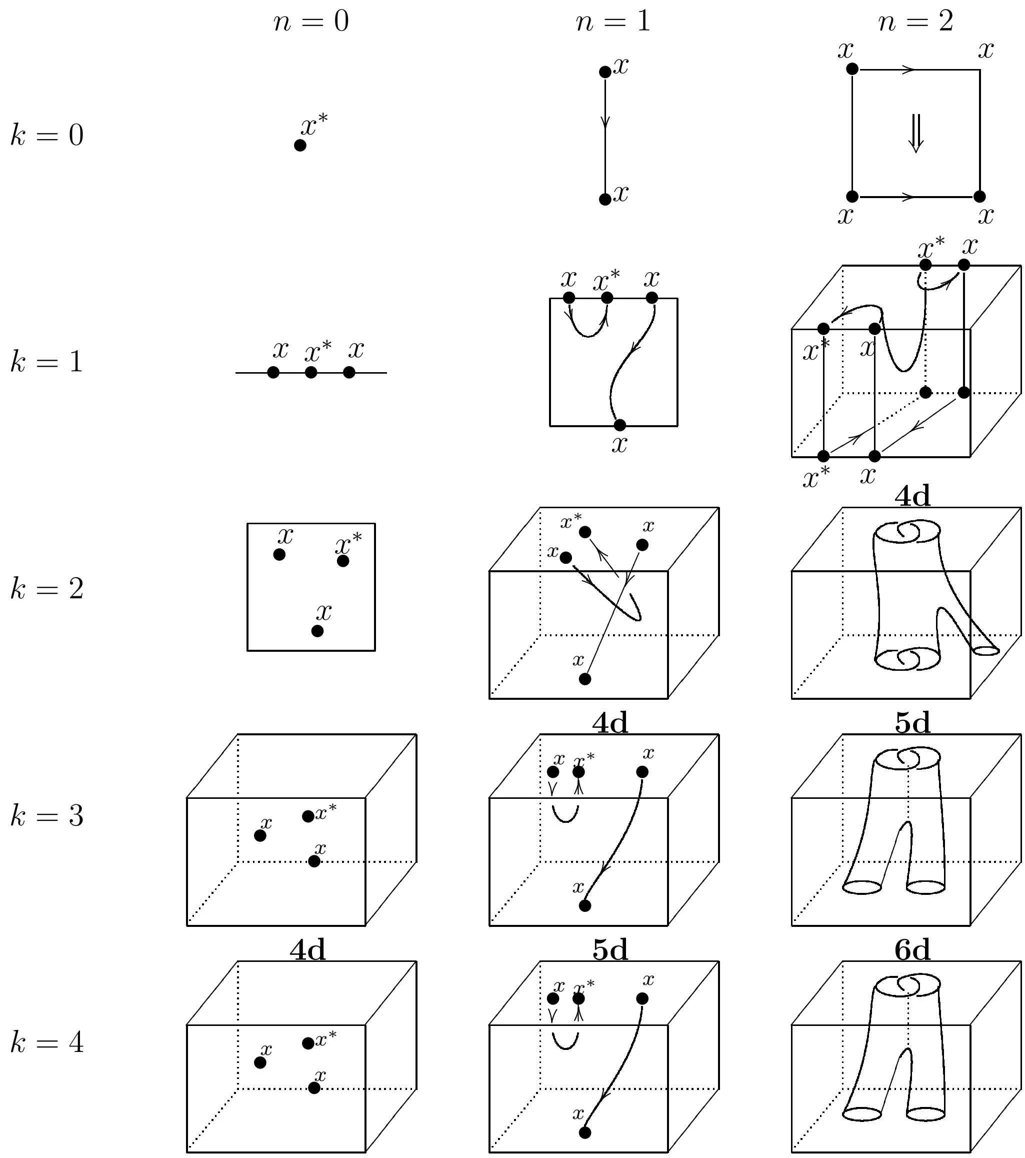
# Why $n$-Categories?

John C. Baez




many figures by Aaron Lauda

## Every Interesting Equation is a Lie!

$x = x$ true, but boring

$x = y$ potentially interesting—
but says two *different*
things are *the same*!

Any interesting equation is really a summary of an interesting *process*. For example:

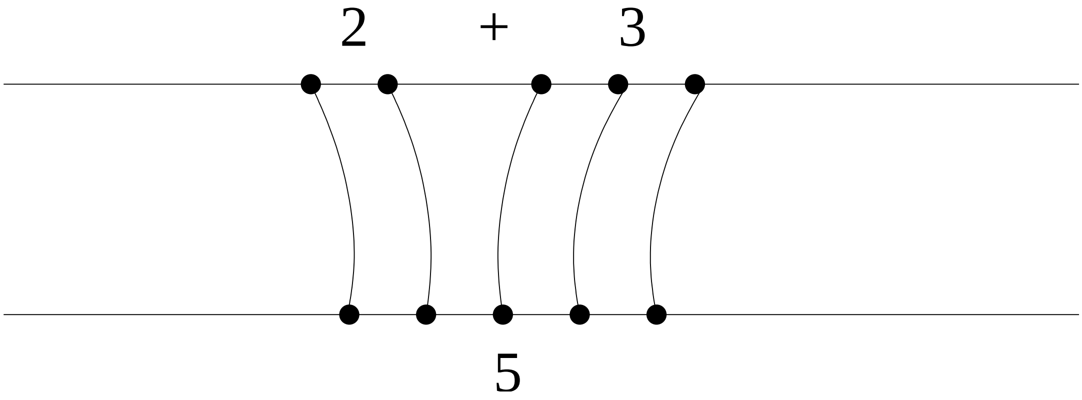
$$\begin{array}{c} 2+3 \\ \| \\ 5 \end{array}$$

is short for:

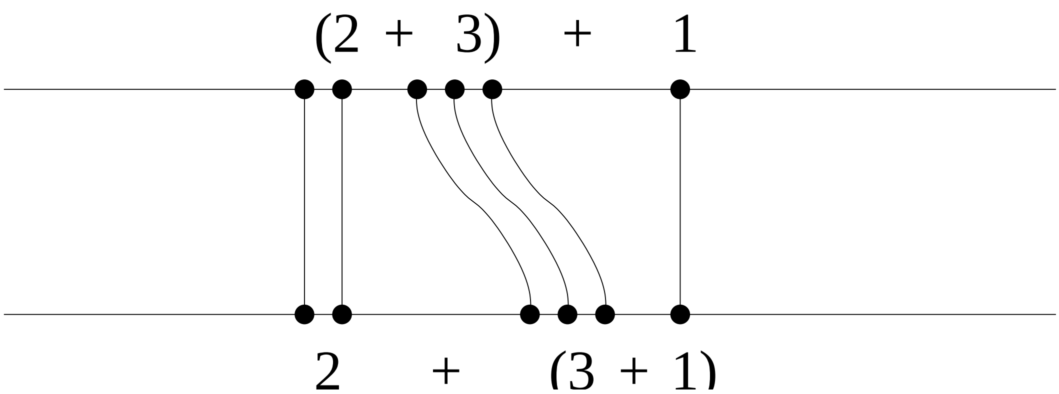
$$\begin{tikzcd}
2 \arrow[dr] \arrow[ddr, "\forall"'] & & 3 \arrow[dl] \arrow[ddl, "\forall"] \\
& 5 \arrow[d, dotted, "\exists!"'] & \\
& X &
\end{tikzcd}$$

## Codimension 1:
## Composition, Associator,...

We add by putting 0-dimensional rocks in a 1-dimensional line:

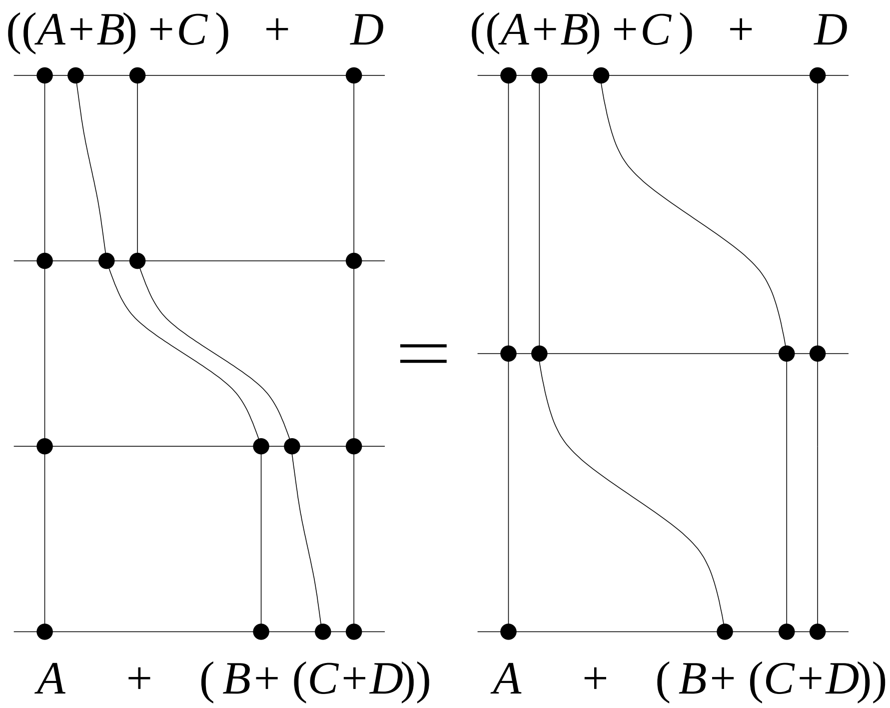



Proving associativity takes time:




We call this proof the **associator**: note 1-dimensional 'worldlines' in 2-dimensional 'spacetime', hence again codimension 2-1 = 1.

The associator satisfies the **pentagon identity**:

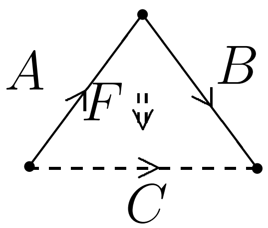



But the process of proving this traces out a 2d surface in 3 dimensions: the **pentagonator**!

And so on....

# Higher Associative Laws: a Simplicial Viewpoint

The hierarchy of ‘higher associative laws’ can also be formalized using *simplices*:




If every way of filling the triangular ‘horn’ factors through $F$, we may call it *a process of composing* $A$ and $B$, and call $C$ *a composite*. This applies to addition of sets:

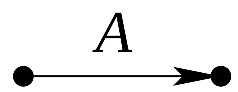
$$\begin{tikzcd}
A \arrow[dr] \arrow[ddr, "\forall"'] & & B \arrow[dl] \arrow[ddl, "\forall"] \\
& C \arrow[d, dotted, "\exists!"'] & \\
& X &
\end{tikzcd}$$

and many other examples, especially *composition of paths in a topological space.*

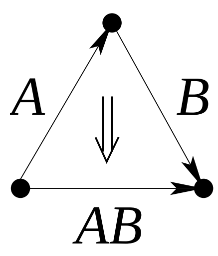
Considering higher-dimensional horns, we get this hierarchy:

Object:

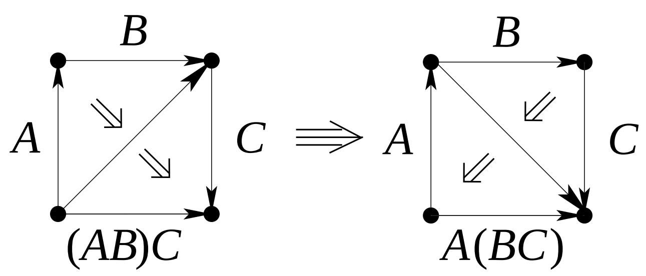
Morphism:



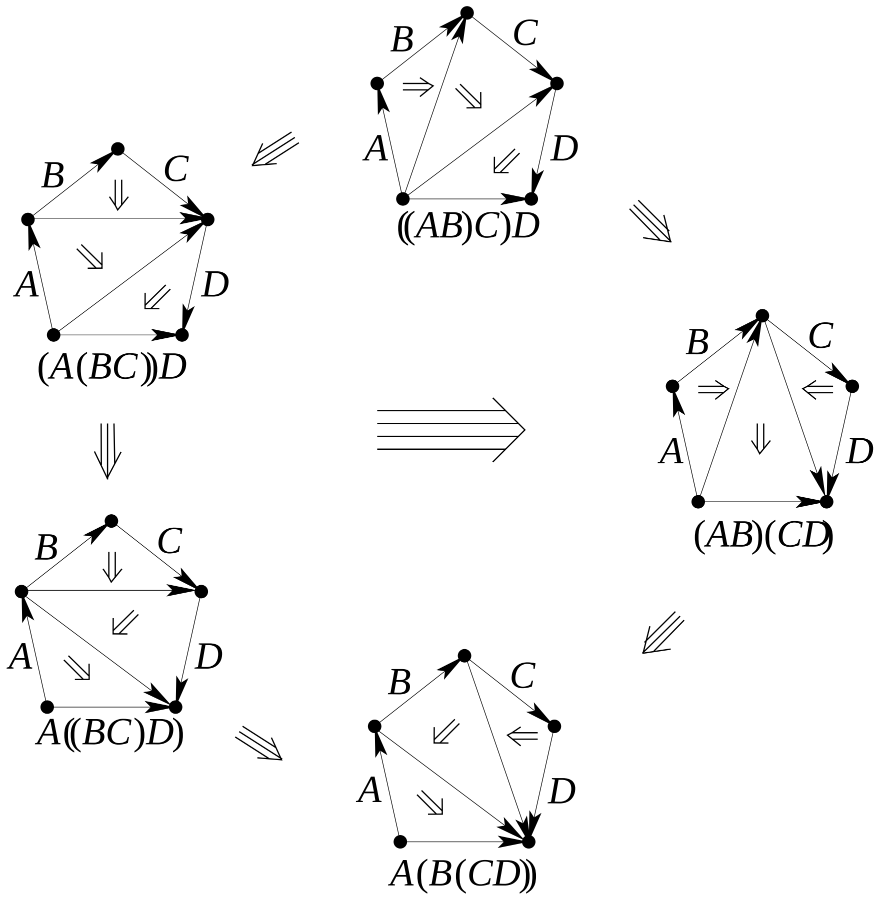

Composition:




Associator:



Pentagonator:




... and so on forever: the Stasheff associahedra!

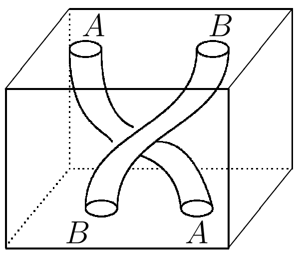
(But there's a subtlety in higher dimensions, appearing already in the next associahedron: the *commutative* law gets involved!)

## Codimension 2: Braiding, Yang–Baxterator,...

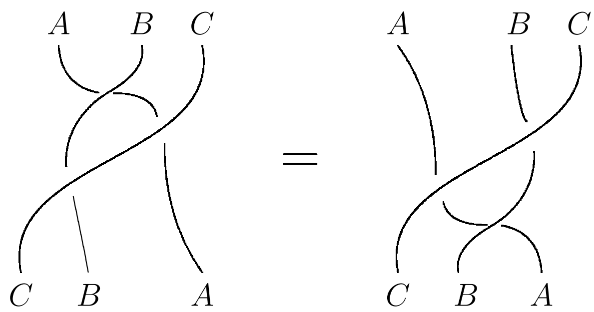
If space is at least 2-dimensional, we can prove the commutative law:

$$A + B = B + A$$

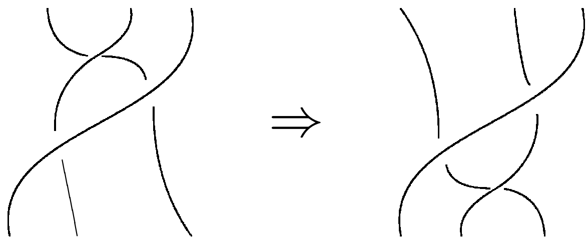
by sliding two piles of rocks around each other. But the proof takes time:



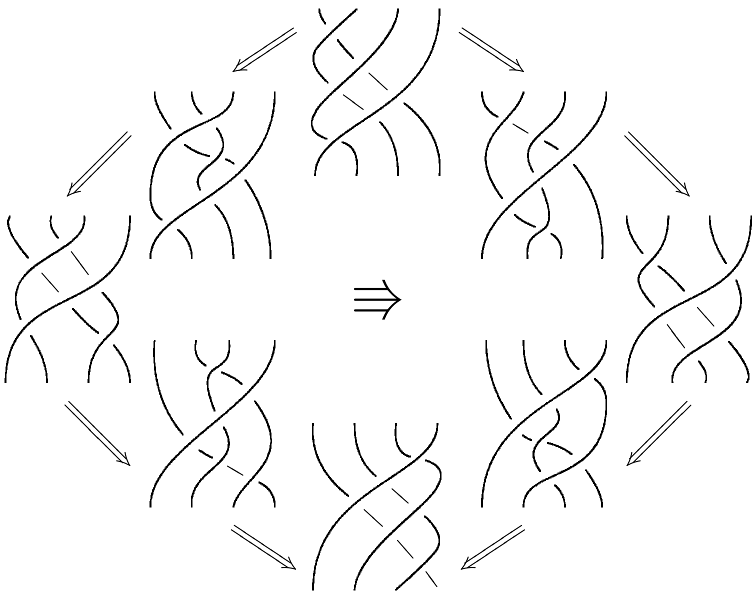

Note 1-dimensional 'worldlines' in 3-dimensional 'spacetime': the **braiding**. The braiding in turn satisfies the **Yang–Baxter equation**:



The process of proving the Yang–Baxter equation traces out a 2d surface in 4 dimensions, the **Yang–Baxterator**:



This in turn satisfies the **Zamolodchikov tetrahedron equation**:

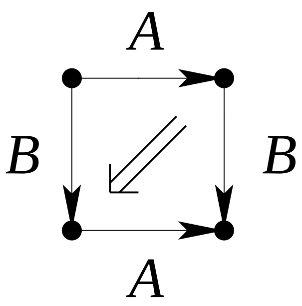


but the proof of this traces out a 3d surface in 5 dimensions... *and so on!*

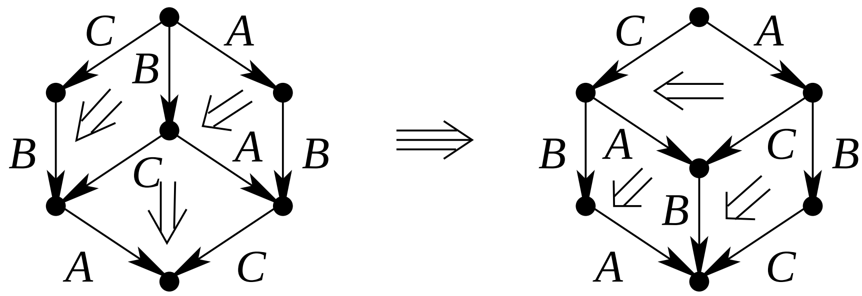
## Higher Commutative Laws: a Cubical Viewpoint

The hierarchy of 'higher commutative laws' can also be formalized using *cubes.*

Braiding:



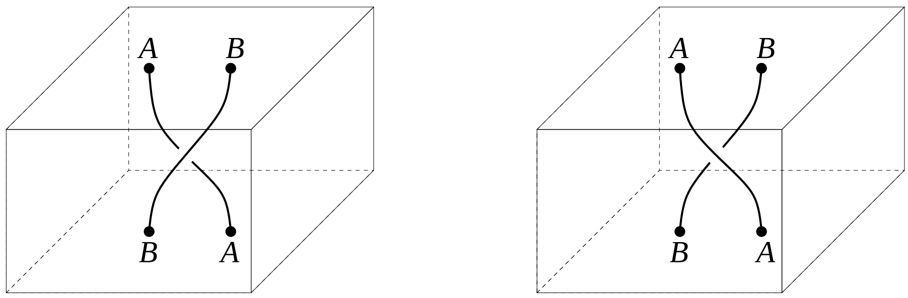

Yang-Baxterator:



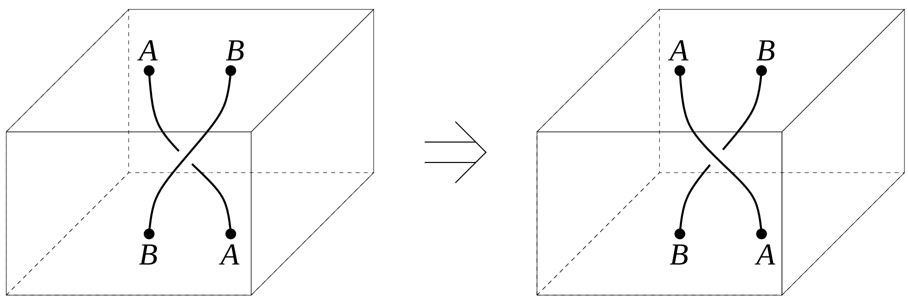

Similarly, the Zamolodchikov tetrahedron equation relates the 'front' and 'back' of a 4-cube, each of which is built from 4 Yang–Baxterator 3-cubes... *and so on!*

# Codimension 3: Syllepsis,...

If space is 2-dimensional, there are two fundamentally different proofs that $A + B = B + A$: the **braiding** versus the **inverse braiding**:




since these are nonisotopic braids in 3d spacetime. But if space is at least 3-dimensional, one proof can be continuously deformed to the other:




since all braids in 4d spacetime are isotopic! This process traces out a 2d surface in 5 dimensions: the **syllepsis**.

The syllepsis satisfies a law of its own... but the proof of this traces out a 3d surface in 6 dimensions... *and so on!*

**And so on for higher codimensions!**

For example, in codimension 4 we get an isotopy between the syllepsis and the 'inverse syllepsis', which are 2d surfaces in 6 dimensions. This isotopy traces out a 3d surface in 7 dimensions, and satisfies a law whose proof traces out a 4d surface in 8 dimensions, etc....

**In short: a hierarchy of 'higher braidings', one for each codimension $k \geq 2$, each satisfying a hierarchy of laws.**

Warning: this is a drastically **simplified** version of the story!

# Why $n$-Categories?

We've seen how beautiful but overwhelmingly complex structures arise when we *treat every equation as a summary of a process.* I've only *begun* to describe these structures! To keep track of them, we need $n$-categories - and not just a definition, but a detailed theory of them.

In particular:

Let a **$k$-tuply monoidal $n$-category** be an $(n+k)$-category that is trivial below dimension $k$ - viewed as an $n$-category with $k$ ways to multiply objects.

Everything I've said so far should be summarized by some theorem relating $k$-tuply monoidal $n$-categories to '$n$-braids in codimension $k$'. A bit more precisely...

**The Braid Hypothesis:** The free $k$-tuply monoidal $n$-category on one object is $\boldsymbol{n}\mathbf{Braid}_{\boldsymbol{k}}$, where:

- objects are finite collections of points in $\mathbb{R}^k$, i.e. elements of

$$X_k = \coprod_{j=0}^{\infty} \frac{\{(x_1, \ldots, x_j)\colon\ x_i \in \mathbb{R}^k,\ x_i \text{ distinct}\}}{S_j}$$

- morphisms are paths in $X_k$,
- 2-morphisms are paths of paths in $X_k$,
- etc...
- $n$-morphisms are *homotopy classes* of paths of paths of paths... in $X_k$.

# THE PERIODIC TABLE

## We expect $k$-tuply monoidal $n$-categories go like this:

| | $n = \mathbf{0}$ | $n = \mathbf{1}$ | $n = \mathbf{2}$ |
|---|---|---|---|
| $k = \mathbf{0}$ | **sets** | **categories** | **2-categories** |
| $k = \mathbf{1}$ | **monoids** | **monoidal categories** | **monoidal 2-categories** |
| $k = \mathbf{2}$ | **commutative monoids** | **braided monoidal categories** | **braided monoidal 2-categories** |
| $k = \mathbf{3}$ | ‘’ | **symmetric monoidal categories** | **sylleptic monoidal 2-categories** |
| $k = \mathbf{4}$ | ‘’ | ‘’ | **symmetric monoidal 2-categories** |
| $k = \mathbf{5}$ | ‘’ | ‘’ | ‘’ |
| $k = \mathbf{6}$ | ‘’ | ‘’ | ‘’ |

- $n$ acts like the *dimension*.
- $k$ acts like the *codimension*.

## The Braid Hypothesis: Examples

The free monoid on one generator is $0\mathrm{Braid}_1$, the **natural numbers**: isotopy classes of collections of points on the line.

The free braided monoidal category on one generator is $1\mathrm{Braid}_2$, the **braid groupoid**: collections of points in the plane and isotopy classes of braids in 3d going between these.

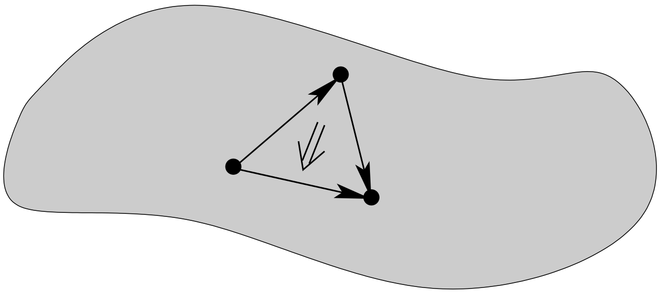
The free symmetric monoidal category on one generator is $1\mathrm{Braid}_3$, the **groupoid of finite sets**: collections of points in $\mathbb{R}^3$ and isotopy classes of braids in 4d going between these. *We live here!*

The free braided monoidal 2-category on one generator is $2\mathrm{Braid}_4$, the **2-braid 2-groupoid**: colllections of points in the plane, braids in 3d between these, and isotopy classes of 2-braids in 4d between these.

All these are just different *views* of a single concept: 'the true natural numbers'.

## How To Understand $n$-Categories

Topology lights the way, since every space $X$ has a 'fundamental $\omega$-groupoid', $\Pi_\infty(X)$. In the simplicial framework:



it's the simplicial set whose $j$-cells are just maps

$$F\colon \Delta^j \to X.$$

Technically this is a **Kan complex**: every horn has a filler!

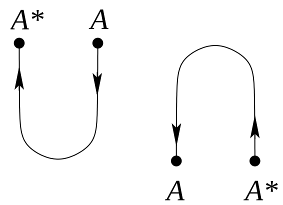
**The Homotopy Hypothesis** (baby version): equivalence classes of $\omega$-groupoids are the same as **homotopy types**: homotopy equivalence classes of locally nice spaces (e.g. CW complexes).

$n\mathrm{Braid}_k$ corresponds to the homotopy type where we take $X_k$, the space of finite collections of points in $\mathbb{R}^k$, and 'kill homotopy groups above $\pi_n$' by attaching cells.

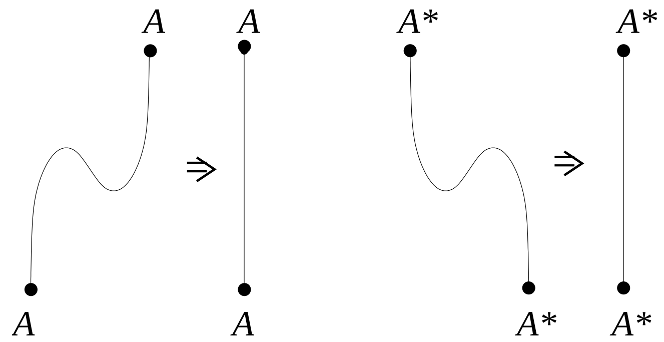
## This is Just the Beginning...

... though not of my talk, you'll be glad to know.

More interesting than $n$-groupoids are '$n$-categories with duals', where all $j$-morphisms have, not weak inverses, but 'duals' or 'adjoints':



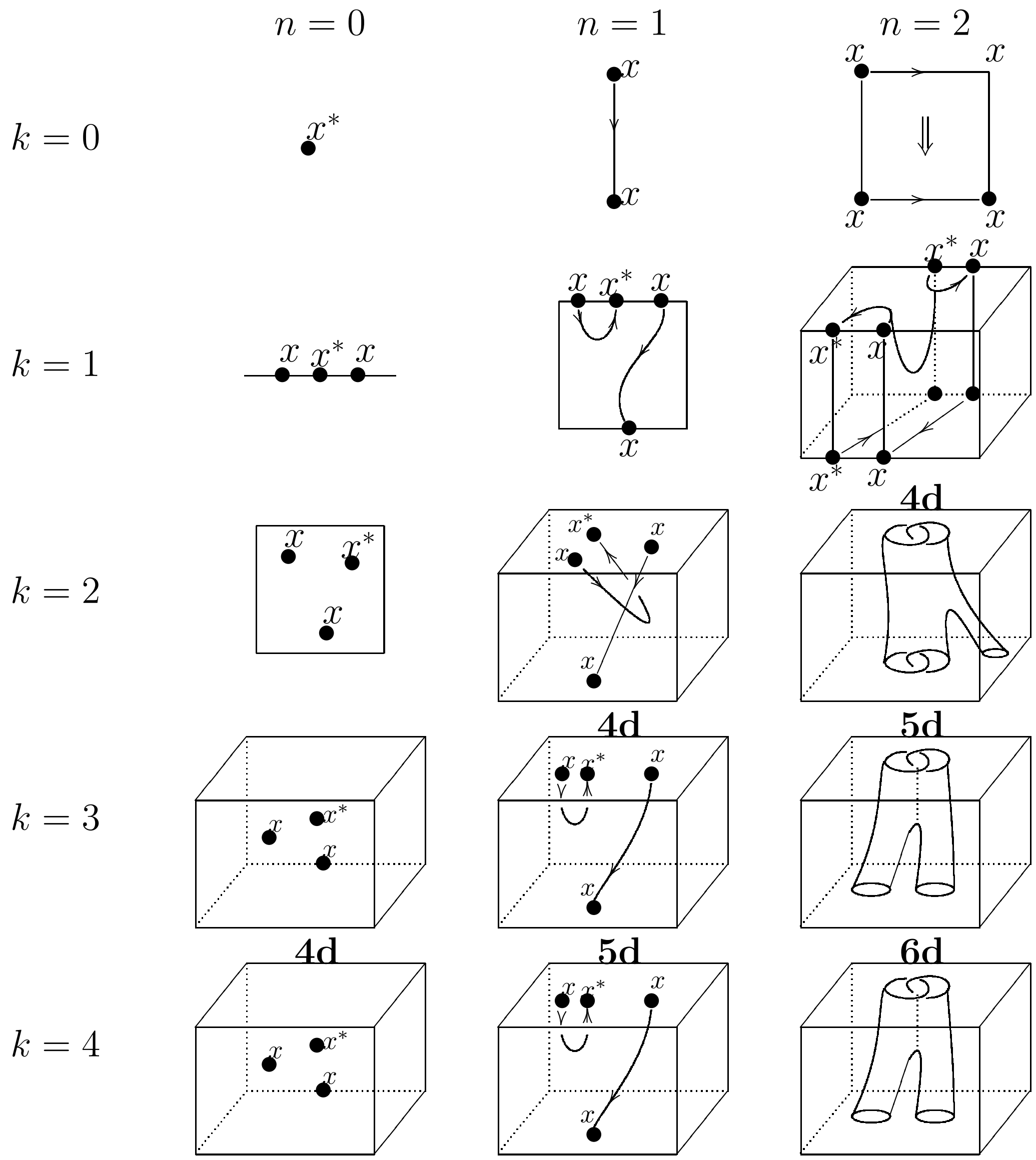

satisfying weakened 'zigzag identities':




which satisfy laws of their own... and so on.

This allows a form of 'subtraction', so it gives us some new views of 'the true integers'....

**The Tangle Hypothesis**: The free $k$-tuply monoidal $n$-category with duals on one generator is $n\mathrm{Tang}_k$: top-dimensional morphisms are $n$-dimensional framed tangles in $n+k$ dimensions.



# Algebraic Structures and the Free Such Structures on One Generator

| | |
|---|---|
| **sets** | $\mathbf{1}$ |
| **monoids** | $\mathbb{N}$ |
| **groups** | $\mathbb{Z}$ |
| $k$**-tuply monoidal** $n$**-categories** | $n\mathrm{Braid}_k \simeq \Pi_{n+k}(X_k)$ |
| $k$**-tuply monoidal** $\omega$**-categories** | $X_k$ |
| $k$**-tuply groupal** $n$**-groupoids** | $\Pi_{n+k}(S^k)$ |
| $k$**-tuply groupal** $\omega$**-groupoids** | $S^k$ |
| **strict** $k$**-tuply groupal** $\omega$**-groupoids** | $K(\mathbb{Z}, k)$ |
| $k$**-tuply monoidal** $n$**-categories with duals** | $n\mathrm{Tang}_k$ |

Thom-Pontryagin map:

$$n\mathrm{Tang}_k \to \Pi_{n+k}(S^k)$$

From homotopy to homology:

$$\Pi_{n+k}(S^k) \to K(\mathbb{Z}, k)$$