

Quantifiers as adjoints

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Propositional logic

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$$\text{Prop}(a \vee b, z) \simeq \text{Prop}^2((a, b), (z, z))$$

Conjunction is *right adjoint* to duplication.

$$\frac{(z \Rightarrow a) \wedge (z \Rightarrow b)}{z \Rightarrow (a \wedge b)}$$

$$\text{Prop}^2((z, z), (a, b)) \simeq \text{Prop}(z, a \wedge b)$$

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Existential quantification is left adjoint to weakening.

$$\frac{\exists b. P(a, b) \Rightarrow Q(a)}{P(a, b) \Rightarrow Q^*(a, [b])}$$

$$2^A(\exists_\pi(P), Q) \simeq 2^{A \times B}(P, \pi^*(Q))$$

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Universal quantification is right adjoint to weakening.

$$\frac{Q^*(a, [b]) \Rightarrow P(a, b)}{Q(a) \Rightarrow \forall b. P(a, b)}$$

$$2^{A \times B}(\pi^*(Q), P) \simeq 2^A(Q, \forall_\pi(P))$$

Generalized quantification

We can quantify over *any function*.

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Theorem

Let $f : X \rightarrow Y$ in \mathbf{Set} , with $U \subset X$ and $V \subset Y$. Define:

$$\exists_f(U) = \{y \mid \exists x. f(x) = y \wedge x \in U\}$$

$$\forall_f(U) = \{y \mid \forall x. f(x) = y \Rightarrow x \in U\}$$

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Then $\mathcal{P} := \mathbf{Set}(-, 2) : \mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Pos}$ gives triple adjoints:

$$X \xrightarrow{f} Y \quad \mapsto \quad \mathcal{P}(X) \begin{array}{c} \xrightarrow{\exists_f} \\ \xleftarrow{f^*} \\ \xrightarrow{\forall_f} \end{array} \mathcal{P}(Y)$$

Generalized quantification

Proof.

$$\exists_f(U) \subset V$$

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$$\begin{array}{lcl} \exists_f(U) & \subset & V \\ \forall y. \exists_f(U)(y) & \Rightarrow & V(y) \end{array}$$

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$$\begin{aligned}\exists_f(U) &\subset V \\ \forall y. \exists_f(U)(y) &\Rightarrow V(y) \\ \forall y. \{y \mid \exists x. f(x) = y \wedge x \in U\}(y) &\Rightarrow V(y)\end{aligned}$$

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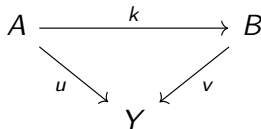
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|-----------|--------|-------------------|
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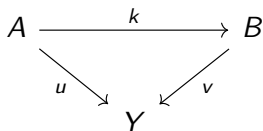
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Generalized elements form more than a preorder - the **slice category** \mathbf{Set}/Y consists of morphisms into Y and commuting triangles:



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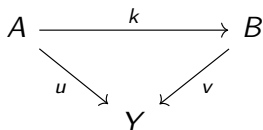
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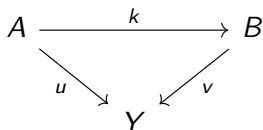


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Generalized elements of Y are Y -typed contexts, and morphisms are changes in context.

Reindexing

As preimage f^* serves to define $\mathcal{P} : \mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Pos}$, more generally *pullback* defines $\mathbf{Set}/- : \mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Cat}$ by an action known as “change of base”:

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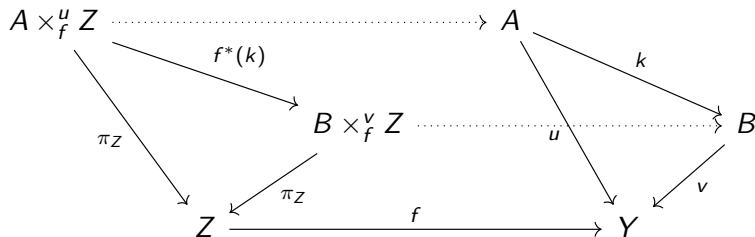
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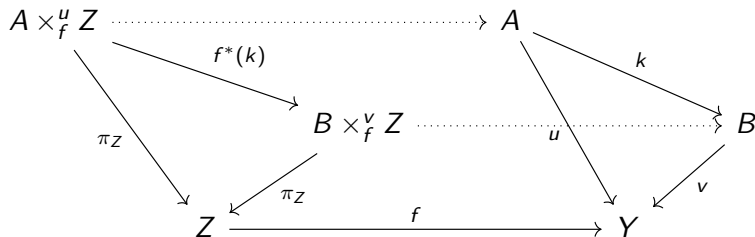
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$$\{A_y := f^*(y) \mid y \in Y\} \mapsto \{A_{f(z)} \mid z \in Z\}$$

$$k : A \rightarrow B \quad \mapsto \quad f^*(k)(a, z) = (k(a), z)$$

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These satisfy a universal property which is the indexed form of co/product.

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All of this reasoning extends well beyond Set.

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\mathcal{C} is locally cartesian closed iff f^* has a right adjoint Π_f .



Saunders Mac Lane and Ieke Moerdijk, *Sheaves in Geometry and Logic*, Springer-Verlag, New York 1992.