Quantifiers as adjoints

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$$\mathrm{Prop}(a \vee b, z) \simeq \mathrm{Prop}^2((a, b), (z, z))$$

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Conjunction is *right adjoint* to duplication.

$$\frac{(z \Rightarrow a) \wedge (z \Rightarrow b)}{z \Rightarrow (a \wedge b)}$$

$$\mathrm{Prop}^2((z, z), (a, b)) \simeq \mathrm{Prop}(z, a \wedge b)$$

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Existential quantification is left adjoint to weakening.

$$\frac{\exists b.P(a,b) \Rightarrow Q(a)}{P(a,b) \Rightarrow Q^*(a,[b])}$$

$$2^A(\exists_\pi(P), Q) \simeq 2^{A\times B}(P, \pi^*(Q))$$

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Universal quantification is right adjoint to weakening.

$$\frac{Q^*(a,[b]) \Rightarrow P(a,b)}{Q(a) \Rightarrow \forall b.P(a,b)}$$

$$2^{A\times B}(\pi^*(Q), P) \simeq 2^A(Q, \forall_\pi(P))$$

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Theorem

Let $f : X \to Y$ *in* Set, *with* $U \subset X$ *and* $V \subset Y$. *Define:*

$$\exists_f(U) = \{y \mid \exists x.\ f(x) = y \wedge x \in U\}$$

$$\forall_f(U) = \{y \mid \forall x.\ f(x) = y \Rightarrow x \in U\}$$

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Then $\mathcal{P} := \mathrm{Set}(-, 2) : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Pos}$ *gives triple adjoints:*

$$X \xrightarrow{f} Y \qquad \mapsto \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, shift left=2, "\exists_f"] \arrow[r, shift right=2, "\forall_f"'] & \mathcal{P}(Y) \arrow[l, "f^*" description] \end{tikzcd}$$

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Proof.

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U &\subset f^*(V)
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$$\begin{array}{lcl} \text{products} & \sim & \text{indexed sums} \\ \text{exponents} & \sim & \text{indexed products.} \end{array}$$

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Generalized elements form more than a preorder - the **slice category** Set/Y consists of morphisms into Y and commuting triangles:

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Generalized elements of Y are Y-typed contexts, and morphisms are changes in context.

Reindexing

As preimage f^* serves to define $\mathcal{P} : \mathrm{Set}^{\mathtt{op}} \to \mathrm{Pos}$, more generally *pullback* defines $\mathrm{Set}/- : \mathrm{Set}^{\mathtt{op}} \to \mathrm{Cat}$ by an action known as "change of base":

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$$\{A_y := f^*(y) \mid y \in Y\} \quad \mapsto \quad \{A_{f(z)} \mid z \in Z\}$$

$$k : A \to B \quad \mapsto \quad f^*(k)(a,z) = (k(a), z)$$

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These satisfy a universal property which is the indexed form of co/product.

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$\mathcal{C}$ is locally cartesian closed iff f^* has a right adjoint Π_f.

References

Saunders Mac Lane and Ieke Moerdijk, *Sheaves in Geometry and Logic*, Springer-Verlag, New York 1992.