What n-Categories Should Be Like

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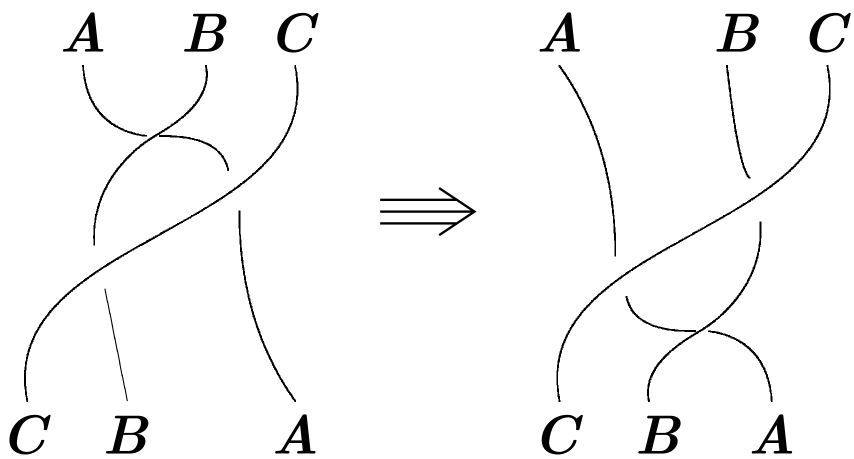
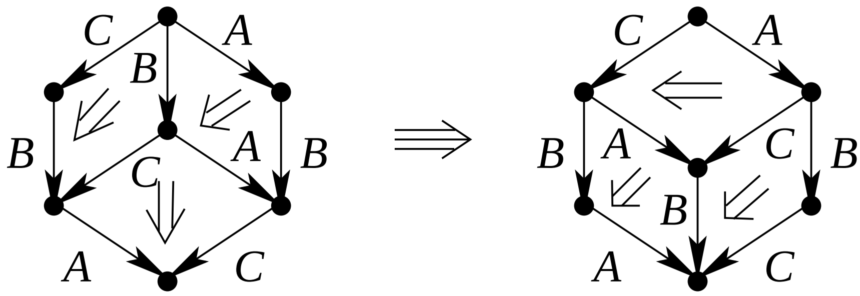
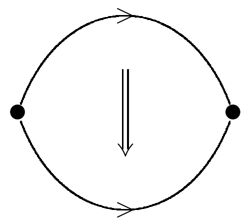




many figures by Aaron Lauda

Many theorems about categories should generalize to n-categories. Let's focus on how n-categories are *different*. I'll try to state results in a formalism-independent way, but sometimes I'll phrase things in 'globular' terms:



There should be an $(n+1)$-category $\boldsymbol{n}\mathbf{Cat}$ whose:

- objects are n-categories,
- 1-morphisms are functors between these,
- 2-morphisms are natural transformations between these,
- 3-morphisms are modifications between these, ...

etc!

3 Ways to Make n-Categories Nicer

1. An **n-groupoid** is an n-category where all morphisms are equivalences (have weak inverses).
2. A **k-tuply monoidal n-category** is an $(n+k)$-category that is trivial below dimension k.
3. A **strict** n-category is one where all laws hold 'on the nose', as equations. (Formalism-dependent? Let's consider globular strict skeletal n-categories.)

Let's consider the effect of these assumptions separately and in combination!

There's more to say about the first two...

There should be an $(n{+}1)$-category $\boldsymbol{n}$**Typ** whose:

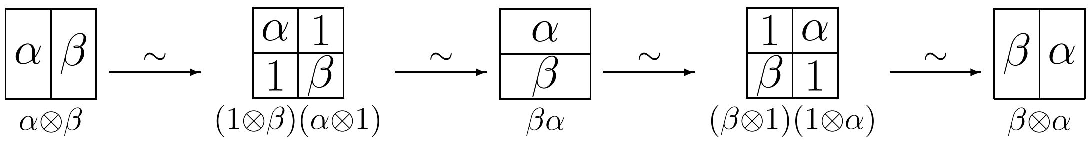
- objects are **homotopy** n**-types**: nice spaces (say CW complexes) with vanishing homotopy groups above the nth,
- 1-morphisms are continuous maps,
- 2-morphisms are homotopies between continuous maps,
- *etc...*
- $(n{+}1)$-morphisms are *homotopy classes* of homotopies between homotopies between ... continuous maps.

The Homotopy Hypothesis: If $\boldsymbol{n}$**Gpd** is the full sub-$(n+1)$-category of nCat whose objects are n-groupoids, there is an equivalence

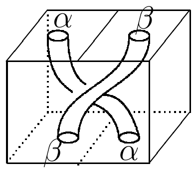
$$\Pi_n \colon n\mathrm{Typ} \to n\mathrm{Gpd}.$$

k-tuply Monoidal n-Categories

A k-tuply monoidal n-category has k ways to multiply objects, satisfying interchange laws up to equivalence. Increasing k increases the 'abelianness', by the Eckmann–Hilton argument. For example, when $k = 2$:



gives us a braiding:



$(n+k)$Cat should have a full $(n+k+1)$-subcategory $\boldsymbol{n}\mathbf{Cat}_{\boldsymbol{k}}$ whose objects are k-tuply monoidal n-categories.

THE PERIODIC TABLE

k-tuply monoidal n-categories

	$n = \mathbf{0}$	$n = \mathbf{1}$	$n = \mathbf{2}$
$k = \mathbf{0}$	**sets**	**categories**	**2-categories**
$k = \mathbf{1}$	**monoids**	**monoidal categories**	**monoidal 2-categories**
$k = \mathbf{2}$	**commutative monoids**	**braided monoidal categories**	**braided monoidal 2-categories**
$k = \mathbf{3}$	‘’	**symmetric monoidal categories**	**sylleptic monoidal 2-categories**
$k = \mathbf{4}$	‘’	‘’	**symmetric monoidal 2-categories**
$k = \mathbf{5}$	‘’	‘’	‘’
$k = \mathbf{6}$	‘’	‘’	‘’

Playing Hopscotch on the Periodic Table

There are many ways to hop around the periodic table. For example, every $(n-1)$-category is an n-category with only identity morphisms, giving **discrete categorification**:

$$(n-1)\mathrm{Cat}_k \xrightarrow{\mathrm{Disc}} n\mathrm{Cat}_k$$

Conversely, we can **decategorify** by discarding n-morphisms and taking isomorphism classes of $(n-1)$-morphisms:

$$(n-1)\mathrm{Cat}_k \xleftarrow{\mathrm{Decat}} n\mathrm{Cat}_k$$

These aren't adjoints, but

$$\mathrm{Decat}(\mathrm{Disc}(C)) \simeq C$$

For $C \in n\mathrm{Gpd} \simeq n\mathrm{Typ}$, forming $\mathrm{Decat}(C)$ is called 'killing the nth homotopy group' — filling n-dimensional holes.

Another process is **looping**:

$$(n-1)\mathrm{Cat}_{k+1} \xleftarrow{\;\Omega\;} n\mathrm{Cat}_k$$

defined for $k > 0$ by $\Omega(C) = \mathrm{End}(1)$. This should have a left adjoint called **delooping**:

$$(n-1)\mathrm{Cat}_{k+1} \xrightarrow{\;B\;} n\mathrm{Cat}_k$$

An $(n+k)$-groupoid trivial below dimension k is a ***k*-tuply groupal *n*-groupoid**. The corresponding homotopy n-type should be a ***k*-fold loop space**: a space of loops in the space of loops in ... some pointed space. Ω and B are then familiar in homotopy theory.

Another process is **forgetting monoidal structure**:

$$\begin{array}{c} n\mathrm{Cat}_k \\ \uparrow F \\ n\mathrm{Cat}_{k+1} \end{array}$$

which should have a left adjoint, **stabilization**:

$$\begin{array}{c} n\mathrm{Cat}_k \\ S\downarrow \\ n\mathrm{Cat}_{k+1} \end{array}$$

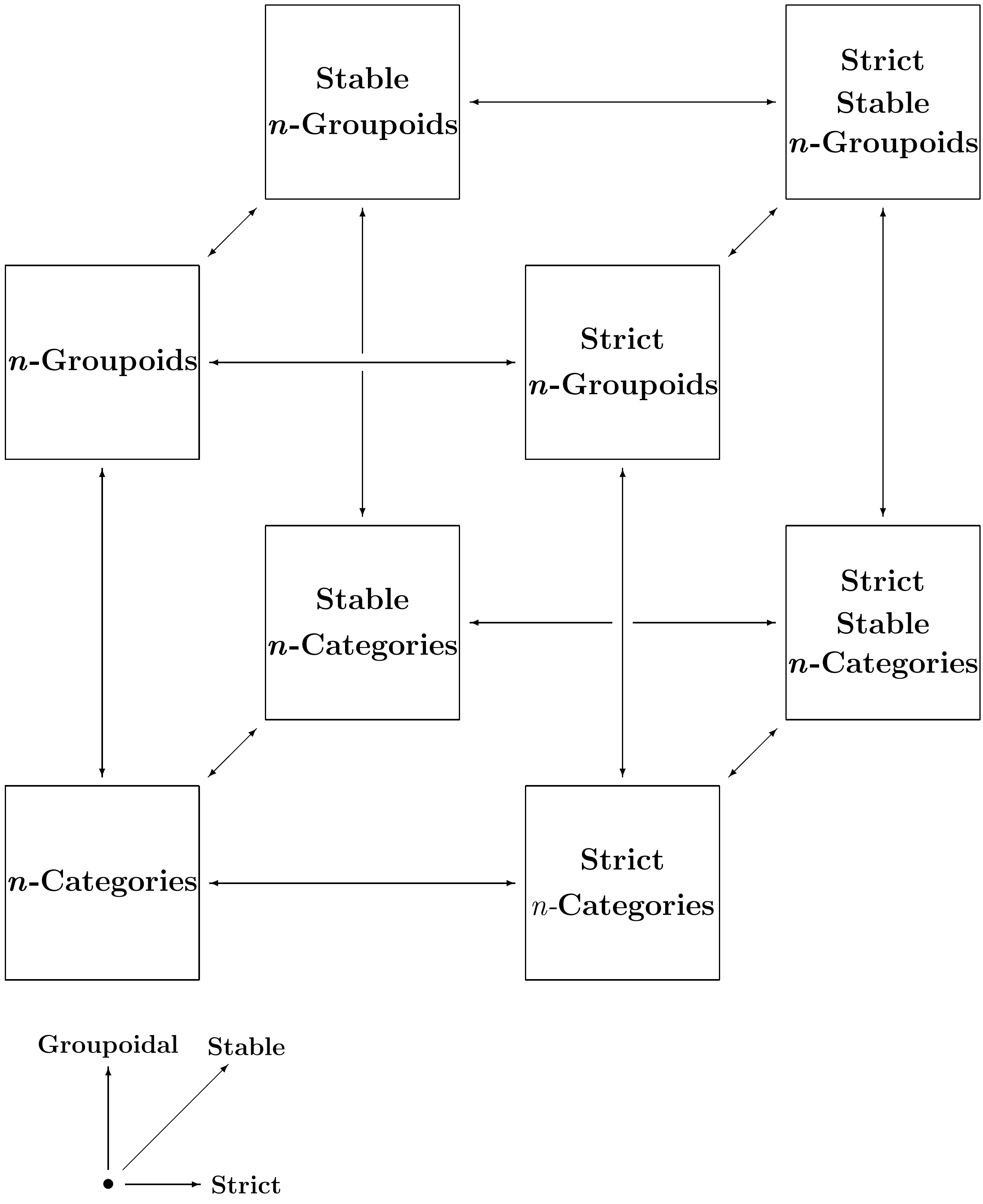
The Stabilization Hypothesis:

$$S\colon n\mathrm{Cat}_k \to n\mathrm{Cat}_{k+1}$$

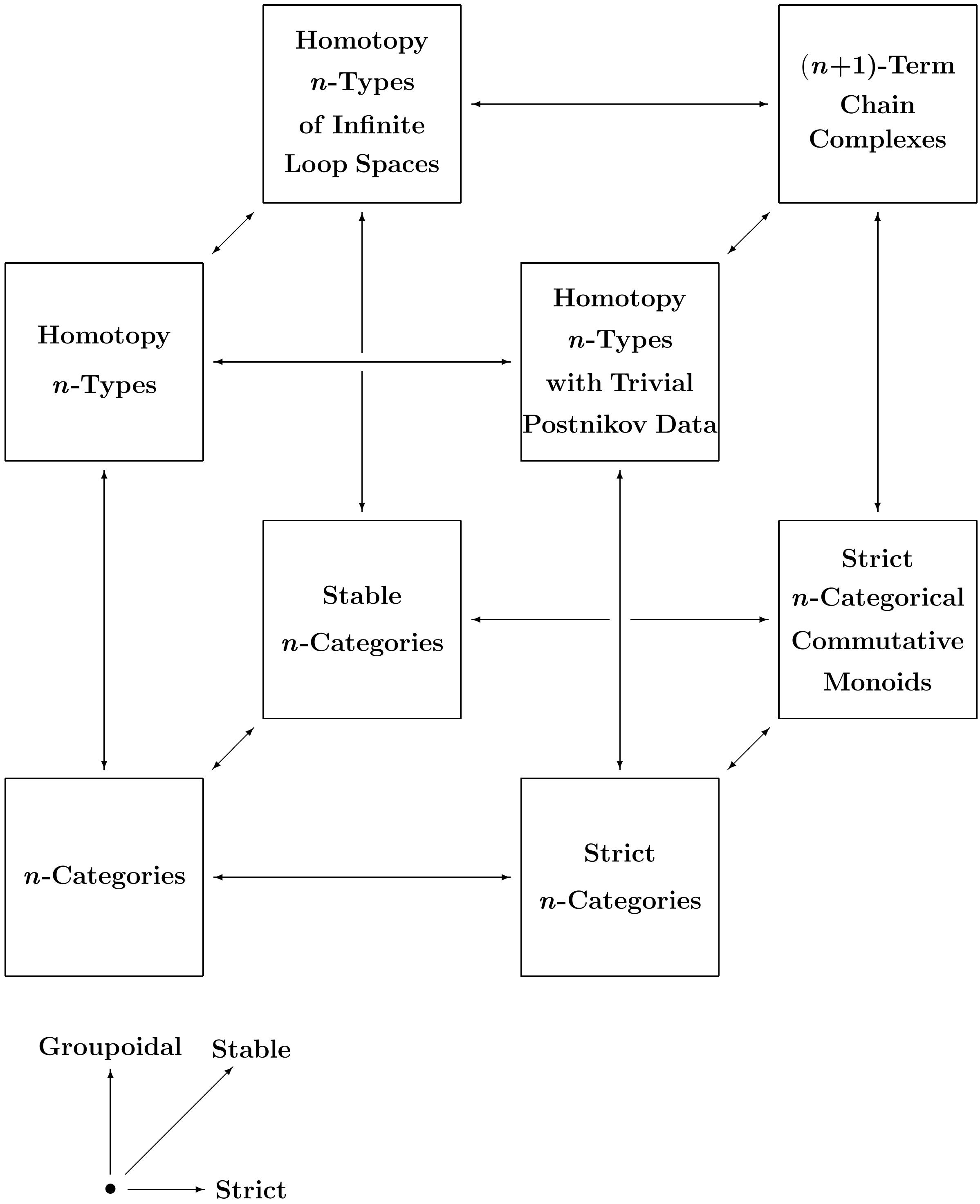
is an equivalence of $(n+k+2)$-categories if $k \geq n+2$.

A **stable *n*-category** is a k-tuply monoidal n-category for any $k \geq n+2$.

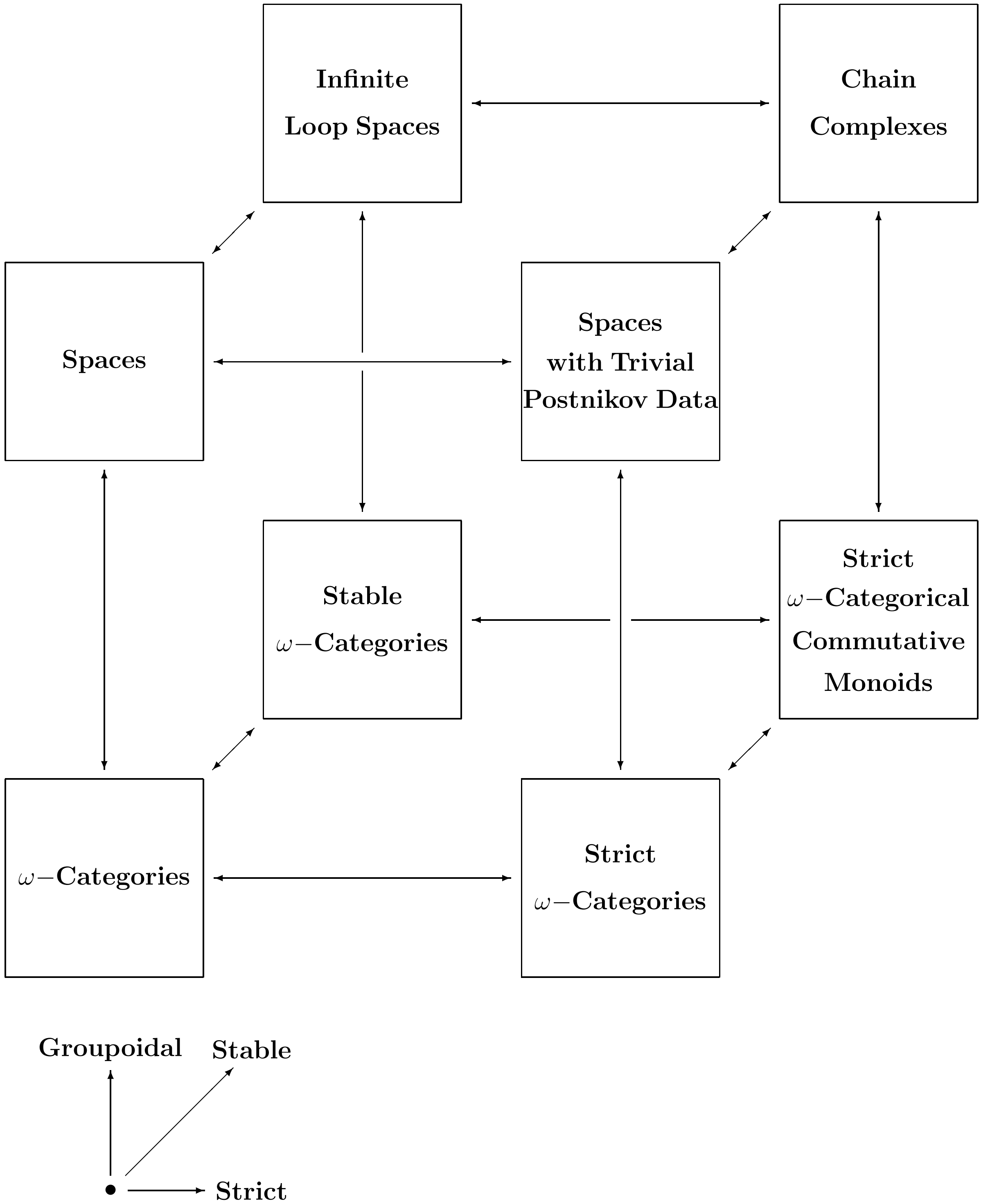
A Weakly Commuting Cube of $(n+1)$-Functors



A Weakly Commuting Cube of $(n+1)$-Functors, Revisited



A Weakly Commuting Cube of $\omega-$Functors



Algebraic Structures and the Free Such Structures on One Generator

sets	$\mathbf{1}$
monoids	$\mathbb{N}$
groups	$\mathbb{Z}$
k**-tuply monoidal** n**-categories**	$n\mathrm{Braid}_k$
k**-tuply monoidal** ω**-categories**	Braid_k
stable ω**-categories**	$\lvert\mathrm{FinSet}_0\rvert$
k**-tuply monoidal** n**-categories with duals**	$n\mathrm{Tang}_k$
stable n**-categories with duals**	$n\mathrm{Cob}$
k**-tuply groupal** n**-groupoids**	$\Pi_{n+k}(S^k)$
k**-tuply groupal** ω**-groupoids**	S^k
stable ω**-groupoids**	$\Omega^\infty S^\infty$
strict k**-tuply groupal** ω**-groupoids**	$K(\mathbb{Z}, k)$