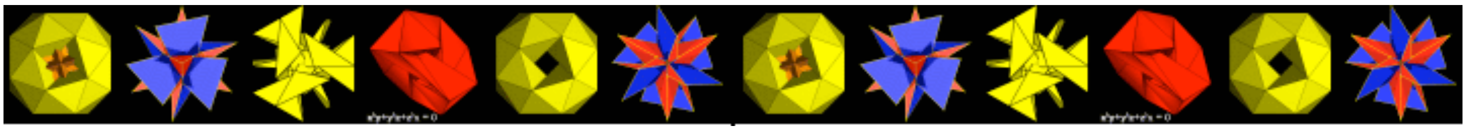
Pure and Applied Geometry



Polyhedral models of Felix Klein´s Quartic

Introduction

Felix Klein's quartic, also called Klein's curve, Klein's regular map or Klein's group PSL (2,7) is one of the most famous mathematical objects, or, as A.M. Macbeath formulated ([L], p. 104): ``It is a truly central piece of mathematics."

Felix Klein discovered this finite group of order 168 in 1879 [K], and since then its properties were investigated, generalized, applied and discussed in hundreds of papers.
The recent book ``The eightfold way" [L] contains several survey articles by prominent experts, which collect and discuss the essentials of Klein's quartic from various aspects. This book was

issued on the occasion of the installation of a nice geometric model of Klein's quartic made of Carrara marble by the artist H. Ferguson and put up at the campus of Berkeley.
The idea to visualize Klein's quartic by geometric models is not new. Felix Klein himself gave a planar and a 3-dimensional model in [K]. The planar one is the general and unsurpassable Poincar\'{e} model (cf. [G] or [L] p. 115), wellknown from classical complex analysis. The 3-dimensional one comes from the fact that Klein's quartic can be realized as a Riemannian manifold or as a regular map on an oriented 2-manifold of genus 3 and with octahedral symmetry ([L] p. 127). It is not metrically ``correct'', but is shows the algebraic and combinatorial properties of Klein's group PSL(2,7). The motivation for such 3-dimensional models is to find realizations as close as possible to the Platonic solids, hence built up of planar (and convex) polygons and with maximal possible symmetry. Polyhedral realizations of groups or regular maps can also be considered as contributions to H.S.M. Coxeter's general concept of ``groups and geometry'' (cf. e.g.[C] and [CM]). In this paper we describe and show the basic polyhedral realizations of Klein's quartic, two of them ``old'' and two new. For this we need some basic properties of Klein's quartic, which can be found in literature (cf. e.g.[K], [L], [CM], [MS] or [SW1]).

Basic facts

Klein's quartic is the algebraic curve with equation

$$x^3y + y^3z + z^3x = 0$$

in homogeneous coordinates. It can be realized as a regular map on an oriented 2-manifold of genus 3; either with 24 heptagons, three meeting at each of its 56 vertices, or with 56 triangles, seven meeting at each of its 24 vertices. The first one is usually written as $\{7,3\}_8$, the second one $\{3,7\}_8$. Here the subscript 8 denotes the length of the Petrie polygons.

A Petrie polygon is a skew polygon where every two but no three consecutive edges belong to the same face of the polyhedron. On a regular map, all possible Petrie polygons have same length. For Klein's map this is 8, and this explains the title ``The eightfold way''. So $\{3,7\}_8$ and $\{7,3\}_8$ are the two dual versions or realizations of Klein's group; in the same way as the regular icosahedron $\{3,5\}_{10}$ and dodecahedron $\{5,3\}_{10}$ are the two dual realizations of the icosahedral group with Petrie polygons of length 10.
The ordered triplets of vertices, edges and faces, briefly called flags of the icosahedron or dodecahedron are all equivalent under the group actions, i.e. the group acts transitively on the flags. As all group actions correspond to geometric symmetries, the icosahedron and dodecahedron (and the other Platonic solids) are considered to be perfect or beautiful or divine. This analogy (and other analogies) to the Platonic solids and their groups is the motivation to find 3-dimensional models of Klein's group. The icosahedral rotation group has order 60, including reflections one gets the full order 120. In the same way the full order of Klein's group is 336, but we consider its subgroup of order 168 and index 2, as Klein himself did.

Curved models

Different from the Platonic solids not all group actions of Klein's group can be realized by a geometric rotation or a reflection of the model. For regular maps of genus g >= 2 with p-gons and q-valent vertices there is the famous Riemann-Hurwitz identity, which relates all relevant numbers, in particular the genus g and the order A of the (automorphism) group:

$$A = 2(g-1)\left(\frac{1}{2} - \frac{1}{p} - \frac{1}{q}\right)^{-1}$$

From p=3, q=7 (or vice versa) and A = 168 follows g=3. As a consequence follows that such groups have maximal order 84(g-1), and Klein's group is the first one of these rare ``Hurwitz groups''.
The maximal geometric symmetry of Klein's group and hence of its geometric models is the octahedral rotation group of order 24. As 168 = 7 · 24, the other 7 operations are ``hidden symmetries''.
So Klein's second model is 3-dimensional with octahedral symmetry, curved, with selfintersections and non compact (cf. [K] or [L], p. 127). It can be described as ``three hyperboloids whose axes meet at right angles, which is certanily appealing'' (J. Gray).

It can easily be shown that any 3-dimensional model with maximal (i.e. octahedral) symmetry has selfintersections, so in order to avoid selfintersections Ferguson's model has next lower symmetry, i.e. tetrahedral rotation symmetry of order 12.
Ferguson's model is the realization of Klein's map $\{7,3\}_8$ on the standard model of an oriented smooth surface of genus 3 with tetrahedral symmetry. It shows the 24 heptagons and hence it corresponds to the regular dodecahedron $\{5,3\}_{10}$. Ferguson's model is curved and so the heptagons are nonplanar.

Polyhedral models

Figure 1

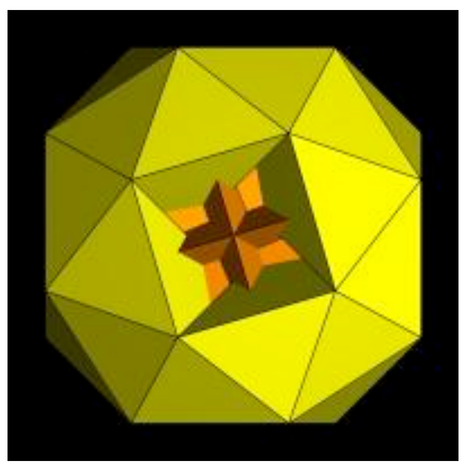


Figure 2

We come back to the natural question, if one can get a closer analogue to the Platonic solids and find models with planar polygons as facets. This question was answered by E. Schulte and J.M. Wills in 1985 [SW1] and 1987 [SW2], where they gave a polyhedral embedding with tetrahedral symmetry (figure 1) and a polyhedral immersion with octahedral symmetry (figure 2). Both are models of $\{3,7\}_8$ with 56 triangles, hence they correspond to the icosahedron $\{3,5\}_{10}$.

The octahedral model has maximal symmetry and the advantage that the symmetry group acts transitively on its 24 vertices. So the vertices are all alike. The vertices can be chosen that their convex hull is the snub cube, hence one of the 13 Archimedean solids. As a consequence 32 of the 56 triangles are even regular.
The three intersecting tunnels of this model correspond to Klein's three intersecting hyperboloids. Altogether this polyhedral model is the simplest one to understand the structure of Klein's group PSL (2,7).

The tetrahedral model of $\{3,7\}_8$ is even more attractive as it can be realized as an embedding, i.e. without selfintersections. Each of the four holes have a strong twist and it is a priori not clear that this can be done without selfintersections. The 24 vertices split into two orbits of 12 vertices under the tetrahedral rotation group. The outer orbit of 12 vertices can be realized again by the vertices of an Archimedean solid, namely the truncated tetrahedron. Several models from cardbord and metal and computer films were made of this realization. (cf. also [BW] and Conway's comment before the title). From its symmetry and embedding properties it corresponds to Ferguson's model, but it is eight years older. H.S.M. Coxeter's comment (Dec. 3, 1984) on this model: ``....a wonderful result''.
The constructions and incidences can be found in detail in [SW1] and [SW2]. For more details cf. [SSW] where one can find also models with integer coordinates.

Dual models

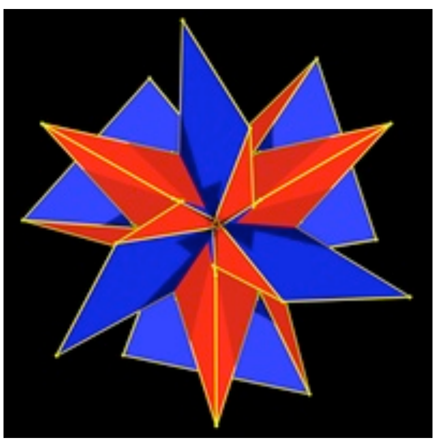


Figure 3a

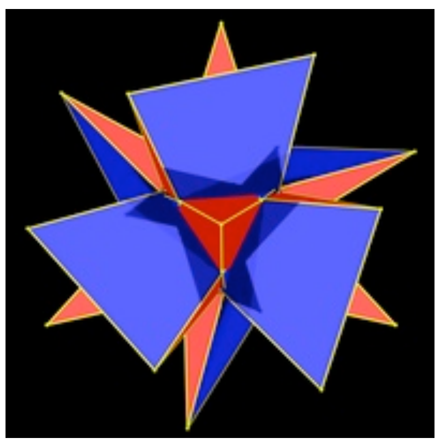


Figure 3b

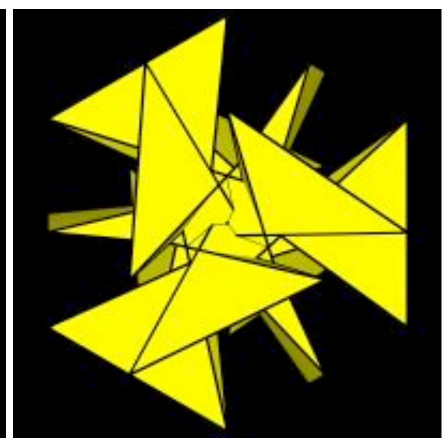


Figure 4

E. Schulte and J.M.Wills never tried to realize the dual map $\{7,3\}_8$ neither with tetrahedral nor with octahedral symmetry. It was intuitively clear that they are much more complicate and hence less useful to understand Klein's group. With modern computer programs a construction of these models is possible and the results are shown in figures 3 and 4.
The complicated shape of these models underlines the simplicity of their duals.

Figure 3 displays $\{7,3\}_8$ with tetrahedral symmetry and figure 4 with octahedral symmetry. Figure 3 is an immersion. Its 24 heptagons lie on two orbits (red and blue) of the tetrahedral group. In figure 4 the heptagonal faces have selfintersections, as some of the classical Kepler-Poinsot star bodies have. Because of the selfintersections both figures are too complicated to be understood at once. It is worth mentioning that the bizarre model with octahedral symmetry is face-transitive, i.e. all its faces are congruent. It is more complicated than its dual in figure 2,

which is built up of triangles.
It might be surprising that the realizations of a pair of dual maps of the same group can be so different. But the answer is quite simple: In the triangulations the facets are (of course) triangles, hence the simplest polygons which are convex and free of selfintersections. All topological complications as twists and curvature are hidden in the vertices whose shape is flexible.
In the dual case with 3-valent vertices all complications have to be stored in the heptagons, which makes the models star-shaped and bizarre. This phenomenon for nonconvex realizations of dual maps is wellknown and described by Grünbaum and Shephard [GS]. For details of the constructions of $\{7,3\}_8$ we refer to [SSW].

Conclusions

The models of Klein's quartic with closest relation to the Platonic solids are the polyhedral embedding of $\{3,7\}_8$ with tetrahedral symmetry and the polyhedral immersion of $\{3,7\}_8$ with octahedral symmetry. Both are built up of planar triangles, so they correspond to the regular icosahedron. Their convex hulls are Archimedean solids. The polyhedral realizations of the dual map $\{7,3\}_8$ are starshaped. They correspond to the regular dodecahedron and remind of the classical Kepler--Poinsot polyhedra, Coxeter's regular complex polyhedra and other generalizations.

So these models can be considered as footnotes to Felix Klein's and H.S.M. Coxeter's general idea to bring algebra and geometry closer together.

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SW1

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