1 Chopping $e_8$ into 31 pieces of dimension 8

In what follows we work with the complex form of $E_8$ and its Lie algebra $e_8$. Choose a root system $\Delta$ for $e_8$. Each root $\psi$ in $\Delta$ gives a root vector $e_\psi$ in $e_8$, and $e_8$ is then a direct sum of its Cartan subalgebra $h$ and the spans of these vectors $e_\psi$. The Cartan subalgebra is 8-dimensional, and there are 240 roots, so the dimension of $e_8$ is 248.

Choose simple roots $\alpha_1, \ldots, \alpha_8$. There is a special element $w$ in $h$ such that

$$w \cdot \alpha_i = 1$$

for $i = 1, \ldots, 8$. This gives a 1-parameter subgroup $\exp(sw)$ in $E_8$, where $s \in \mathbb{C}$. An element of $E_8$ is called principal if it is conjugate to one of the form $\exp(sw)$. There is a unique element of order 2 of the form $\exp(sw)$, so we get a distinguished conjugacy class of order-2 elements in $E_8$, namely the principal ones. There’s a copy of the group $(\mathbb{Z}/2)^5$ sitting inside this conjugacy class. The group $(\mathbb{Z}/2)^5$ has 32 characters, that is, homomorphisms

$$\chi : (\mathbb{Z}/2)^5 \longrightarrow U(1).$$

31 of these characters are nontrivial. Note that

$$248 = 31 \times 8.$$

The nontrivial characters $\chi_j (j = 1, \ldots, 31)$ have an interesting property. Let

$$h_j = \{ x \in e_8 : \text{Ad}(a)x = \chi_j(a)x \text{ for all } a \in (\mathbb{Z}/2)^5 \}$$

Then $h_j$ is a Cartan subalgebra of $e_8$! And,

$$e_8 = h_1 \oplus \ldots \oplus h_{31}.$$

So, the 248-dimensional Lie algebra $e_8$ is a direct sum of 31 8-dimensional Cartan algebras.
2 How $SL(2, 32)$ acts to permute the 31 pieces of $e_8$

$E_8$ has a finite subgroup called the Dempwolf group, $F_{Demp}$, with the above $(Z/2)^5$ as a normal subgroup. In fact we have an exact sequence:

$$1 \rightarrow (Z/2)^5 \rightarrow F_{Demp} \rightarrow SL(2, 32) \rightarrow 1.$$ 

Starting from the above $(Z/2)^5$ subgroup of $E_8$, we get a chain of subgroups

$$(Z/2) \subset (Z/2)^2 \subset (Z/2)^3 \subset (Z/2)^4 \subset (Z/2)^5.$$ 

The centralizer of $Z/2$ in $E_8$ is $D_8 = Spin(16)$. The centralizer of $(Z/2)^2$ is $D_4 \times D_4$, or more precisely, $Spin(8) \times Spin(8)$ mod the diagonal copy of $Z/2$.

We have a vector space decomposition

$$e_8 = (so(8) \oplus so(8)) \oplus V_8 \otimes V_8 \oplus S_8^+ \otimes S_8^+ \oplus S_8^- \otimes S_8^-$$

where $V_8$, $S_8^+$ and $S_8^-$ are the 8-dimensional "vector", "right-handed spinor" and "left-handed spinor" representations of $Spin(8)$, respectively. These three representations are related by triality.

The elements of the Dempwolf group permute the 64-dimensional subspaces $V_8 \otimes V_8$, $S_8^+ \otimes S_8^+$ and $S_8^- \otimes S_8^-$ of $e_8$.

Griess and Ryba have been studying finite subgroups of $E_8$, for example:


Serre has too:


The 5-dimensional vector space $(Z/2)^5$ can be made into a field, $F_{32}$, and invertible $2 \times 2$ matrices with determinant 1 having entries in this field form the group $SL(2, 32)$. This whole group sits inside $E_8$. It contains the aforementioned copy of $(Z/2)^5$ as matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Proving this was hard!

The group $SL(2, 32)$ has cardinality

$$3 \times 11 \times 31 \times 32$$
corresponding to something analogous to an Iwasawa decomposition $G = KAN$: the subgroup $K$ has cardinality $3 \times 11$, the subgroup $A$ has cardinality $31$, and the subgroup $N$ (consisting of the upper triangular matrices shown above) has cardinality $32$.

The solvable subgroup of $SL(2,32)$ has one irreducible $31$-dimensional representation. The adjoint action of $SL(2,32)$ on $e_8$ acts to permute its $31$ Cartan subalgebras according to this representation!

3 A product of two copies of the Standard Model gauge group in $E_8$

The gauge group of the Standard model is usually said to be $SU(3) \times SU(2) \times U(1)$, but this group has a $Z/6$ subgroup that acts trivially on all known particles. The quotient $(SU(3) \times SU(2) \times U(1))/(Z/6)$ is isomorphic to $SU(3) \times SU(2)) - that is, the subgroup of $SU(5)$ consisting of block diagonal matrices with a $3 \times 3$ block and a $2 \times 2$ block. So, this could be called the “true” gauge group of the Standard Model. Each generation of fermions transforms according to the obvious representation of $SU(3) \times SU(2))$ on the exterior algebra $\Lambda C^5$. This idea is the basis of the $SU(5)$ grand unified theory.

There is an element of order $11$ in $SL(2,32) \subset E_8$. The centralizer of this element in $E_8$ is a product of two copies of the Standard Model gauge group:

$$S(U(3) \times U(2)) \times (SU(3) \times SU(2))$$

Let $g_i$ be the subspace of $e_8$ consisting of linear combinations of root vectors $e_{\phi} i$ where the root $\phi$ is a sum of $i$ simple roots. Let $g_{-i}$ be the analogous thing defined using the negatives of the simple roots. Then, the Lie algebra of the $S(U(3) \times U(2)) \times (SU(3) \times SU(2))$ subgroup of $E_8$ is

$$g_{-22} \oplus g_{-11} \oplus g_{11} \oplus g_{22}$$

(There is also an element of order $3$ in $SL(2,32)$, whose centralizer in $E_8$ is $SU(9)$.)

4 A product of two copies of $su(5)$ in $e_8$

There’s also a copy of $(Z/5)^3$ in $E_8$. If we think of this as a $3$-dimensional vector space containing “lines” (1-dimensional subspaces), then it contains

$$1 + 5 + 5^2 = 31$$

lines. The centralizer in $E_8$ of any such line is the group

$$(SU(5) \times SU(5))/\langle Z/5 \rangle$$
This group is 48-dimensional. It has a 248-dimensional representation coming from the adjoint action on $e_8$. This is the direct sum of the 48-dimensional subrepresentation coming from $su(5) \oplus su(5) \subset e_8$ and a representation of dimension

$$248 - 48 = 200$$

This 200-dimensional representation is

$$5 \otimes 10 \oplus 5^* \otimes 10^* \oplus 10 \otimes 5^* \oplus 10^* \otimes 5$$

where 5 is the defining representation of $SU(5)$, 10 is its second exterior power, and $5^*$ and $10^*$ are the duals of these.

Is any of this useful in particle physics? In particular, what could the “doubling” mean?