1-morphism

the **associator**

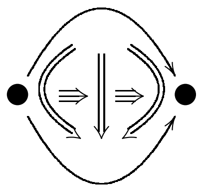
For any pointed n-groupoid, these operations give a group π_2, an action of π_1 on π_2, and a cohomology class

$$[a] \in H^3(\pi_1, \pi_2) \qquad \text{(the associator)}$$

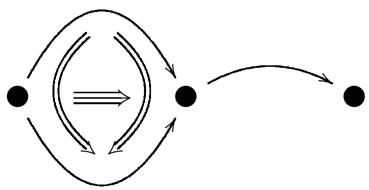
Together with π_1, these *classify* connected 2-groupoids up to equivalence.

Dimension 3 and Beyond

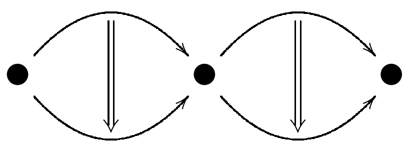
Can we go on? These cell colonies give interesting 3-morphisms:



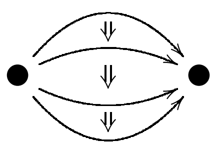
composition of 3-morphisms



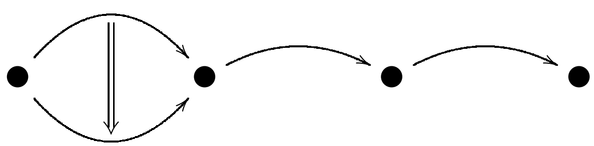
whiskering a 3-morphism by a 1-morphism



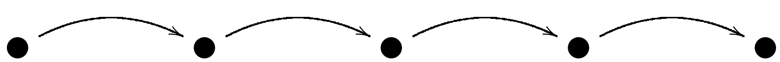
the **braiding** for 2-morphisms



the **associator** for 2-morphisms



pseudonaturality of the associator for 1-morphisms



the **pentagonator** for 1-morphisms

How can we use these to *classify* connected 3-groupoids?

And how about n-groupoids for higher n?

Most homotopy theorists consider the combinatorics of homotopy types a complicated morass. ***Maybe globular n-categories can help!***

Also: the homotopy hypothesis says that any sub-∞-groupoid of an ∞-category corresponds to a homotopy type. So, we can use homotopy theory to study the *coherence laws* that hold — up to further coherence laws — in an ∞-category.

In short:

The homotopy hypothesis may or may not help homotopy theory — but it's already helped n-category theory, and will surely continue to do so!