Diffeomorphism-Invariant Spin Network States

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Abstract

We extend the theory of diffeomorphism-invariant spin network states from the real-analytic category to the smooth category. Suppose that G is a compact connected semisimple Lie group and $P \to M$ is a smooth principal G-bundle. A 'cylinder function' on the space of smooth connections on P is a continuous complex function of the holonomies along finitely many piecewise smoothly immersed curves in M. We construct diffeomorphism-invariant functionals on the space of cylinder functions from 'spin networks': graphs in M with edges labeled by representations of G and vertices labeled by intertwining operators. Using the 'group averaging' technique of Ashtekar, Marolf, Mourão and Thiemann, we equip the space spanned by these 'diffeomorphism-invariant spin network states' with a natural inner product.

Introduction

In the 'new variables' approach to quantizing gravity, the kinematical Hibert space of the theory should consist of functions on some completion of the space of connections on a principal SU(2) bundle over the smooth 3-manifold representing space. Defining the inner product in this Hilbert space requires a measure on the completed space of connections. Starting from this Hilbert space, one should then solve constraint equations corresponding to gauge-invariance, diffeomorphism-invariance and invariance under time evolution to obtain the space of physical states. The deepest problems with this program are those associated with invariance under time evolution, i.e. the Hamiltonian constraint. However, the proper treatment of the other two constraints also presents problems, some of which have been avoided by assuming M is real-analytic and only demanding invariance under real-analytic diffeomorphisms. Working in the real-analytic category, Ashtekar, Lewandowski, Marolf, Mourão and Thiemann have constructed a Hilbert space of gauge- and diffeomorphism-invariant states spanned by 'spin networks' [3]. Here we do the same in the smooth category.

Before describing our result more precisely, let us briefly review the state of the art [1, 2, 4, 5, 7]. First, there is a natural way to complete the space of connections on any smooth principal G-bundle $P \to M$ when G is a compact connected Lie group. This goes as follows. Let $\mathcal{A}$ be the space of smooth connections on P, and define a 'cylinder function' on $\mathcal{A}$ to be a continuous function of the holonomies along finitely many piecewise immersed paths in M. Taking the sup norm completion of the algebra of cylinder functions, one obtains a commutative C*-algebra, and the spectrum of this C*-algebra is a compact Hausdorff space $\overline{\mathcal{A}}$ having $\mathcal{A}$ as a dense subset. Any cylinder function uniquely extends to a continuous function on $\overline{\mathcal{A}}$.

Second, there is a natural Borel measure on $\overline{\mathcal{A}}$, the 'uniform measure' μ_0. With respect to this measure, the probability distribution of the holonomies along the edges of a graph embedded in M is given by Haar measure on a product of copies of G. In other words, suppose that $e_1, \ldots, e_n$ are smoothly embedded copies of the unit interval in M which intersect, if at all, only at their endpoints. In this situation we say that e_i are the edges of a graph. Let F be the cylinder function

$$F(A) = f(\mathrm{T}\exp \int_{e_1} A, \ldots, \mathrm{T}\exp \int_{e_n} A)$$

where f is a continuous complex-valued function on G^n and $\mathrm{T}\exp \int_{e_i} A$ is the holonomy of the connection A along e_i, regarded as a group element by means of an arbitrary trivialization of P at the endpoints of this curve. Then the integral of F with respect to the uniform measure is given by

$$\int_{\overline{\mathcal{A}}} F(A)\, d\mu_0(A) = \int_{G^n} f(g_1, \ldots, g_n)\, dg_1 \cdots dg_n$$

where the right-hand integral is taken with respect to normalized Haar measure on G^n.

In the real-analytic category the above property is sufficient to characterize the uniform measure, because any cylindrical function can be expressed as above in terms of the holonomies of curves forming a graph. This no longer holds in the smooth category, making it a bit trickier to fully characterize uniform measure. The reason is that smoothly embedded curves can intersect each other in extremely complicated ways, for example in a Cantor set. In a previous paper [5] we dealt with this issue by generalizing graphs to 'webs'. Like a graph, a web consists of finitely many curves embedded in M, but in a web these curves are allowed to intersect each other in certain specified ways. In Section 1 we recall the concept of a web and characterize uniform measure in the smooth category using webs.

Using uniform measure one can define the 'kinematical Hilbert space' $L^2(\overline{\mathcal{A}})$. The group of gauge transformations has a unitary representation on this Hilbert space, coming from its action on $\mathcal{A}$, and there is a large subspace of $L^2(\overline{\mathcal{A}})$ consisting of functions invariant under all gauge transformations. We denote this 'gauge-invariant Hilbert space' by $L^2(\overline{\mathcal{A}/\mathcal{G}})$, since its elements may also be regarded as square-integrable functions on a certain completion of the space of connections modulo gauge transformations. In our previous paper we constructed an explicit set of functions spanning the gauge-invariant Hilbert space, the 'spin web states'. Each such state is determined by a 'spin web': a web with edges labeled by representations of G and vertices labeled by intertwining operators. We review the theory of spin webs in Section 2.

The simplest spin web states are the 'spin network states', corresponding to webs that are simply graphs. In the real-analytic category, spin network states are enough to span the gauge-invariant Hilbert space. This is no longer true in the smooth category. However, we show in Section 2 that any spin network state is orthogonal to any spin web state that cannot be written as a spin network state with the same underlying graph. This result is crucial for the next step: constructing the diffeomorphism-invariant Hilbert space.

Naively one might try to define the diffeomorphism-invariant Hilbert space as the subspace of $L^2(\overline{\mathcal{A}/\mathcal{G}})$ consisting of functions invariant under all diffeomorphisms of M, or at least those in the identity component of the diffeomorphism group. Moreover, one might naively be inclined to obtain such functions by 'averaging over the action of the diffeomorphism group'. However, things are not so simple: there appears to be no 'Haar measure' on the diffeomorphism group, and there are typically very few diffeomorphism-invariant functions in $L^2(\overline{\mathcal{A}/\mathcal{G}})$. The point is that one should seek diffeomorphism-invariant elements, not of $L^2(\overline{\mathcal{A}/\mathcal{G}})$, but of some larger space of 'generalized functions' on $\overline{\mathcal{A}/\mathcal{G}}$. Working in the real-analytic category, Ashtekar, Lewandowski, Marolf, Mourão and Thiemann [3] constructed such diffeomorphism-invariant generalized functions by a clever procedure which amounts to averaging spin network states over the action of the diffeomorphism group.

Using the orthogonality result of the previous section, in Section 3 we carry out a similar group averaging procedure in the smooth category, obtaining 'diffeomorphism-invariant spin network states' labeled by diffeomorphism equivalence classes of spin networks. Completing the space spanned by these in its natural inner product, we obtain the diffeomorphism-invariant Hilbert space.

One might imagine extending this diffeomorphism-invariant Hilbert space to include more general diffeomorphism-invariant spin web states. Unfortunately, spin webs that are not spin networks behave badly. In general two spin webs may not be orthogonal even if their underlying webs do not have the same range. Further, there are infinitely many diffeomorphisms taking a typical spin web state to spin web states that are not orthogonal to it, even after the obvious quotients. We give examples of both these phenomena in Section 4. Thus it appears difficult to find an explicit orthonormal basis of the full gauge-invariant Hilbert space, and difficult to construct diffeomorphism-invariant states from spin webs.

We should note that when M is real-analytic, the previously studied real-analytic spin networks are a special case of our smooth spin networks. Furthermore, any smooth manifold can be given a real-analytic structure, which is unique up to smooth diffeomorphism. However, there are many more smooth spin networks than real-analytic ones, even modulo smooth diffeomorphisms, because there are smooth vertex types unrealizable by analytic curves.

1 Uniform measure

We begin with a terse review of uniform measure on the space of connections. For the most part we follow the treatment in our previous paper [5], but we simplify the setup using a result of Lewandowski and Thiemann [6].

Fix a connected compact semisimple Lie group G, a smooth (paracompact) manifold M, and a smooth principal G-bundle $P \to M$. By a *curve* we mean a piecewise smooth map from an interval $[a, b] \subset \mathbb{R}$ to M that is an immersion on each piece. Two curves are considered equivalent if one is the composition of the other with an orientation-preserving diffeomorphism between their domains (so that one is just a reparametrization on the other). A *family* is a finite set of curves with a chosen ordering $c_1, \ldots, c_n$. If C is such a family, we define $\mathrm{Range}(C)$, the *range* of C, to be the union of the ranges of the individual curves. A point p in $\mathrm{Range}(C)$ is a *regular point* if it is not the image of an endpoint or nondifferentiable point of C, and there is a neighborhood of it whose intersection with $\mathrm{Range}(C)$ is an embedded interval. A family C is *parametrized consistently* if each curve is parametrized so that $c_i(t) = c_j(s)$ implies $t = s$. Thus each of the curves is actually an embedding, and each point p in the range of the family is associated to a unique value of the parameter, which we call $t(p)$. If a family $\{c_1, \ldots, c_n\}$ is parametrized consistently and p is a point in $\mathrm{Range}(C)$, define the *type* of a regular point p, τ_p, to be the Lie subgroup of G^n consisting of all n-tuples $(g_1, \ldots, g_n)$ such that for some $g \in G$ we have $g_i = g$ if p lies on c_i, and $g_i = 1$ otherwise.

A family T is a *tassel based on* $p \in \mathrm{Range}(T)$ if:

(a) $\mathrm{Range}(T)$ lies in a contractible open subset of M.

(b) T can be consistently parametrized in such a way that $c_i(0) = p$ is the left endpoint of every curve c_i.

(c) Two curves in T that intersect at a point other than p. intersect at a point other than p in every neighborhood of p.

(d) Any type which occurs at some point in $\mathrm{Range}(T)$ occurs in every neighborhood of p.

(e) No two curves in T have the same range.

A *web* ω is a finite collection of tassels $\omega^1, \ldots, \omega^k$ such that for $i \neq j$:

(a) Any curve in the tassel ω^i intersects any curve in ω^j, if at all, only at their endpoints

(b) There is a neighborhood of each such intersection point whose intersection with $\mathrm{Range}(\omega^i \cup \omega^j)$ is an embedded interval.

(c) $\mathrm{Range}(\omega^i)$ does not contain the base of ω^j.

We may apply concepts defined for families to webs, since every web ω has an associated family $\omega^1 \cup \cdots \cup \omega^k$. We define an *edge* of a web ω to be a curve in one of the families ω^i, and define a *vertex* of ω to be a point of M that is an endpoint of some edge of ω. Since the edges of a web are oriented, we may speak of the *source* and *target* of any edge, these being its initial and final endpoints.

Using webs one can characterize uniform measure as follows. First, note that any web ω with edges $e_1, \ldots, e_n$ together with trivializations of P at the vertices of ω determines a map from $\mathcal{A}$ to G^n given by:

$$A \mapsto (\mathrm{T}\exp \int_{e_1} A, \ldots, \mathrm{T}\exp \int_{e_n} A).$$

This map extends uniquely to a continuous map

$$p_\omega \colon \overline{\mathcal{A}} \to G^n.$$

We may push forward any Borel measure on $\overline{\mathcal{A}}$ to a Borel measure on G^n by this map p_ω. We then have:

Proposition 1. *There exists a unique Borel measure on $\overline{\mathcal{A}}$, the* uniform measure *μ_0, such that if ω is a web with n edges, the pushfoward of μ_0 by p_ω is normalized Haar measure on G^n.*

Proof: In our previous paper we showed that the range of p_ω is a Lie subgroup of G^n and that μ_0 is uniquely characterized by the property that its pushforward is normalized Haar measure on this subgroup for every web ω. Previously we did not include clause (e) in the definition of a tassel, but we may assume this without loss of generality, since two consistently oriented curves with the same range have the same holonomy for every connection. Using this clause and the fact that G is semisimple, Lewandowski and Thiemann [6] subsequently showed that p_ω is onto. ■

2 Spin webs and spin networks

Using uniform measure one can define the kinematical Hilbert space $L^2(\overline{\mathcal{A}})$. This in turn allows us to construct the gauge-invariant Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}})$, which consists of all functions in $L^2(\overline{\mathcal{A}})$ that are invariant under gauge transformations. In this section we provide a detailed description of the gauge-invariant Hilbert space in terms of spin webs and spin networks.

A *spin web* is a triple $W = (\omega, \rho, \iota)$ consisting of:

(a) a web ω

(b) a labeling ρ of each edge e of ω with a nontrivial irreducible representation ρ_e of G

(c) a labeling ι of each vertex v of ω with an intertwining operator ι_v from the tensor product of the ρ_e for which v is the target of e to the tensor product of the ρ_e for which v is the source of e.

Given a spin web $W = (\omega, \rho, \iota)$, the *spin web state* Ψ_W is the cylinder function on $\mathcal{A}$ constructed as follows:

$$\Psi_W(A) = [\bigotimes_e \rho_e(\mathrm{T}\exp \int_e A)] \cdot [\bigotimes_v \iota_v],$$

where '$\cdot$' stands for contracting, at each vertex v of ω, the upper indices of the matrices corresponding to the incoming edges, the lower indices of the matrices assigned to the outgoing edges, and the corresponding indices of the intertwiner ι_v.

Proposition 2. *Finite linear combinations of spin web states are dense in $L^2(\overline{\mathcal{A}/\mathcal{G}})$.*

Proof: This is a slight rephrasing of a result in our previous paper. There we called every state of the form Ψ_W a 'spin network state', but here we reserve that term for a special case (see below). Also, here we assume without loss of generality that the underlying web of W has no two edges with the same range and no edge labeled by a trivial representation. ∎

We say a subset of M is a *graph in M* if it is the union of a finite collection of embedded copies of the unit interval that intersect, if at all, only at their endpoints. We say that a web is a *graph* if its range is a graph in M. Note that for every graph G in M there is a web with G as its range, and this web is unique up to inserting and deleting bivalent vertices and reversing orientations of edges.

We define a *spin network* to be a spin web $\Gamma = (\gamma, \rho, \iota)$ whose underlying web γ is a graph. In this case we call the spin web state Ψ_Γ a *spin network state.* This definition of spin network state is a bit different from the usual one [4]. However, apart from the fact that our graphs have smooth rather than real-analytic edges, the differences are purely superficial. To see this, suppose we have a spin network $\Gamma = (\gamma, \rho, \iota)$ as defined above. Then the range of γ is the union of the ranges of a finite set of curves intersecting only at their endpoints. Subdividing these curves if necessary, we may assume that each edge of ω is a product of these curves and their inverses. Call this set of curves E and the set of their endpoints V. Then, just as in the usual definition of spin network state, we may write

$$\Psi_\Gamma(A) = [\bigotimes_{e\in E} \rho_e(\mathrm{T}\exp \int_e A)] \cdot [\bigotimes_{v\in V} \iota_v],$$

for some choice of representations ρ_e and intertwining operators ι_v.

We say that a spin web state Ψ is *supported* on a web ω if it equals Ψ_W for some spin web W with ω as its underlying web. We also say that Ψ is supported on the range of ω, especially when W is a spin network, so that the range of ω is a graph in M. Note that many different graphs γ have the same graph in M as their range: we can change γ without changing its range by introducing and deleting bivalent vertices on embedded intervals, and also by reversing the orientation of edges. If a spin network state is supported on γ it is also supported on all other graphs with the same range. However, it is supported on a unique graph in M. The following is a restatement of arguments in earlier work on spin networks [4].

Proposition 3. *If two spin network states have nonzero inner product, they are supported on the same graph in M.*

Proof: Let Γ and Γ' be spin networks with underlying graphs γ and γ', respectively. If the ranges of γ and γ' are not the same, consider an open segment of an edge contained in one and not in the other. We may choose a web on which both Ψ_Γ and $\Psi_{\Gamma'}$ are supported [5], and having one edge lying entirely in this open segment. Uniform measure for this web gives the holonomy of this edge Haar measure distribution, independent as a random variable from the holonomies of the other edges. In computing the inner product of Ψ_Γ and $\Psi_{\Gamma'}$, this variable will appear once, represented in the nontrivial irreducible representation labeling that edge. Since the integral against Haar measure of the nontrivial irreducible representation of a group-valued variable is zero, the whole inner product is zero. Two spin network states with nonzero inner product must therefore be supported on the same graph G in M. ∎

The above proposition gives an essentially complete description of the inner product on the portion of $L^2(\overline{\mathcal{A}/\mathcal{G}})$ spanned by spin network states, since the description of the inner product of two spin network states supported on the same graph is well-understood and involves only elementary group representation theory [4]. It remains to understand the inner product of a spin network state with a general spin web state and the inner product of two arbitrary spin web states. The latter question is quite subtle and appears to admit no simple answer (see Section 4). The former proves to be tractable, and is the subject of the next theorem, the key technical result of this paper.

Theorem 1. *If the inner product of a spin network state with a spin web state is nonzero, then they are both spin network states supported on the same graph in M.*

Proof: Let Γ be a spin network with underlying graph γ, and W a spin web with underlying web ω. We assume the inner product of Ψ_Γ and Ψ_W is nonzero and show that the base of any tassel ω^i of ω has a neighborhood N such that $\mathrm{Range}(\omega^i) \cap N \subseteq \mathrm{Range}(\gamma)$. This implies that the range of ω is a graph in M, so that Ψ_W is a spin network state. The rest of the theorem follows from Proposition 3.

To compute the inner product of Ψ_Γ and Ψ_W, note from our previous paper that there is a web ω' such that every curve in ω or γ is a product of curves in ω' and their inverses. Moreover we may assume that the base of every tassel in ω is the base of a tassel in ω'. The inner product is the integral with respect to uniform measure of some function of the holonomies of the edges of ω'. These holonomies are independent group-valued random variables distributed according to Haar measure. Thus if any edge of ω' does not lie entirely in the range of γ but lies in the range of ω it will appear in the inner product computation once, represented in some nontrivial irreducible representation, and therefore will make the whole inner product zero.

Writing any edge e of ω^i as a product of edges in ω' and their inverses, the rightmost term in this product will be an edge of ω' whose range lies within that of e in some neighborhood of the base of ω^i. By the previous paragraph, if the inner product of Ψ_Γ and Ψ_W is nonzero, this edge of ω' must lie entirely in the range of γ. This proves the claim of the first paragraph, and hence the theorem. ■

Thus $L^2(\overline{\mathcal{A}/\mathcal{G}})$ decomposes into an uncountable orthogonal direct sum, with one countable-dimensional summand for each graph in M, spanned by spin network states supported on that graph in M, and one summand containing all the spin web states that are not spin network states.

3 The diffeomorphism-invariant Hilbert space

The next step is to construct the diffeomorphism-invariant Hilbert space. Since there are very few diffeomorphism-invariant states in $L^2(\overline{\mathcal{A}/\mathcal{G}})$, we look for diffeomorphism-invariant vectors in a larger space. A good choice for this larger space is the topological dual $\mathcal{C}^*$, where $\mathcal{C}$ is the space of gauge-invariant cylinder functions. One may think of elements of $\mathcal{C}^*$ as 'generalized functions' on $\overline{\mathcal{A}}/\mathcal{G}$. We construct diffeomorphism-invariant elements of $\mathcal{C}^*$ essentially by averaging spin network states over the action of the diffeomorphism group, following the technique of Ashtekar, Lewandowski, Marolf, Mourão and Thiemann [3]. We also follow their method to define an inner product on the resulting 'diffeomorphism-invariant spin network states', which allows us to construct the diffeomorphism-invariant Hilbert space.

In general, only those diffeomorphisms of M in the connected component of the identity lift to automorphisms of the bundle $P \to M$. However, all diffeomorphisms of M lift to automorphisms of 'natural' bundles such as trivial bundles, the frame bundle, or other bundles built from the tangent bundle using functorial constructions. In quantum gravity it remains controversial whether one should impose invariance under all diffeomorphisms or only those in the identity component. Luckily we do not need to resolve this issue here. In what follows, by a *diffeomorphism* we mean an element of some fixed subgroup $\mathcal{D} \subseteq \mathrm{Diff}(M)$, all of whose elements lift to automorphisms of P. Note that with this definition all diffeomorphisms act on $\mathcal{A}/\mathcal{G}$, $L^2(\overline{\mathcal{A}/\mathcal{G}})$, $\mathcal{C}$, $\mathcal{C}^*$, and so on.

Given a spin network $\Gamma = (\gamma, \rho, \iota)$, the range of γ is a graph in M, say G. We may write G in a unique way as a disjoint union of finitely many points, embedded open intervals and circles, such that none of the points has an neighborhood in G diffeomorphic to an interval embedded in M. Let $\mathcal{D}_\Gamma$ be the group of diffeomorphisms mapping each of these points, intervals and circles onto itself in an orientation-preserving way. Let $\mathcal{D}/\mathcal{D}_\Gamma$ be the quotient of $\mathcal{D}$ on the right by $\mathcal{D}_\Gamma$. Note that two diffeomorphisms in the same equivalence class of this quotient act the same way on the spin network state Ψ_Γ, so we can speak of the orbit $(\mathcal{D}/\mathcal{D}_\Gamma)\Psi_\Gamma$.

Proposition 4. *Given spin network states $\Psi_\Gamma, \Psi_{\Gamma'}$, the set of elements of $(\mathcal{D}/\mathcal{D}_\Gamma)\Psi_\Gamma$ having nonzero inner product with $\Psi_{\Gamma'}$ is finite.*

Proof: By Theorem 1, the inner product of $g\Psi_\Gamma$ and $\Psi_{\Gamma'}$ is zero unless g takes the graph G in M on which Γ is supported to the graph G' in M on which Γ' is supported. It follows that if we write G and G' as above as a union of points, intervals and circles, g establishes a one-to-one correspondence between the points, intervals and circles of G and those of G'. Moreover, $[g] \in \mathcal{D}/\mathcal{D}_\Gamma$ is determined by this one-to-one correspondence. Since finitely such one-to-one correspondences are possible, there are finitely many $[g]$ for which $g\Psi_\Gamma$ and $\Psi_{\Gamma'}$ have nonzero inner product. ∎

Thus it makes sense to define the quantity

$$\langle\langle \Psi_\Gamma, \Psi_{\Gamma'} \rangle\rangle = \sum_{\Phi \in (\mathcal{D}/\mathcal{D}_\Gamma)\Psi_\Gamma} \langle \Phi, \Psi_{\Gamma'} \rangle$$

for spin network states Ψ_Γ and $\Psi_{\Gamma'}$. We may then extend this by sesquilinearity to all finite linear combinations of spin network states. It is not a priori clear that the extension is well-defined, but in fact it is. To see this, consider a function Φ that can be written as a finite linear combination of spin network states. Consider such a decomposition, and for each graph G in M let Φ_G be the sum of all spin network states appearing in the decomposition that are supported on G, weighted by their coefficients. Thus $\Psi = \sum_G \Phi_G$. Consider another such decomposition $\Phi = \sum_G \Phi'_G$. By Proposition 3 the ordinary inner product satisfies

$$\langle \Phi_G, \Phi_G \rangle = \langle \Phi, \Phi_G \rangle = \langle \Phi'_G, \Phi_G \rangle = \langle \Phi'_G, \Phi \rangle = \langle \Phi'_G, \Phi'_G \rangle$$

and thus $\Phi_G = \Phi'_G$. Thus, while the exact decomposition into spin network states is not unique, the terms Φ_G are. But clearly if Φ_G and $\Phi'_{G'}$ are linear combinations of spin networks supported on the graphs G and G' in M, respectively, then

$$\langle\langle \Phi_G, \Phi'_{G'} \rangle\rangle = \sum_i \langle g_i \Phi_G, \Phi'_{G'} \rangle$$

where the g_i are representatives of equivalence classes of diffeomorphisms taking G to G'. This is independent of the choice of decomposition of Φ_G and $\Phi'_{G'}$ into spin network states. From this it follows that $\langle\langle\cdot,\cdot\rangle\rangle$ is well-defined on finite linear combinations of spin network states.

For any spin network Γ, the linear functional $\langle\langle\Psi_\Gamma,\cdot\rangle\rangle$ extends from finite linear combinations of spin network states to all of $\mathcal{C}$. The space $\mathcal{C}$ is the union over all families C of the spaces of gauge-invariant cylinder functions depending on the holonomies along the curves in C. Since each of these spaces is a Banach space in the sup norm, one can make $\mathcal{C}$ into a topological vector space with the inductive limit topology. One can check that with this topology, the *diffeomorphism-invariant spin network state* $\langle\langle\Psi_\Gamma,\cdot\rangle\rangle$ is an element of the topological dual $\mathcal{C}^*$. That $\langle\langle\Psi_\Gamma,\cdot\rangle\rangle$ is really diffeomorphism-invariant follows from:

Theorem 2. *$\langle\langle\cdot,\cdot\rangle\rangle$ is a positive-semidefinite, conjugate symmetric, sesquilinear form on finite linear combinations of spin network states. The quotient by the null space is exactly the quotient by the action of the diffeomorphism group.*

Proof: The conjugate symmetry and sesquilinearity is obvious. To see that it is positive semidefinite, consider $\Phi = \sum_G \Phi_G$, with notation as above. We have

$$\langle\langle\Phi,\Phi\rangle\rangle = \sum_{G,G'} \langle\langle\Phi_G,\Phi_{G'}\rangle\rangle,$$

Note however that if g is a diffeomorphism which takes G to G', then $g\Phi_G$ is supported on G' and

$$\langle\langle\Phi_G,\Phi_{G'}\rangle\rangle = \langle\langle g\Phi_G,\Phi_{G'}\rangle\rangle.$$

Thus, if we divide the graphs G in M into equivalence classes of graphs in M that are all diffeomorphic to each other, choose a representative of each class, and choose diffeomorphisms connecting each to the representative, we can can replace Φ with $\Phi' = \sum_G \Phi'_G$ where now distinct G cannot be mapped to each other by diffeomorphisms, and

$$\langle\langle\Phi,\Phi\rangle\rangle = \langle\langle\Phi',\Phi'\rangle\rangle = \sum_G \langle\langle\Phi'_G,\Phi'_G\rangle\rangle.$$

To see that $\langle\langle\Phi'_G,\Phi'_G\rangle\rangle \geq 0$, choose representatives g_i of the equivalences classes of diffeomorphisms which map G to itself, and note that

$$\sum_{i,j} \langle g_i\Phi'_G, g_j\Phi'_G\rangle = \sum_{i,j} \langle g_j^{-1}g_i\Phi'_G,\Phi'_G\rangle = \sum_i n_G\langle g_i\Phi'_G,\Phi'_G\rangle = n_G\langle\langle\Phi'_G,\Phi'_G\rangle\rangle,$$

where n_G is the number of diffeomorphisms g_i. Since the original inner product is positive-definite, $\langle\langle\Phi'_G,\Phi'_G\rangle\rangle \geq 0$, and it is zero exactly when each Φ'_G has $\sum_i g_i\Phi'_G = 0$. But this condition says exactly that the Φ'_G (and hence the original Φ_G) are a sum of elements of the form $1/n_G \sum_i(\Phi'_G - g_i\Phi'_G)$. From this the last statement follows. ∎

If we quotient the space of finite linear combinations of spin network states by the kernel of $\langle\langle\cdot,\cdot\rangle\rangle$ and then complete it in this inner product, we obtain the *diffeomorphism-invariant Hilbert space* $\mathcal{H}_{\mathrm{diff}}$. Any spin network state Ψ_Γ determines a diffeomorphism-invariant state $[\Psi_\Gamma] \in \mathcal{H}_{\mathrm{diff}}$, and also a continuous linear functional $\langle\langle \Psi_\Gamma, \cdot \rangle\rangle$ on the space of cylinder functions. The map

$$[\Psi_\Gamma] \mapsto \langle\langle \Psi_\Gamma, \cdot \rangle\rangle$$

extends uniquely to a continuous linear map from $\mathcal{H}_{\mathrm{diff}}$ to $\mathcal{C}^*$. Since this map is one-to-one, we may think of $\mathcal{H}_{\mathrm{diff}}$ as a subspace of the space of diffeomorphism-invariant vectors in $\mathcal{C}^*$.

4 Problems with spin webs

We now give examples of:

1. Spin web states whose inner product is nonzero but whose underlying webs do not have the same range.

2. A spin web state Ψ_W whose orbit $(\mathcal{D}/\mathcal{D}_W)\Psi_W$ contains infinitely many distinct spin web states whose inner product with Ψ_W is nonzero. (Here $\mathcal{D}_W$ is the set of diffeomorphisms fixing Ψ_W.)

The examples are generated out of the standard smooth function constructed in most introductory analysis classes, whose domain and range are $[0, 1]$, which is positive on $(0, 1)$, and whose value and all order derivatives are 0 at 0 and 1. Choose one such and call it $\alpha(x)$. Let $\alpha_{a,b}$ be α composed with a linear function so that its domain is now $[a, b]$, and let $x_i = 1/2(1 + \mathrm{sgn}(i)(1 - 2^{-i}))$, an order-preserving map of the integers into the unit interval with 0 and 1 as accumulation points. Now let

$$\beta_i^{\pm} = \pm 2^{-4^{|i|}} \alpha_{x_i, x_{i+1}}.$$

This unintuitive formula describes a doubly infinite sequence of disjoint (except for their endpoints) 'blips' above and below the x-axis between 0 and 1, converging to both endpoints in such a fashion that any choice of signs for each integer i indicates a collection of functions which can be pasted together to get a smooth function on the unit interval whose graph is an embedded curve in the plane.

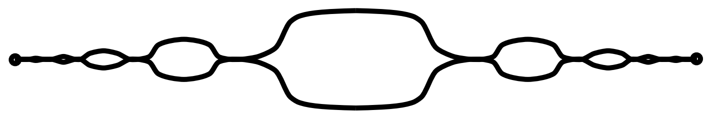

Figure 1: The four curves $c_1, \ldots, c_4$

Now let c_1 be the curve gotten by taking all plus signs, c_2 by taking all minus signs, c_3 by taking a plus sign when i is even and a minus when i is odd, and c_4 by taking a plus when i is odd and a minus when i is even. The range of these four curves is shown in Figure 1.

Fix a trivial SU(2) bundle over the plane. Label each curve c_i with the same representation ρ_i, namely the spin-1/2 representation (i.e. the 2-dimensional irreducible representation), and assign both endpoints the canonical invariant element of $\rho_1 \otimes \rho_2$ tensored with the canonical element of $\rho_3 \otimes \rho_4$, the subscripts indicating to which curve the representation corresponds. The family of four curves is not itself a web, but if we use the labelings to define a function Ψ of connections in the usual way, it is easy to check by cutting each curve in half at $x = 0$ that Ψ is a spin web state. It is also easy to check that the holonomies of the four curves are independent random variables with Haar measure distribution with respect to uniform measure.

Now let Φ be defined the same way, only pick some odd i and make c_2 and c_3 take the plus rather than the minus route at i, so that β_i^- is not in the range of the web supporting Φ. Thus the range of the web supporting Φ is a proper subset of that for Ψ. A calculation shows that the inner product of Ψ and Φ is nonzero. Thus Φ gives an example of the first observation. In fact, the same construction gives infinitely many such Φ.

For the second observation, we think of the curves c_i as living in the xy plane in $\mathbb{R}^3$. We consider the same Ψ, and for each i consider a diffeomorphism g_i which interchanges the curves β_i^+ and β_i^- and leaves the other $\beta_j^\pm$ fixed. The inner product of Ψ and $g_i\Psi$ is nonzero even though these states are distinct. Thus the spin web states $g_i\Psi$ are an infinite class of different elements of the orbit of Ψ having nonzero inner product with Ψ.

Based on these examples, it would seem quite difficult to give an effective procedure for constructing an orthonormal basis of the full $L^2(\overline{\mathcal{A}/\mathcal{G}})$ or to give a version of 'averaging over the action of the diffeomorphism group' that would apply to spin webs that are not spin networks.

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