Fractional Index Theory

$$\mathrm{Index}_a(\not{\partial}^+) = \int_Z \widehat{A}(Z) \quad \in \mathbb{Q}$$

Workshop on Geometry and Lie Groups

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THE UNIVERSITY of ADELAIDE

References

[MMS4]

V. Mathai, R.B. Melrose and I.M. Singer,

Equivariant and fractional index of projective elliptic operators,

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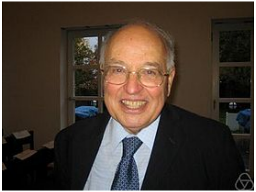

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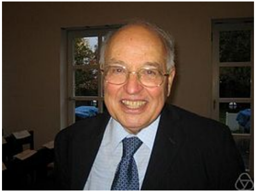

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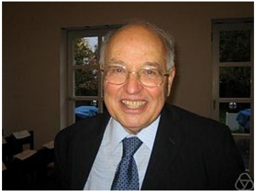

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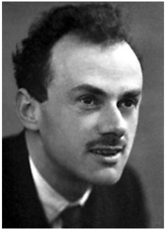

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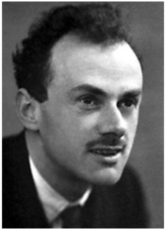

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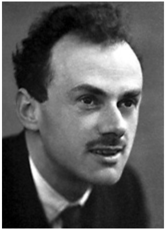

Dirac defined an operator $\partial\!\!\!/$ on $\mathbb{R}^n$ that solved the **square root problem** for the Laplacian on $\mathbb{R}^n$, that is, $\partial\!\!\!/^2 = \Delta$. The construction was novel as it used **Clifford algebras and spinors** in an essential way.

Dirac operators

More precisely, if $\{\gamma_j\}_{j=1}^{n}$ denote Clifford multiplication by an orthonormal basis of $\mathbb{R}^n$, then the Clifford algebra relations are $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{ij}$.

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$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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$$\boxed{\partial\!\!\!/ = \sum_{j=1}^{n} \gamma_j \frac{\partial}{\partial x_j}.}$$

It turns out that this operator plays a fundamental role in quantum mechanics, and is known as the **Dirac operator**.

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By construction,

$$\not{\partial}^2 = \Delta.\mathrm{Id}.$$

The Index Theorem for Dirac operators

Atiyah and Singer extended the definition of the **Dirac operator**, $\partial\!\!\!/^{+}$ on any compact **spin** manifold Z of even dimension, and computed the analytic index,

$$\begin{aligned}\mathrm{Index}_a(\partial\!\!\!/^{+}) &= \dim(\mathrm{nullspace}\,\partial\!\!\!/^{+}) - \dim(\mathrm{nullspace}\,\partial\!\!\!/^{-}) \\ &= \int_Z \widehat{A}(Z) \quad \in \mathbb{Z}\end{aligned}$$

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where RHS is the A-hat genus of the manifold Z. In terms of the Riemannian curvature Ω_Z of Z, $\widehat{A}(Z) = \sqrt{\det\left(\frac{\frac{1}{4\pi}\Omega_Z}{\sinh(\frac{1}{4\pi}\Omega_Z)}\right)}$.

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?Question?: Since $\int_Z \widehat{A}(Z) \notin \mathbb{Z}$ continues to make sense for non-spin manifolds Z, what corresponds to the analytic index in this situation, since the usual Dirac operator does not exist?

Outline of talk

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1. In **[MMS3]**, we generalize the notion of "ψdo", to "projective ψdo". In particular, on an oriented even dimensional Riemannian manifold, we define the notion of **projective spin Dirac operator**. We define its fractional analytic index, and prove an **index theorem** showing that it equals the $\widehat{A}$-genus (proof sketched in the talk).

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2. On the oriented orthonormal frame bundle of such a manifold, we show in **[MMS4]** that there also always exists a Spin-equivariant transversally elliptic Dirac operator. The relation between the **fractional analytic index** of the projective Dirac operator and the **equivariant index** of the associated Spin-equivariant transversally elliptic Dirac operator is explained there and sketched in the talk.

Projective vector bundles

A projective vector bundle on a manifold Z is **not** a global bundle on Z, but rather it is a vector bundle $E \to Y$, where

$$PU(n) \longrightarrow Y \xrightarrow{\phi} Z$$

is a principal $PU(n)$-bundle,

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where $\mathcal{L} = U(n) \times_{U(1)} \mathbb{C} \to PU(n)$ is the primitive line bundle,

$$\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2} \cong \mathcal{L}_{g_1.g_2}, \qquad g_i \in PU(n).$$

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The identification (1) gives a projective action of $PU(n)$ on E, i.e. an action of $U(n)$ on E s.t. the center $U(1)$ acts as scalars.

The **Dixmier-Douady invariant** of Y,

$$DD(Y) = \delta(Y) \in \mathrm{Torsion}(H^3(Z, \mathbb{Z}))$$

is the obstruction to lifting the principal $PU(n)$-bundle Y to a principal $U(n)$-bundle. (The construction also works for any principal G bundle P over Z, together with a central extension $\widehat{G}$ of G.)

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$$\mathcal{A} = Y \times_{PU(n)} M_n(\mathbb{C})$$

is called the associated **Azumaya bundle**.

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Also $End(\mathcal{S}) \cong \psi^* Cl(E)$.

Schwartz kernel theorem

For a compact manifold, Z, and vector bundles E and F over Z, the **Schwartz kernel theorem** gives a 1-1 correspondence,

continuous linear operators, $C^\infty(Z, E) \longrightarrow C^{-\infty}(Z, F)$

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where $\mathrm{Hom}(E, F)_{(z,z')} = F_z \boxtimes E^*_{z'}$ is the 'big' homomorphism bundle over Z^2 and Ω_R the density bundle from the right factor.

When restricted to pseudodifferential operators, $\Psi^m(Z, E, F)$, get an isomorphism with the space of **conormal distributions** with respect to the diagonal, $I^m(Z^2, \Delta; \mathrm{Hom}(E, F))$. i.e.

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When further restricted to differential operators $\mathrm{Diff}^m(Z,E,F)$ (which by definition have the property of being local operators) this becomes an isomorphism with the space of conormal distributions, $I^m_\Delta(Z^2,\Delta;\mathrm{Hom}(E,F))$, with respect to the diagonal, **supported within the diagonal**, Δ. i.e.

$$\mathrm{Diff}^m(Z,E,F) \iff I^m_\Delta(Z^2,\Delta;\mathrm{Hom}(E,F))$$

Projective differential operators/ΨDOs

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Since a projective vector bundle E is not global on Z, one **cannot** make sense of sections of E, let alone operators acting between sections! However, it still makes sense to talk about Schwartz kernels even in this case, as we explain.

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The restriction $\Delta^* \mathrm{Hom}(E, F) = \mathrm{hom}(E, F)$ to the diagonal is an ordinary vector bundle, it is therefore reasonable to expect that $\mathrm{Hom}(E, F)$ also restricts to an ordinary vector bundle in a tubular nbd N_ϵ of the diagonal.

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In [MMS3], it is shown that there is a canonical such choice, $\mathrm{Hom}^{\mathcal{A}}(E, F)$ of such that the composition properties hold.

This allows us to define the space of projective pseudo-differential operators $\Psi_\epsilon^\bullet(Z; E, F)$ with Schwartz kernels supported in an ϵ-neighborhood N_ϵ of the diagonal Δ in Z^2, with the space of conormal distributions, $I_\epsilon^\bullet(N_\epsilon, \Delta; \mathrm{Hom}^{\mathcal{A}}(E, F))$.

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This allows us to define the space of projective pseudo-differential operators $\Psi^{\bullet}_{\epsilon}(Z; E, F)$ with Schwartz kernels supported in an ϵ-neighborhood N_{ϵ} of the diagonal Δ in Z^2, with the space of conormal distributions, $I^{\bullet}_{\epsilon}(N_{\epsilon}, \Delta; \mathrm{Hom}^{\mathcal{A}}(E, F))$.

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Despite **not** being a space of operators, this has precisely the same local structure as in the standard case and has similar composition properties provided supports are restricted to appropriate neighbourhoods of the diagonal.

The space of **projective smoothing operators**, $\Psi_\epsilon^{-\infty}(Z; E, F)$ is defined as the smooth sections, $C_c^\infty(N_\epsilon; \mathrm{Hom}^{\mathcal{A}}(E, F) \otimes \pi_R^* \Omega)$.

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The space of all projective differential operators, $\mathrm{Diff}^{\bullet}(Z; E, F)$ is defined as those conormal distributions that are supported within the diagonal Δ in Z^2,

$$\mathrm{Diff}^{\bullet}(Z; E, F) \quad := \quad I_{\Delta}^{\bullet}(N_{\epsilon}, \Delta; \mathrm{Hom}^{\mathcal{A}}(E, F)).$$

In fact, $\mathrm{Diff}^{\bullet}(Z; E, F)$ is even a **ring** when $E = F$.

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Recall also that $\hom(\mathcal{S}, \mathcal{S}) \cong Cl(Z)$, has an extension to $\tilde{C}l(Z)$ in a tubular neighbourhood of the diagonal Δ, with an induced connection ∇.

The **projective spin Dirac operator** is defined as the distributional section

$$\not{\partial} = cl \cdot \nabla_L(\kappa_{Id}), \qquad \kappa_{Id} = \delta(z - z')Id_S$$

Here ∇_L is the connection ∇ restricted to the left variables with cl the contraction given by the Clifford action of T^*Z on the left. As in the usual case, the projective spin Dirac operator $\not{\partial}$ is **elliptic** and odd wrt $\mathbb{Z}_2$ grading of $\mathcal{S}$.

The **principal symbol map** is well defined for conormal distributions, leading to the globally defined symbol map,

$$\sigma : \Psi^m_\epsilon(Z; E, F) \longrightarrow C^\infty(T^*Z, \pi^* \hom(E, F)),$$

homogeneous of degree m; here $\hom(E, F)$, is a globally defined, **ordinary vector bundle** with fibre, $\hom(E, F)_z = F_z \otimes E^*_z$. Thus **ellipticity** is well defined, as the **invertibility of this symbol**.

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Equivalently, $A \in \Psi^m_{\epsilon/2}(Z; E, F)$ is elliptic if there exists a parametrix $B \in \Psi^{-m}_{\epsilon/2}(Z; F, E)$ and smoothing operators $Q_R \in \Psi^{-\infty}_\epsilon(Z; E, E)$, $Q_L \in \Psi^{-\infty}_\epsilon(Z; F, F)$ such that

$$BA = I_E - Q_R, \qquad AB = I_F - Q_L$$

The **trace functional** is defined on projective smoothing operators $\mathrm{Tr} : \Psi_{\epsilon}^{-\infty}(Z; E) \to \mathbb{C}$ as

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It vanishes on commutators, i.e. $\mathrm{Tr}(QR - RQ) = 0$, if $Q \in \Psi_{\epsilon/2}^{-\infty}(Z; F, E)$, $R \in \Psi_{\epsilon/2}^{-\infty}(Z; E, F)$ which follows from Fubini's theorem.

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The **fractional analytic index** of the projective elliptic operator $A \in \Psi_\epsilon^\bullet(Z; E, F)$ is defined in the essentially analytic way as,

$$\mathrm{Index}_a(A) = \operatorname{Tr}([A, B]) \in \mathbb{R}$$

where B is a parametrix for A, and the RHS is the notation for $\operatorname{Tr}_F(AB - I_F) - \operatorname{Tr}_E(BA - I_E)$.

For $A \in \Psi^m_{\epsilon/4}(Z; E, F)$, the Guillemin-Wodzicki **residue trace** is,

$$\mathrm{Tr}_R(A) = \lim_{z \to 0} z \,\mathrm{Tr}(AD(z))$$

where $D(z) \in \Psi^z_{\epsilon/4}(Z; E)$ is an entire family of ΨDOs of complex order z which is elliptic and such that $D(0) = I$. The residue trace is independent of the choice of such a family.

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1. The residue trace Tr_R vanishes on all ΨDOs of sufficiently negative order.
2. The residue trace Tr_R is also a trace functional, that is,

$$\mathrm{Tr}_R([A, B]) = 0,$$

for $A \in \Psi^m_{\epsilon/4}(Z; E, F),\ B \in \Psi^{m'}_{\epsilon/4}(Z; F, E)$.

The **regularized trace**, is defined to be the residue,

$$\mathrm{Tr}_D(A) = \lim_{z \to 0} \frac{1}{z} \left(z\,\mathrm{Tr}(AD(z)) - \mathrm{Tr}_R(A) \right).$$

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Therefore the fractional analytic index is also given by,

$$\mathrm{Index}_a(A) = \operatorname{Tr}_D([A, B])$$

for a projective elliptic operator A, and B a parametrix for A.

The regularized trace Tr_D is **not** a trace function, but however it satisfies the **'trace defect formula'**,

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2. the **multiplicativity property** of of the index,

$$\mathrm{Index}_a(A_2A_1) = \mathrm{Index}_a(A_1) + \mathrm{Index}_a(A_2),$$

where A_i for $i = 1, 2$ are projective elliptic ΨDOs.

An analogue of the **McKean-Singer formula** holds,

$$\mathrm{Index}_a(\not{\partial}_E^+) = \lim_{t \downarrow 0} \mathrm{Tr}_s(H_\chi(t))$$

where $H_\chi(t) = \chi(H_t)$ is a globally defined, **truncated heat kernel**, both in space (in a nbd of the diagonal) and in time.

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The local index theorem can then be applied, thanks to the McKean-Singer formula, to obtain the **index theorem** for projective spin Dirac operators.

Index of projective spin Dirac operators

Theorem ([MMS3])

The projective spin Dirac operator on an even-dimensional compact oriented manifold Z, has fractional analytic index,

$$\mathrm{Index}_a(\partial\!\!\!/^+) = \int_Z \widehat{A}(Z) \in \mathbb{Q}.$$

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The projective spin Dirac operator on an even-dimensional compact oriented manifold Z, has fractional analytic index,

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Recall that $Z = \mathbb{C}P^{2n}$ is an oriented but **non-spin** manifold such that $\int_Z \widehat{A}(Z) \notin \mathbb{Z}$, justifying the title of the talk. e.g.

$$Z = \mathbb{C}P^2 \quad \Longrightarrow \quad \mathrm{Index}_a(\partial\!\!\!/^+) = -1/8.$$

$$Z = \mathbb{C}P^4 \quad \Longrightarrow \quad \mathrm{Index}_a(\partial\!\!\!/^+) = 3/128.$$

Equivariant transversally elliptic Dirac operator

Recall that projective half spinor bundles $\mathcal{S}^{\pm}$ on Z can be realized as *Spin*(n)-equivariant honest vector bundles, $(\tilde{\mathcal{S}}^{+}, \tilde{\mathcal{S}}^{-})$, over the total space of the oriented frame bundle $\mathcal{P}$ and in which the center, $\mathbb{Z}_2$, acts as ± 1, as follows:

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One can define the *Spin*(n)-equivariant transversally elliptic Dirac operator $\tilde{\partial\!\!\!/}^{\pm}$ using the Levi-Civita connection on Z together with the Clifford contraction, where **transverse ellipticity** means that the principal symbol is invertible when restricted to directions that are conormal to the fibres.

The nullspaces of $\tilde{\not\partial}^{\pm}$ are infinite dimensional unitary representations of $Spin(n)$. The transverse ellipticity implies that the characters of these representations are **distributions** on the group $Spin(n)$. In particular, the **multiplicity** of each irreducible unitary representation in these nullspaces is **finite**, and grows at most polynomially.

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The ***Spin*(*n*)-equivariant index** of $\tilde{\not{\partial}}^{+}$ is defined to be the following distribution on $Spin(n)$,

$$\mathrm{Index}_{Spin(n)}(\tilde{\not{\partial}}^{+}) = \mathrm{Char}(\mathrm{Nullspace}((\tilde{\not{\partial}}^{+})) - \mathrm{Char}(\mathrm{Nullspace}(\tilde{\not{\partial}}^{-}))$$

An alternate, **analytic** description of the *Spin*(n)-equivariant index of $\tilde{\eth}^+$ is: for a function of compact support $\chi \in \mathcal{C}^\infty_c(G)$, the action of the group induces a graded operator

$$T_\chi : C^\infty(\mathcal{P}; \tilde{\mathcal{S}}) \longrightarrow C^\infty(\mathcal{P}; \tilde{\mathcal{S}}), \quad T_\chi u(x) = \int_G \chi(g) g^* u dg,$$

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$$T_\chi \circ (\tilde{\partial\!\!\!/}^+ \circ Q - I_-) \in \Psi^{-\infty}(\mathcal{P};\tilde{\mathcal{S}}^-); \quad T_\chi \circ (Q \circ \tilde{\partial\!\!\!/}^+ - I_+) \in \Psi^{-\infty}(\mathcal{P};\tilde{\mathcal{S}}^+)$$

are smoothing operators. The ***Spin*(n)-equivariant index** of $\tilde{\partial\!\!\!/}^+$, evaluated at $\chi \in \mathcal{C}_c^\infty(G)$, is also given by:

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$$\mathrm{Index}_{Spin(n)}(\tilde{\partial\!\!\!/}^+)(\chi) = \mathrm{Tr}(T_\chi \circ (\tilde{\partial\!\!\!/}^+ \circ Q - I_-)) - \mathrm{Tr}(T_\chi \circ (Q \circ \tilde{\partial\!\!\!/}^+ - I_+))$$

Relation between the two types of Dirac operators

Theorem ([MMS4])

Let $\pi : \mathcal{P}^2 \to Z^2$ denote the projection. The pushforward map, π_*, maps the Schwartz kernel of the Spin(n)-transversally elliptic Dirac operator to the projective Dirac operator: That is,

$$\pi_*(\tilde{\partial\!\!\!/}^{\pm}) = \partial\!\!\!/^{\pm}.$$

Relation between the two solutions

We will now relate these two pictures.

An easy argument shows that the support of the equivariant index distribution is contained within the center $\mathbb{Z}_2$ of *Spin*(n).

Theorem

Let $\phi \in \mathcal{C}^\infty(Spin(n))$ *be such that :*

1. $\phi \equiv 1$ *in a neighborhood of e;*
2. $-1 \notin \mathrm{supp}(\phi)$. *Then*

$$\mathrm{Index}_{Spin(n)}(\tilde{\not{\partial}}^+)(\phi) = \mathrm{Index}_a(\not{\partial}^+)$$

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Informally, the fractional analytic index, of the projective Dirac operator $\partial\!\!\!/^+$, is the coefficient of the delta function (distribution) at the identity in $Spin(n)$ of the $Spin(n)$-equivariant index for the associated transversally elliptic Dirac operator $\tilde{\partial\!\!\!/}^+$ on $\mathcal{P}$.