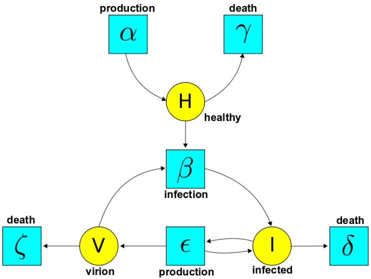
The Foundations of Applied Mathematics

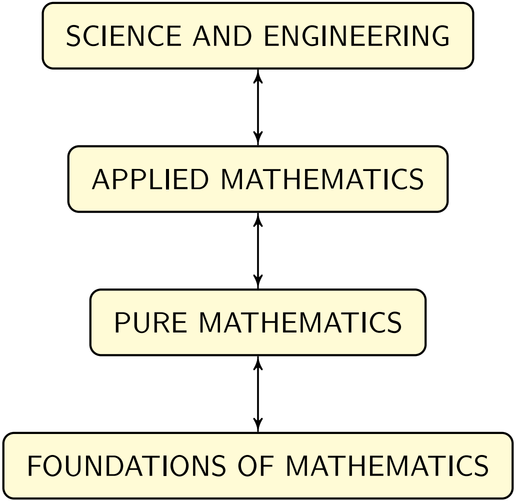


John Baez

Category-Theoretic Foundations of Mathematics Workshop

May 5, 2013

We often picture the flow of information about mathematics a bit like this:



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But the picture is close enough to true that deviations are interesting.

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For example: pure mathematicians tried to eliminate infinitesimals, but applied mathematicians kept using them... leading to nonstandard analysis and synthetic differential geometry. The latter approach drops the law of excluded middle!

Computer science is the biggest example of “applied math” that grew directly out of work in logic, where new ideas directly impact foundations:

- uncomputability, undecidability,...
- computer-aided proofs: what is a proof?
- category-theoretic logic.

But I want to talk about some *other* applications of mathematics that seem to call for category-theoretic foundations.

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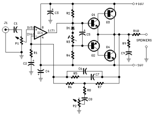

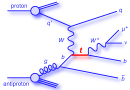

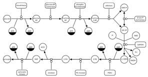

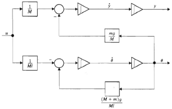

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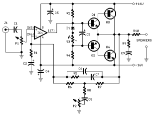

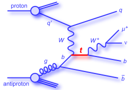

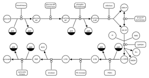

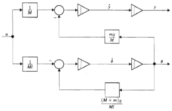

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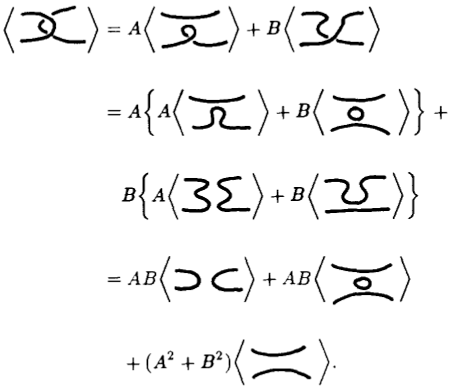
Let me sketch some applications of these diagrams... in the order in which I met them (a sign of old age).

In the 1980s, string diagrams became important at the interface of knot theory and quantum field theory:

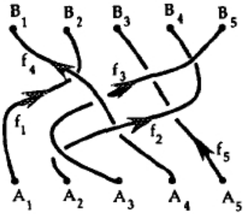
Proof. (a)



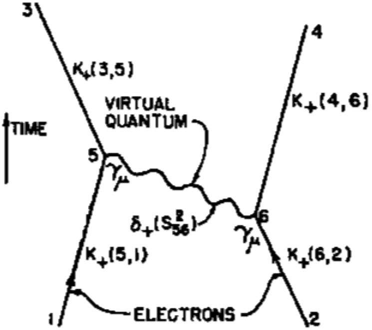
Part (b) is left for the reader.

By 1986, these diagrams were seen to describe morphisms in braided monoidal categories:

Such an arrow can be viewed as the braid α labelled by $f_1, \ldots, f_n$ as, for example:

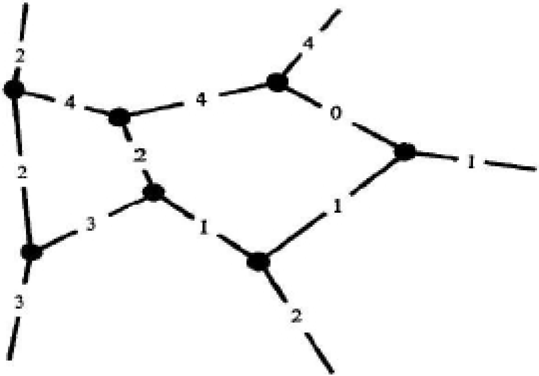


It became clear that Feynman diagrams, developed back in the 1940s, fit nicely into this theory:



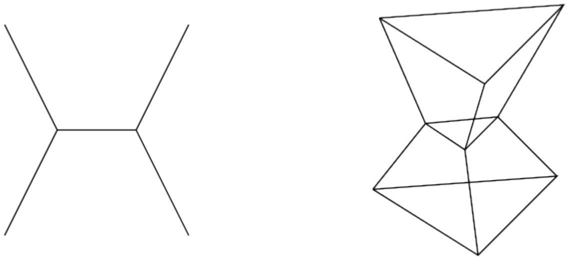
They describe morphisms in certain symmetric monoidal categories.

Penrose's 'spin networks', going back to 1971, also fit nicely into this theory:



By 1995, they were adopted by loop quantum gravity to describe quantum states of the geometry of space.

In 1997, higher-dimensional diagrams called 'spin foams' were introduced to describe the geometry of spacetime:



1. Feynman diagram versus spin foam

These are connected to higher categories, where we have objects, morphisms, morphisms between morphisms, etc.

There is by now a huge flowering of work on higher categories and their applications to physics, computer science, many areas of pure mathematics...

... and also the *foundations* of mathematics, as in

- higher topos theory,
- homotopy type theory, and
- univalent foundations.

which are all closely connected.

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$$x = x$$

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is interesting, but not the whole truth. It really means

$$\exists f : x \xrightarrow{\sim} y$$

"There is some reversible process f taking us from x to y."

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Higher categorical foundations are starting to give us an outlook in which computation, proof and the passage of time are intimately linked.

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There are, however, uses of string diagrams in applied math, science and engineering that have yet to be reckoned with!

Most challenging are the diagrams in biology, which often describe *qualitative* — that is, *non-numerical* — information.

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The Systems Biology Graphical Notation project is trying to standardize these diagrams. They are developing 3 diagram languages:

- process diagrams
- entity relationship diagrams
- activity flow diagrams

Process Diagrams show how entities change from one type to another over time:

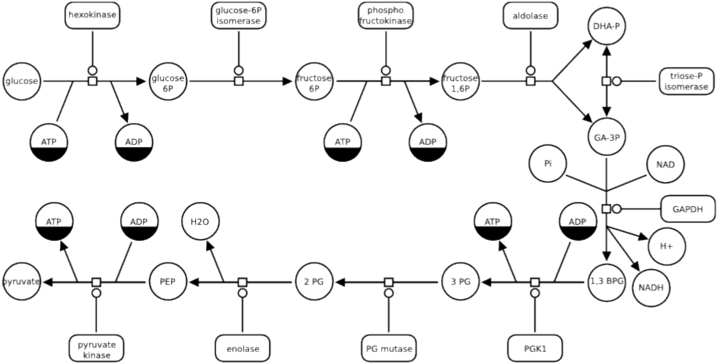

Figure A.1: *Glycolysis. This example illustrates how SBGN can be used to describe metabolic pathways.*

Entity Relationship Diagrams show how entities influence the behavior of each other:

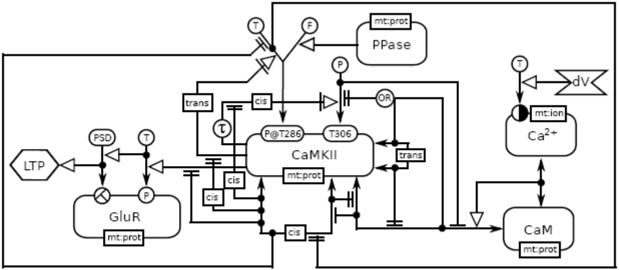

Figure A.2: *Regulation of calcium/calmoduline kinase II effect on synaptic plasticity.*

Activity Flow Diagrams show the flow of information between entities:

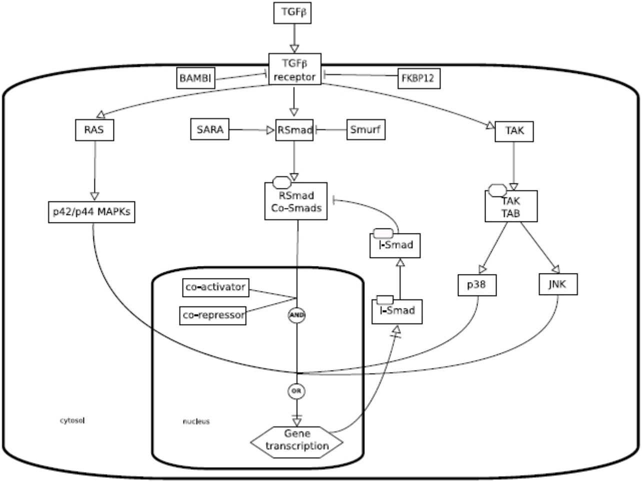

Figure A.3: *Transforming Growth Factor beta signaling pathway.*

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While biology is one of the biggest and most exciting branches of science today, I know of no mathematicians, logicians or philosophers studying these questions!

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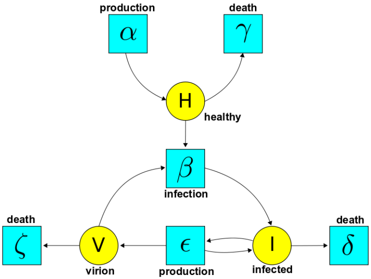
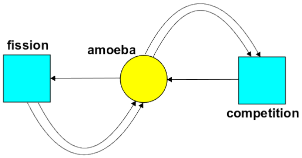
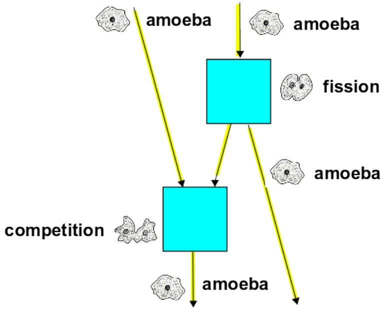
I've been working on easier examples. Petri nets are presentations of symmetric monoidal categories free on some objects and morphisms. They've been much studied in computer science, but in biology and chemistry we mostly need 'stochastic' Petri nets, where each generating morphism is equipped with a 'rate constant' in $(0, \infty)$:



This one describes the interaction between white blood cells and the virus that causes AIDS.

Stochastic Petri nets give a probabilistic analogue of quantum field theory, which deserves a book.





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So, not just categories but *bicategories* pervade applied math!

This is especially clear in control theory, which uses 'signal flow graphs' to describe physical systems:

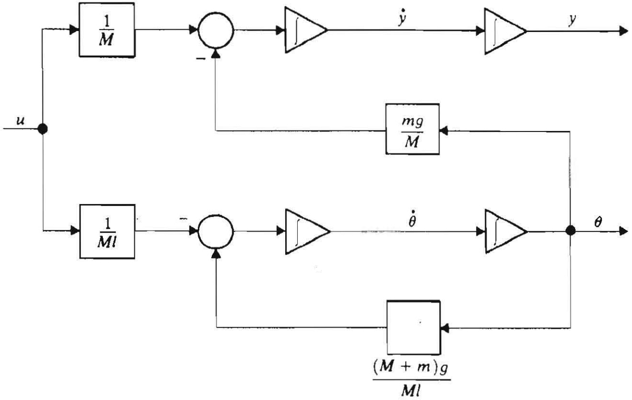

Figure 2.11 Block diagram of dynamics of inverted pendulum on moving cart.

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But control theorists use different signal flow graphs to describe the same relation. The same process can be implemented in different ways! So, they are dealing with *bicategory*, where two signal flow graphs are *isomorphic* if they give the same relation.

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Perhaps research on categorical foundations of mathematics should look to applied mathematics for inspiration here.