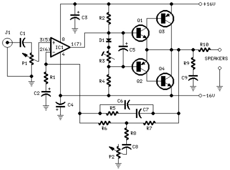
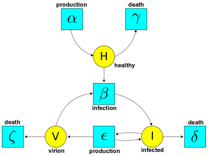
# STRUCTURED COSPANS AND DOUBLE CATEGORIES

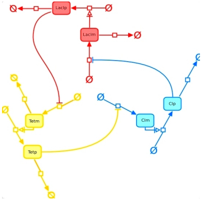
**John Baez, Kenny Courser, Christina Vasilakopoulou**

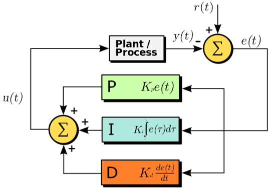
1 April 2020

Throughout science and engineering, people use *networks*, drawn as boxes connected by wires:



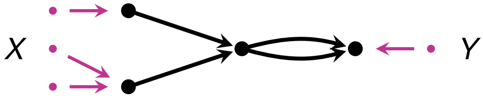




So, they're using categories! Which categories are these?

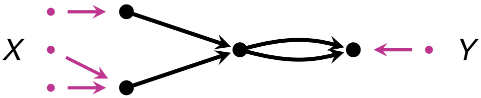
Networks of a particular kind, with specified inputs and outputs, can be seen as morphisms in a particular symmetric monoidal category:




Such networks let us describe "open systems", meaning systems where:

- stuff can flow in or out;
- we can combine systems to form larger systems by composition and tensoring.

We can describe networks with inputs and outputs using cospans with extra structure. For example, this:




is really a cospan of finite sets:

$$X \xrightarrow{i} S \xleftarrow{o} Y$$

where $S$ is decorated with extra structure: edges making $S$ into the vertices of a graph.

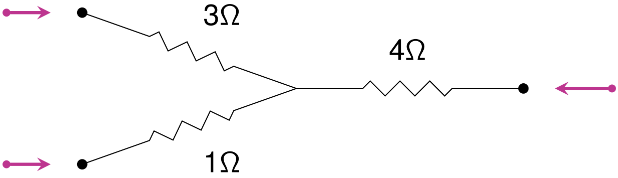
Fong invented ‘decorated cospans’ to make this precise:

- Brendan Fong, Decorated cospans, arXiv:1502.00872.

We’ve used them to study many kinds of networks.

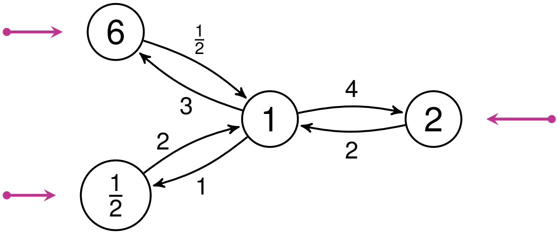
Electrical circuits:

- Brendan Fong, JB, A compositional framework for passive linear networks, arXiv:1504.05625.



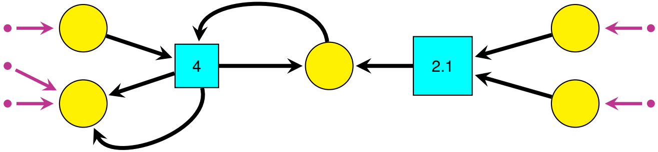
Markov processes:

- Brendan Fong, Blake Pollard, JB, A compositional framework for Markov processes, arXiv:1508.06448.



Petri nets with rates:

- Blake Pollard, JB, A compositional framework for reaction networks, arXiv:1704.02051.



Now Kenny Courser has developed a simpler formalism: ‘structured cospans’.

We have redone most of the previous work using structured cospans:

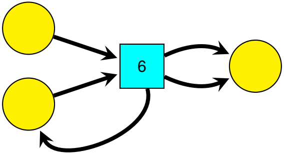
- JB and Kenny Courser, Structured cospans, arXiv:1911.04630.
- Kenny Courser, *Open Systems: A Double Categorical Perspective*, https://tinyurl.com/courser-thesis.

Let’s see how structured cospans work in an example: Petri nets with rates.

A **Petri net with rates** is a diagram like this:

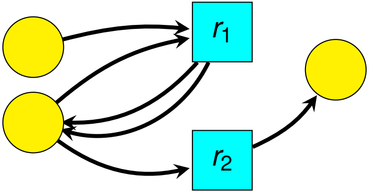
$$(0,\infty) \xleftarrow{\ r\ } T \overset{s}{\underset{t}{\rightrightarrows}} \mathbb{N}[S]$$

where $S$ and $T$ are finite sets, and $\mathbb{N}[S]$ is the set of finite formal sums of elements of $S$.

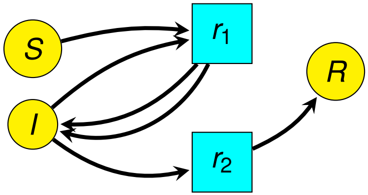



We call elements of $S$ **places** ,
elements of $T$ **transitions** ,
and $r(t)$ the **rate constant** of the transition $t \in T$.

Given a Petri net with rates, we can write down a **rate equation** describing dynamics. Example: the **SIR model** of infectious disease:



Given a Petri net with rates, we can write down a **rate equation** describing dynamics. Example: the **SIR model** of infectious disease:

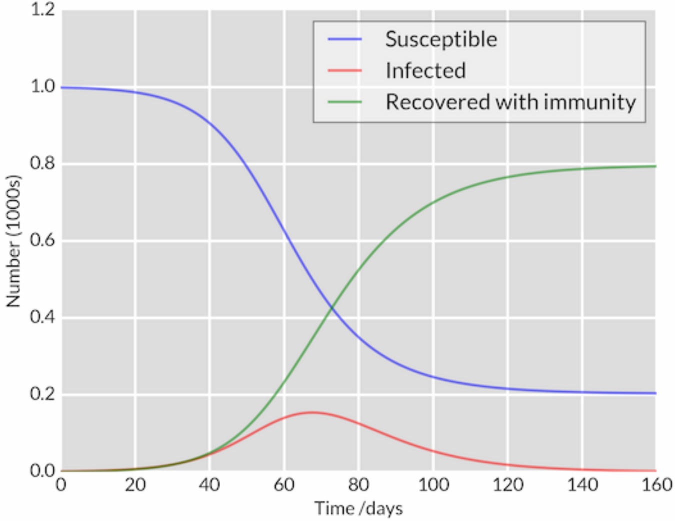



gives this rate equation:

$$\frac{dS}{dt} = -r_1 \, SI$$

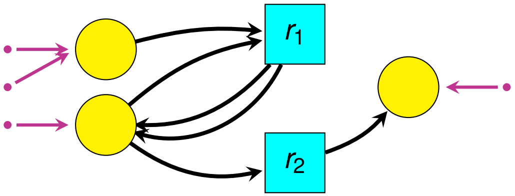
$$\frac{dI}{dt} = r_1 \, SI - r_2 \, I$$

$$\frac{dR}{dt} = r_2 \, I$$



Susceptible
Infected
Recovered with immunity
Number (1000s)
Time /days
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100
120
140
160

We can also define an *open* Petri net with rates:



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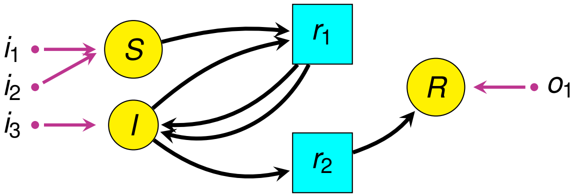



These give rise to an *open* rate equation:

$$\frac{dS}{dt} = -r_1\, SI + i_1 + i_2$$

$$\frac{dI}{dt} = r_1\, SI - r_2\, I + i_3$$

$$\frac{dR}{dt} = r_2\, I - o_1$$

There is a category Open($\mathrm{Petri}_r$) where objects are finite sets and morphisms are open Petri nets.

There is a functor

$$\blacksquare \colon \mathrm{Open}(\mathrm{Petri}_r) \to \mathrm{Dynam}$$

sending each open Petri net to its rate equation, which is treated as a morphism in a category Dynam. Since

$$\blacksquare(PQ) = \blacksquare(P)\,\blacksquare(Q)$$

the process of extracting the rate equation from an open Petri net is ‘compositional’.

How do we build the category $\mathrm{Open}(\mathrm{Petri}_r)$, with open Petri nets as morphisms?

Using the theory of structured cospans!

Given a functor

$$L\colon \mathsf{A} \to \mathsf{X}$$

a **structured cospan** is a diagram

$$L(a) \xrightarrow{\;i\;} x \xleftarrow{\;o\;} L(b)$$

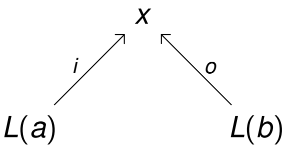
Think of $\mathsf{A}$ as a category of objects with ‘less structure’, and $\mathsf{X}$ as a category of objects with ‘more structure’. $L$ is often a left adjoint.

## Theorem (Kenny Courser, JB)

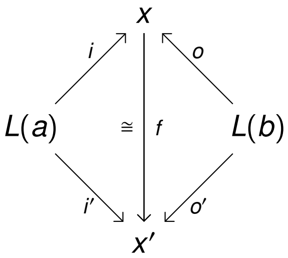
*Let* A *and* X *be categories with finite colimits, and* $L\colon \mathsf{A} \to \mathsf{X}$ *a left adjoint.*

*Then there is a symmetric monoidal category* ${}_L\mathsf{Csp}(\mathsf{X})$ *where:*

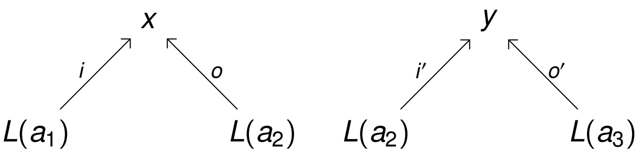
- *an object is an object of* A
- *a morphism is an isomorphism class of structured cospans:*



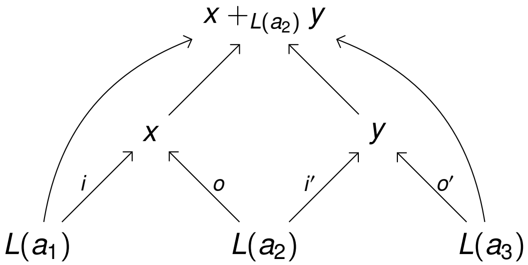
Here two structured cospans are **isomorphic** if there is a commuting diagram of this form:



Given two structured cospans




we compose them by taking a pushout in the category X:



To tensor structured cospans:

$$
\begin{array}{ccc}
 & x & \\
 {}^{i}\nearrow & & \nwarrow^{o} \\
L(a) & & L(b)
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & x' & \\
 {}^{i'}\nearrow & & \nwarrow^{o'} \\
L(a') & & L(b')
\end{array}
$$

we use coproducts in A and X:

$$
\begin{array}{ccccc}
 & & x + x' & & \\
 & {}^{i+i'}\nearrow & & \nwarrow^{o+o'} & \\
 & L(a) + L(a') & & L(b) + L(b') & \\
 {}^{\cong}\nearrow & & & & \nwarrow^{\cong} \\
L(a + a') & & & & L(b + b')
\end{array}
$$

and the fact that $L\colon \mathsf{A} \to \mathsf{X}$ preserves coproducts.

This theorem applies to many examples, giving structured cospan categories whose morphisms are:

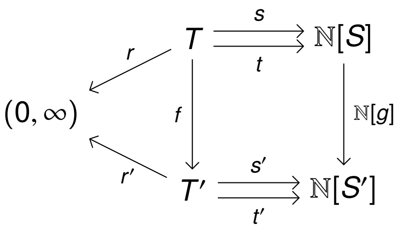
- open electrical circuits
- open Markov processes
- open Petri nets
- open Petri nets with rates

etcetera.

In all these examples A and X have finite colimits and $L \colon \mathsf{A} \to \mathsf{X}$ is a left adjoint, so the theorem applies.

Let's see what it looks like for open Petri nets with rates.

There is a category $\mathrm{Petri}_r$ where objects are Petri nets with rates, and morphisms are diagrams like this:




where the square involving $s$ and $s'$ commutes, as does the square involving $t$ and $t'$.

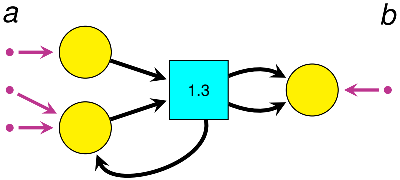
There is a functor $R\colon \mathrm{Petri}_r \to \mathrm{FinSet}$ sending any Petri net with rates to its underlying set of places.

This has a left adjoint $L\colon \mathrm{FinSet} \to \mathrm{Petri}_r$ sending any set to the Petri net with that set of places, and no transitions.

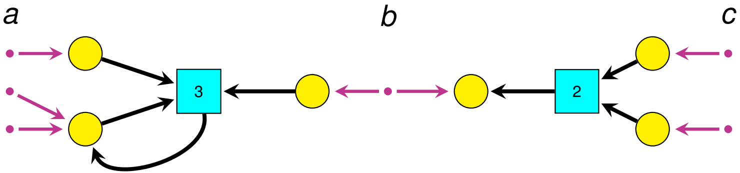
In this example, a structured cospan

$$L(a) \xrightarrow{\;i\;} x \xleftarrow{\;o\;} L(b)$$

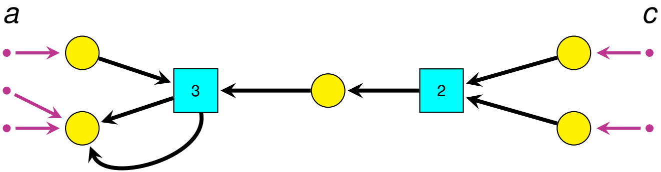
is an **open Petri net with rates**:



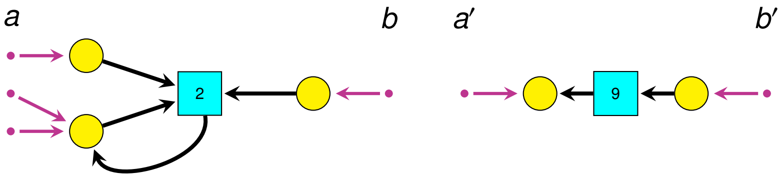
We can compose open Petri nets with rates:



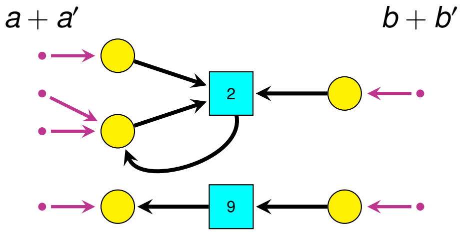

by identifying the outputs of the first with the inputs of the second:



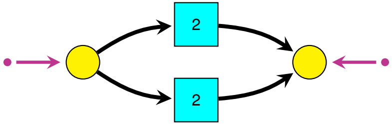
To tensor open Petri nets with rates:




we set them side by side:



What if we want to use actual structured cospans, rather than isomorphism classes? We must do this to point to a *specific* place or transition in an open Petri net:




Then we should use a symmetric monoidal *double* category!

- L. W. Hansen and M. Shulman, Constructing symmetric monoidal bicategories functorially, arXiv:1910.09240.

A double category has figures like this:

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ {\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} \\ C & \xrightarrow[N]{} & D \end{array}$$

So, it has:

- **objects** such as $A, B, C, D$,
- **vertical 1-morphisms** such as $f$ and $g$,
- **horizontal 1-cells** such as $M$ and $N$,
- **2-morphisms** such as $\alpha$.

2-morphisms can be composed vertically and horizontally, and the interchange law holds:

$$
\begin{array}{ccccccc}
A & \xrightarrow{M} & B & \qquad & B & \xrightarrow{M'} & C \\
{\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} & & {\scriptstyle g}\downarrow & \Downarrow \beta & \downarrow{\scriptstyle h} \\
D & \xrightarrow[N]{} & E & & E & \xrightarrow[N']{} & F
\end{array}
$$

$$
\begin{array}{ccccccc}
D & \xrightarrow{N} & E & \qquad & E & \xrightarrow{N'} & F \\
{\scriptstyle f'}\downarrow & \Downarrow \alpha' & \downarrow{\scriptstyle g'} & & {\scriptstyle g'}\downarrow & \Downarrow \beta' & \downarrow{\scriptstyle h'} \\
G & \xrightarrow[O]{} & H & & H & \xrightarrow[P]{} & I
\end{array}
$$

Vertical composition is strictly associative and unital, but horizontal composition is not.

## Theorem (Kenny Courser, JB)

*Let* A *and* X *be categories with finite colimits, and* $L\colon \mathsf{A} \to \mathsf{X}$ *a left adjoint.*

*Then there is a symmetric monoidal double category* ${}_L\mathbb{C}\mathbf{sp}(\mathsf{X})$ *where:*

- *an object is an object of* A
- *a vertical 1-morphism is a morphism of* A
- *a horizontal 1-cell is a structured cospan* $L(a) \xrightarrow{i} x \xleftarrow{o} L(b)$
- *a 2-morphism is a commutative diagram*

$$\begin{array}{ccccc} L(a) & \xrightarrow{i} & x & \xleftarrow{o} & L(b) \\ {\scriptstyle L(f)}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle L(g)} \\ L(a') & \xrightarrow[i']{} & x' & \xleftarrow[o']{} & L(b') \end{array}$$

Horizontal composition is defined using pushouts in X; composing these:

$$
\begin{array}{ccccc}
L(a) & \longrightarrow & x & \longleftarrow & L(b) \\
\downarrow & & \downarrow & & \downarrow \\
L(a') & \longrightarrow & x' & \longleftarrow & L(b')
\end{array}
\qquad\qquad
\begin{array}{ccccc}
L(b) & \longrightarrow & y & \longleftarrow & L(c) \\
\downarrow & & \downarrow & & \downarrow \\
L(b') & \longrightarrow & y' & \longleftarrow & L(c')
\end{array}
$$

gives this:

$$
\begin{array}{ccccc}
L(a) & \longrightarrow & x +_{L(b)} y & \longleftarrow & L(c) \\
\downarrow & & \downarrow & & \downarrow \\
L(a') & \longrightarrow & x' +_{L(b')} y' & \longleftarrow & L(c')
\end{array}
$$

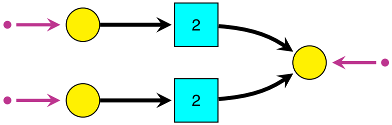
Vertical composition is straightforward.

Tensoring uses binary coproducts in both A and X, and the fact that $L\colon \mathsf{A} \to \mathsf{X}$ preserves these:

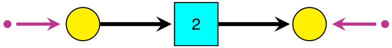
$$\begin{array}{ccccc} L(a_1) & \longrightarrow & x_1 & \longleftarrow & L(b_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a_2) & \longrightarrow & x_2 & \longleftarrow & L(b_2) \end{array} \quad \otimes \quad \begin{array}{ccccc} L(a'_1) & \longrightarrow & x'_1 & \longleftarrow & L(b'_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a'_2) & \longrightarrow & x'_2 & \longleftarrow & L(b'_2) \end{array}$$

$$= \quad \begin{array}{ccccc} L(a_1 + a'_1) & \longrightarrow & x_1 + x'_1 & \longleftarrow & L(b_1 + b'_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a_2 + a'_2) & \longrightarrow & x_2 + x'_2 & \longleftarrow & L(b_2 + b'_2) \end{array}$$

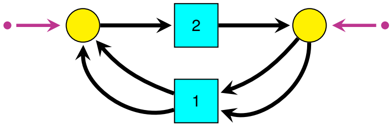
In the case of open Petri nets, a 2-morphism can map this horizontal 1-cell:




to this:




or this:



In summary:

- Symmetric monoidal categories are a good formalism for describing open systems — treating them as morphisms.
- Symmetric monoidal *double* categories are good for describing open systems precisely, not just up to isomorphism. They let us study maps *between* open systems.
- Structured cospans are good for building both of these things.