How to wellorder finite trees and get good ordinal notations Logic seminar - Berkeley

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Why trees?

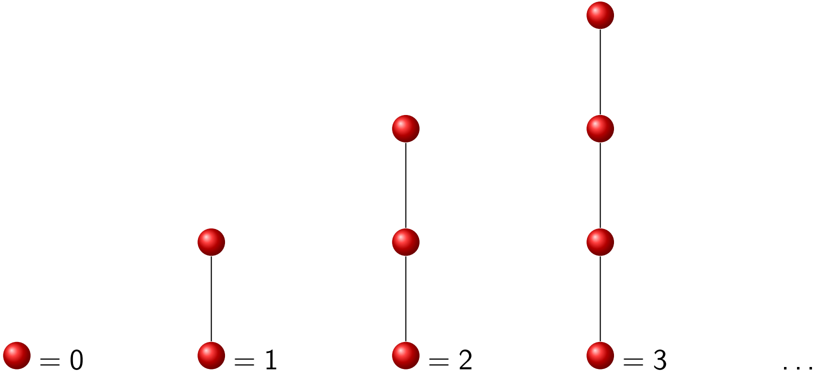
Formulas, proofs, computations, . . . are decorated trees.

Ordinal notations give a way of looking at ordinals as trees.

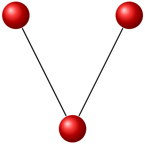
The finite trees

- The trees are finite
- They have root
- The branches are ordered from left to right

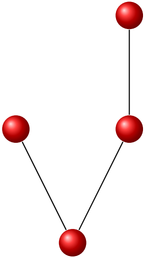
The natural numbers



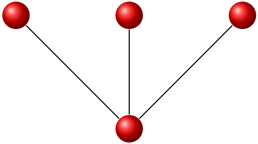
Some infinite ordinals



$= \omega$



$= \omega^{\omega}$



$= \epsilon_0$

Connecting trees and ordinals

Linear ordering of trees – definition

$$\boxed{S < T \Leftrightarrow S \leq \langle T \rangle \vee (\langle S \rangle < T \wedge \langle S \rangle < \langle T \rangle)}$$

- $\langle S \rangle$ — sequence of immediate subtrees
- $S \leq \langle T \rangle$ — there is an element of $\langle T \rangle$ ...
- $\langle S \rangle < T$ — for all elements of $\langle S \rangle$ ...
- $\langle S \rangle < \langle T \rangle$ — lexicographical ordering
  - length of sequences
  - rightmost where they differ

Connecting trees and ordinals

Linear orderings of trees – properties

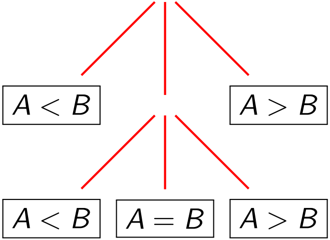
$$\boxed{S < T \Leftrightarrow S \leq \langle T \rangle \vee (\langle S \rangle < T \wedge \langle S \rangle < \langle T \rangle)}$$

$$\boxed{S < T \Leftrightarrow S \leq \langle T \rangle \vee (\neg T \leq \langle S \rangle \wedge \langle S \rangle < \langle T \rangle)}$$

- Linear order
- Equality is the usual equality of trees
- Well order
- 1-1 correspondence with an initial segment of ordinals
- Up to the small Veblen ordinal

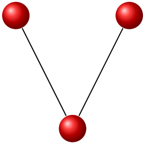
Decision tree

Given two trees A and B we decide the ordering by

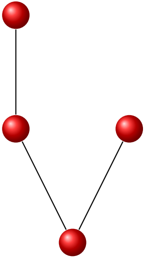


1. Decide whether $A \leq \langle B \rangle$ or $B \leq \langle A \rangle$. If neither go to 2.
2. Decide lexicographical ordering between $\langle A \rangle$ and $\langle B \rangle$.

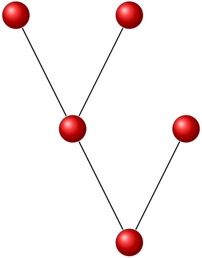
Larger ordinals



$= \omega$

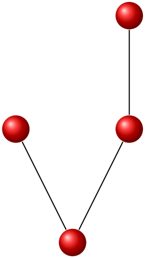


$= \omega \cdot 2$

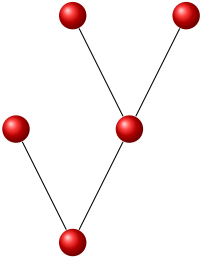


$= \omega^2$

Even larger ordinals

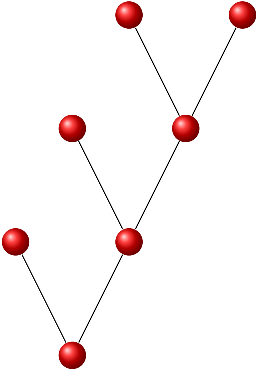


$= \omega^{\omega}$



$= \omega^{(\omega^{\omega})}$

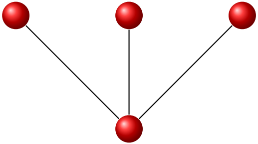
Even larger ordinals



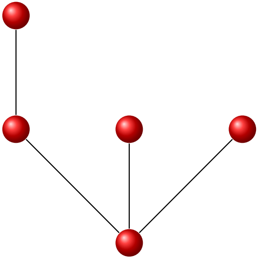
$$= \omega^{(\omega^{(\omega^\omega)})}$$

Much larger ordinals

The ϵ-numbers



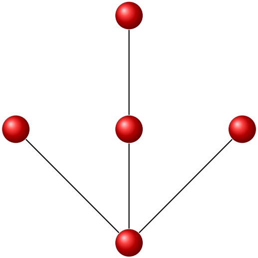
$= \epsilon_0$



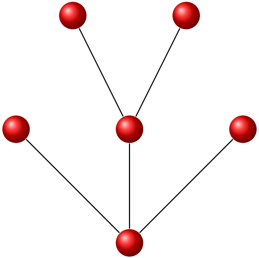
$= \epsilon_1$

Much larger ordinals

The critical ϵ-numbers



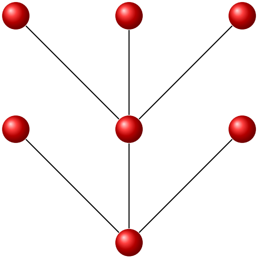
$= \kappa_1$



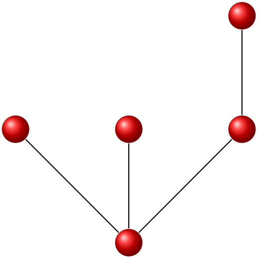
$= \kappa_\omega$

Very much larger ordinals

The critical ϵ-numbers



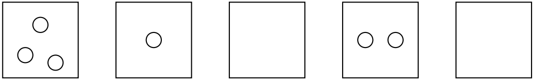
$= \kappa_{\epsilon_0}$



$= \Gamma_0$

Balls in boxes

Finite number of balls in a finite number of boxes.

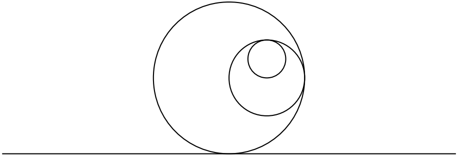


Rule: *Take away 1 ball from a box and replace it with any (finite) number of balls in the boxes to the left.*

The game must terminate. Analyzed using ω^ω.
Delayed decisions.

Primitive recursion

Computation of a primitive recursive function



Similar to the game of balls.
Compile time versus run time.

$$\langle A \rangle < A$$

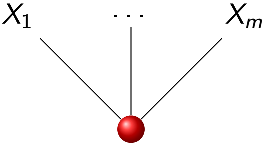
The immediate subtrees of a tree is smaller than the tree.

Approaching a tree from below

Smaller branching

Given a tree A with branching n at the root and $m < n$. Then A is closed under the function

$X_1, \ldots, X_m \mapsto$

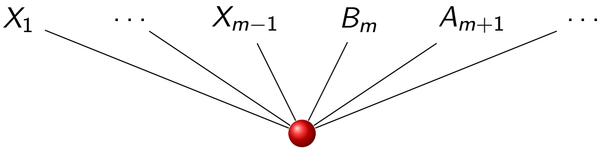


Approaching a tree from below

Lexicographical ordering

Given a tree A with immediate subtrees $A_1, \ldots, A_n$, $m \leq n$, $B_m < A_m$. Then A is closed under the function

$X_1, \ldots, X_{m-1} \mapsto$



Approaching a tree from below

Fundamental set

The fundamental set of A is the sets of all trees we get from

- Immediate subtrees

and closing under

- Smaller branching
- Less lexicographically

Theorem

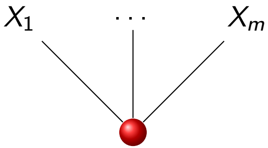
The fundamental set of A is cofinal in A.

Well order

Using bar induction

The function

$X_1, \ldots, X_m \mapsto$



preserve wellfoundedness. This is proved by induction over the height of the tree and the lexicographical ordering of the sequence of wellfoundedness orderings of X_1 up to X_m.

Well order

Using minimal bad sequence

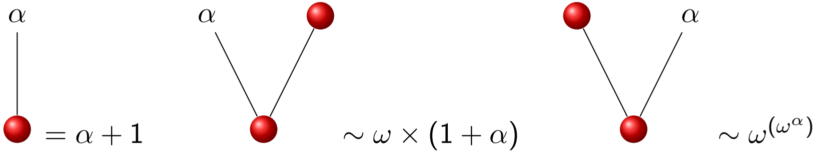
Assume there is a not well founded tree. Then we construct a minimal bad sequence $A_1 > A_2 > A_3 > \cdots$. By minimality we cannot use the condition $\langle A_i \rangle \geq A_{i+1}$ in the ordering. So we must have $\langle A_1 \rangle > \langle A_2 \rangle > \langle A_3 \rangle > \cdots$. But then by standard argument from some element of all sequences must be equally long, and then from some new element one of the elements of the sequences must be descending contradicting that our original sequence where minimal bad.

Well order

Using Kruskals theorem

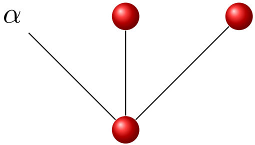
Assume we have an infinite descendig sequence $A_1 > A_2 > A_3 > \cdots$, then by Kruskals theorem there is $i < j$ with A_i homeomorphically embedded in A_j. But then also $A_i \leq A_j$. Contradiction.

Some ordinal functions

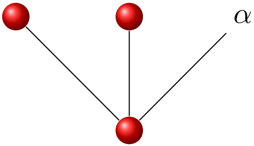


Here $\sim$ indicates that we jump over the fixed points. (The tree functions do not have fixed points.)

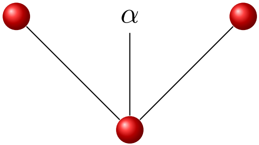
Some more ordinal functions



$\sim \epsilon_\alpha$



$\sim \Gamma_{\alpha-1}$



$\sim \kappa_\alpha$

The finite trees connect up to the Veblen hierarchy — using finite number of arguments and in the enumerations we jump over fix points.

Connecting trees with ordinals

Linear extensions of homeomorphic embeddings

Our ordering is a linear extension of the homeomorhic embedding between finite trees.

Conversely if we consider maximal ordertypes of linear extensions of homeomorphic embedding between finite trees there are essentially two — the one which we consider and its mirror image (taking the lexicographical ordering the other way).

Essentially – we do not have an exact matchup, only one for the important trees.

History

- Oswald Veblen (1880 – 1960) 1908
- Gerhard Gentzen (1909 – 1945) 1936
- Wilhelm Ackermann (1896 – 1962) 1951
- Kurt Schütte (1909 – 1998) 1952
- Gaisi Takeuti (1926 –) around 1960

Going further

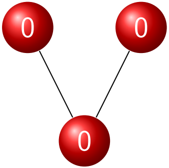
- Go up to the small Veblen ordinal
- Can reach the large Veblen ordinal by using infinite branching with finite support
- Can analyze formal theories of inductive definitions where we do not require the solution to be minimal
- Do not reach the usual inductive definitions where we have minimal solutions
- We need finite labeled trees

Labeled trees

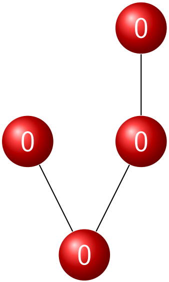
- We have given a wellordered set Λ of labels used for labeling all nodes
- The least labels are 0, 1, 2, ...
- Our old trees correspond to trees labeled with only 0
- Slightly more complicated definition of the new ordering
- The new ordering extends the old one

Some old trees

$= 0$

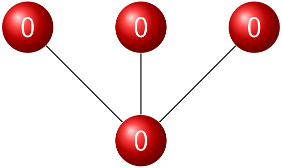


$= \omega$

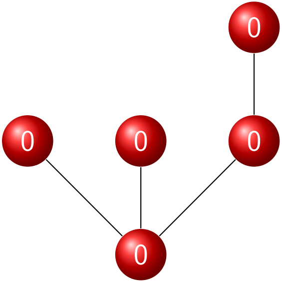


$= \omega^{\omega}$

More old trees



$= \epsilon_0$



$= \Gamma_0$

Some new trees

1 $=$ *Small Veblen ordinal*

2 $=$ *Howard ordinal*

and then the theory goes on and on . . .

Last slide

FIN