10 KINDS OF MATTER

In 1996, Altland and Zirnbauer classified phases of matter into 10 kinds based on charge conjugation ($C$) and time reversal ($T$) symmetry. They can have an antiunitary $C$ symmetry operator with $C^2 = \pm 1$ or lack this symmetry. They can have an antiunitary $T$ operator with $C^2 = \pm 1$ or lack this. This gives $3 \times 3 = 9$ kinds. They can also lack $C$ and $T$ symmetry but still have a unitary operator corresponding to $CT$; by adjusting the phase we may assume $(CT)^2 = 1$.

Kitaev, Moore and Freed showed these 10 kinds of matter correspond to the 10 division superalgebras. There are 3 (associative) division algebras: the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, and quaternions $\mathbb{H}$. A ‘division superalgebra’ is a $\mathbb{Z}_2$-graded generalization, with a bosonic and fermionic part. 8 of these have the same categories of graded representations as the real Clifford algebras called $\text{Cl}_0, \ldots, \text{Cl}_7$ in the chart. The other 2 have the same categories of graded representations as the complex Clifford algebras $\text{Cl}_0$ and $\text{Cl}_1$.

The chart shows how the 10 kinds of matter correspond to the 10 division superalgebras. For example, consider matter with only $T$ symmetry obeying $T^2 = -1$. Since $T$ is antiunitary we have $iT = -T i$. Thus $i, T$ and $iT$ are 3 anticommuting square roots of one, giving a representation of the quaternions $\mathbb{H}$. So, we write $\text{Cl}_4 \simeq \mathbb{H}$. An algebra can sometimes be made into a division superalgebra in more than one way. For example, $\text{Cl}_4 \simeq \mathbb{H}$ where $i, j, k \in \mathbb{H}$ are even, while $\text{Cl}_2 \simeq \mathbb{H}$ where $i, j \in \mathbb{H}$ are odd.

When we combine two physical systems we tensor their Hilbert spaces, and tensor the algebras these Hilbert spaces are representations of. In fact:

$$
\text{Cl}_0 \otimes \text{Cl}_0' \simeq \text{Cl}_{0+0'} \mod 8 \\
\text{Cl}_1 \otimes \mathbb{C} \otimes \text{Cl}_{j+j'} \simeq \text{Cl}_{j+j'} \mod 2 \\
\text{Cl}_i \otimes \mathbb{C} \otimes \text{Cl}_j \simeq \text{Cl}_{i+j} \mod 2
$$

where $\simeq$ means they have equivalent categories of graded representations. This gives the 10-element set $\mathbb{Z}_8 \cup \mathbb{Z}_2$ an associative product and multiplicative identity, but not inverses — so we say it is a ‘monoid’ but not a group.

We have proved that for any field $k$, the associative division superalgebras over $k$ form a monoid, the ‘super Brauer monoid’ of $k$. When $k = \mathbb{R}$ this gives the monoid $\mathbb{Z}_8 \cup \mathbb{Z}_2$, and the rules for combining the 10 kinds of matter.