Lie 2-Groups, Lie 2-Algebras, and Loop Groups

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Internalization

Often a useful first step in the categorification process involves using a technique developed by Ehresmann called ‘internalization.’

How do we do this?

- Given some concept, express its definition completely in terms of commutative diagrams.
- Now interpret these diagrams in some ambient category $K$.

We will consider the notion of a ‘category in $K$’ for various categories $K$.

A **strict 2-group** is a category in Grp, the category of groups.

A **2-vector space** is a category in Vect, the category of vector spaces.
A 2-vector space, $V$, consists of:

- a vector space of objects, $Ob(V)$

- a vector space of morphisms, $Mor(V)$

Together with:

- linear source and target maps
  \[ s, t: Mor(V) \to Ob(V), \]

- a linear identity-assigning map
  \[ i: Ob(V) \to Mor(V), \]

- a linear composition map
  \[ \circ: Mor(V) \times_{Ob(V)} Mor(V) \to Mor(V) \]
such that the following diagrams commute, expressing the usual category laws:

- laws specifying the source and target of identity morphisms:

\[
\begin{array}{ccc}
\text{Ob}(V) & \xleftarrow{i} & \text{Mor}(V) \\
\downarrow^{1\text{Ob}(V)} & \downarrow^{s} & \\
\text{Ob}(V) & \end{array}
\quad
\begin{array}{ccc}
\text{Ob}(V) & \xleftarrow{i} & \text{Mor}(V) \\
\downarrow^{1\text{Ob}(V)} & \downarrow^{t} & \\
\text{Ob}(V) & \end{array}
\]

- laws specifying the source and target of composite morphisms:

\[
\begin{array}{ccc}
\text{Mor}(V) \times_{\text{Ob}(V)} \text{Mor}(V) & \xrightarrow{\circ} & \text{Mor}(V) \\
p_1 \downarrow & & \downarrow^{s} \\
\text{Mor}(V) & \xrightarrow{s} & \text{Ob}(V) \\
\end{array}
\quad
\begin{array}{ccc}
\text{Mor}(V) \times_{\text{Ob}(V)} \text{Mor}(V) & \xrightarrow{\circ} & \text{Mor}(V) \\
p_2 \downarrow & & \downarrow^{t} \\
\text{Mor}(V) & \xrightarrow{t} & \text{Ob}(V) \\
\end{array}
\]
• the associative law for composition of morphisms:

\[
\begin{array}{c}
\text{Mor}(V) \times_{\text{Ob}(V)} \text{Mor}(V) \times_{\text{Ob}(V)} \text{Mor}(V) \\
\downarrow \uparrow \downarrow \uparrow \\
\text{Mor}(V) \times_{\text{Ob}(V)} \text{Mor}(V) \xrightarrow{\circ} \text{Mor}(V) \\
\end{array}
\]

• the left and right unit laws for composition of morphisms:

\[
\begin{array}{c}
\text{Ob}(V) \times_{\text{Ob}(V)} \text{Mor}(V) \xrightarrow{i \times 1} \text{Mor}(V) \times_{\text{Ob}(V)} \text{Mor}(V) \\
\downarrow \uparrow \downarrow \uparrow \\
\text{Mor}(V) \xrightarrow{\circ} \text{Mor}(V) \\
\end{array}
\]
We can also define **linear functors** between 2-vector spaces, and **linear natural transformations** between these, in the obvious way.

**Theorem.** The 2-category of 2-vector spaces, linear functors and linear natural transformations is equivalent to the 2-category of:

- 2-term chain complexes $C_1 \xrightarrow{d} C_0$,
- chain maps between these,
- chain homotopies between these.
A strict Lie 2-group $G$ is a category in LieGrp, the category of Lie groups.

A strict Lie 2-algebra $L$ is a category in LieAlg, the category of Lie algebras.

We can also define strict homomorphisms between each of these and strict 2-homomorphisms between them, in the obvious way. Thus, we have two strict 2-categories: $\text{SLie2Grp}$ and $\text{SLie2Alg}$.

The picture here is very pretty: Just as Lie groups have Lie algebras, strict Lie 2-groups have strict Lie 2-algebras.

**Proposition.** There exists a unique 2-functor

$$d: \text{SLie2Grp} \rightarrow \text{SLie2Alg}$$
Examples of Strict Lie 2-Groups

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra.

- **Automorphism 2-Group**
  
  Objects : $= \text{Aut}(G)$
  Morphisms : $= G \rtimes \text{Aut}(G)$

- **Shifted $U(1)$**
  
  Objects : $= *$
  Morphisms : $= U(1)$

- **Tangent 2-Group**
  
  Objects : $= G$
  Morphisms : $= \mathfrak{g} \rtimes G \cong TG$

- **Poincaré 2-Group**
  
  Objects : $= SO(n, 1)$
  Morphisms : $= \mathbb{R}^n \rtimes SO(n, 1) \cong ISO(n, 1)$
A semistrict Lie 2-algebra consists of:

- a 2-vector space $L$

equipped with:

- a functor called the bracket:

$$\cdot \cdot : L \times L \to L$$

bilinear and skew-symmetric as a function of objects and morphisms,

- a natural isomorphism called the Jacobiator:

$$J_{x,y,z} : [[x, y], z] \to [x, [y, z]] + [[x, z], y],$$

trilinear and antisymmetric as a function of the objects $x, y, z$,

such that:

- the Jacobiator identity holds, meaning the following diagram commutes:
We can define **homomorphisms** between Lie 2-algebras, and **2-homomorphisms** between these.

Given Lie 2-algebras $L$ and $L'$, a **homomorphism** $F: L \rightarrow L'$ consists of:

- a functor $F$ from the underlying 2-vector space of $L$ to that of $L'$, linear on objects and morphisms,

- a natural isomorphism
  $$F_2(x, y): [F(x), F(y)] \rightarrow F[x, y],$$
bilinear and skew-symmetric as a function of the objects $x, y \in L$,

such that:

- the following diagram commutes for all objects $x, y, z \in L$:

\[
\begin{array}{ccc}
[F(x), [F(y), F(z)]] & \xrightarrow{J_{F(x), F(y), F(z)}} & [[F(x), F(y)], F(z)] + [F(y), [F(x), F(z)]] \\
\downarrow_{[1, F_2]} & & \downarrow_{[F_2, 1]+[1, F_2]} \\
[F(x), F[y, z]] & \xrightarrow{F_2} & [F[x, y], F(z)] + [F(y), F[x, z]] \\
\downarrow_{F_2} & & \downarrow_{F_2+F_2} \\
F[x, [y, z]] & \xrightarrow{F(J_{x, y, z})} & F[[x, y], z] + F[y, [x, z]]
\end{array}
\]
**Theorem.** The 2-category of Lie 2-algebras, homomorphisms and 2-homomorphisms is equivalent to the 2-category of:

- 2-term $L_{\infty}$-algebras,
- $L_{\infty}$-homomorphisms between these,
- $L_{\infty}$-2-homomorphisms between these.

The Lie 2-algebras $L$ and $L'$ are **equivalent** if there are homomorphisms

$$f: L \to L', \quad \bar{f}: L' \to L$$

that are inverses up to 2-isomorphism:

$$f \bar{f} \cong 1, \quad \bar{f} f \cong 1.$$

**Theorem.** Lie 2-algebras are classified up to equivalence by quadruples consisting of:

- a Lie algebra $\mathfrak{g}$,
- an abelian Lie algebra (= vector space) $\mathfrak{h}$,
- a representation $\rho$ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$. 
The Lie 2-Algebra $\mathfrak{g}_k$

Suppose $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{R}$. To get a Lie 2-algebra having $\mathfrak{g}$ as objects we need:

- a vector space $\mathfrak{h}$,
- a representation $\rho$ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

Assume without loss of generality that $\rho$ is irreducible. To get Lie 2-algebras with nontrivial Jacobiator, we need $H^3(\mathfrak{g}, \mathfrak{h}) \neq 0$. By Whitehead’s lemma, this only happens when $\mathfrak{h} = \mathbb{R}$ is the trivial representation. Then we have

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

with a nontrivial 3-cocycle given by:

$$\nu(x, y, z) = \langle [x, y], z \rangle.$$ 

The Lie algebra $\mathfrak{g}$ together with the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$ and $k$ times the above 3-cocycle give the Lie 2-algebra $\mathfrak{g}_k$.

In summary: every simple Lie algebra $\mathfrak{g}$ gives a one-parameter family of Lie 2-algebras, $\mathfrak{g}_k$, which reduces to $\mathfrak{g}$ when $k = 0$!

**Puzzle:** Does $\mathfrak{g}_k$ come from a Lie 2-group?
Coherent 2-Groups

A coherent 2-group is a weak monoidal category in which every morphism is invertible and every object is equipped with an adjoint equivalence.

A homomorphism between coherent 2-groups is a weak monoidal functor. A 2-homomorphism is a monoidal natural transformation. The coherent 2-groups $X$ and $X'$ are equivalent if there are homomorphisms

$$f : X \to X', \quad \bar{f} : X' \to X$$

that are inverses up to 2-isomorphism:

$$f \bar{f} \simeq 1, \quad \bar{f} f \simeq 1.$$

Theorem. Coherent 2-groups are classified up to equivalence by quadruplets consisting of:

- a group $G$,
- an abelian group $H$,
- an action $\alpha$ of $G$ as automorphisms of $H$,
- an element $[a] \in H^3(G, H)$.
Suppose we try to copy the construction of \( g_k \) for a particularly nice kind of Lie group. Let \( G \) be a simply-connected compact simple Lie group whose Lie algebra is \( g \). We have

\[
H^3(G, U(1)) \to \mathbb{Z} \to \mathbb{R} \cong H^3(g, \mathbb{R})
\]

Using the classification of 2-groups, we can build a skeletal 2-group \( G_k \) for \( k \in \mathbb{Z} \):

- \( G \) as its group of objects,
- \( U(1) \) as the group of automorphisms of any object,
- the trivial action of \( G \) on \( U(1) \),
- \( [a] \in H^3(G, U(1)) \) given by \( k \iota[\nu] \), which is nontrivial when \( k \neq 0 \).

**Question:** Can \( G_k \) be made into a Lie 2-group?

Here’s the bad news:

**(Bad News) Theorem.** Unless \( k = 0 \), there is no way to give the 2-group \( G_k \) the structure of a Lie 2-group for which the group \( G \) of objects and the group \( U(1) \) of endomorphisms of any object are given their usual topology.
(Good News) **Theorem.** For any $k \in \mathbb{Z}$, there is a Fréchet Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra $\mathcal{P}_k g$ is equivalent to $g_k$.

An object of $\mathcal{P}_k G$ is a smooth path $f : [0, 2\pi] \to G$ starting at the identity. A morphism from $f_1$ to $f_2$ is an equivalence class of pairs $(D, \alpha)$ consisting of a disk $D$ going from $f_1$ to $f_2$ together with $\alpha \in U(1)$:

For any two such pairs $(D_1, \alpha_1)$ and $(D_2, \alpha_2)$ there is a 3-ball $B$ whose boundary is $D_1 \cup D_2$, and the pairs are equivalent when

$$\exp \left( 2\pi i k \int_B \nu \right) = \frac{\alpha_2}{\alpha_1}$$

where $\nu$ is the left-invariant closed 3-form on $G$ with

$$\nu(x, y, z) = \langle [x, y], z \rangle$$

and $\langle \cdot, \cdot \rangle$ is the smallest invariant inner product on $g$ such that $\nu$ gives an integral cohomology class.
\( \mathcal{P}_k G \) and Loop Groups

We can also describe the 2-group \( \mathcal{P}_k G \) as follows:

- An object of \( \mathcal{P}_k G \) is a smooth path in \( G \) starting at the identity.
- Given objects \( f_1, f_2 \in \mathcal{P}_k G \), a morphism
  \[ \hat{\ell} : f_1 \to f_2 \]
  is an element \( \hat{\ell} \in \Omega_k G \) with
  \[ p(\hat{\ell}) = f_2 / f_1 \]
  where \( \Omega_k G \) is the level-\( k \) Kac–Moody central extension of the loop group \( \Omega G \):

\[
1 \longrightarrow U(1) \longrightarrow \Omega_k G \xrightarrow{p} \Omega G \longrightarrow 1
\]

Remark: \( p(\hat{\ell}) \) is a loop in \( G \). We can get such a loop with
\[ p(\hat{\ell}) = f_2 / f_1 \]
from a disk \( D \) like this:
The Lie 2-Group $P_k G$

Thus, $P_k G$ is described by the following where $p \in P_0 G$ and $\hat{\gamma} \in \widehat{\Omega_k G}$:

- A Fréchet Lie group of objects:
  $$\text{Ob}(P_k G) = P_0 G$$

- A Fréchet Lie group of morphisms:
  $$\text{Mor}(P_k G) = P_0 G \ltimes \widehat{\Omega_k G}$$

- source map: $s(p, \hat{\gamma}) = p$

- target map: $t(p, \hat{\gamma}) = p \partial(\hat{\gamma})$ where $\partial$ is defined as the composite
  $$\widehat{\Omega_k G} \overset{p}{\longrightarrow} \Omega \overset{i}{\hookrightarrow} P_0 G$$

- composition: $(p_1, \hat{\gamma}_1) \circ (p_2, \hat{\gamma}_2) = (p_1, \hat{\gamma}_1 \hat{\gamma}_2)$ when $t(p_1, \hat{\gamma}_1) = s(p_2, \hat{\gamma}_2)$, or $p_2 = p_1 \partial(\hat{\gamma}_1)$

- identities: $i(p) = (p, 1)$
Topology of $\mathcal{P}_kG$

The nerve of any topological 2-group is a simplicial topological group and therefore when we take the geometric realization we obtain a topological group:

**Theorem.** For any $k \in \mathbb{Z}$, the geometric realization of the nerve of $\mathcal{P}_kG$ is a topological group $|\mathcal{P}_kG|$. We have

$$\pi_3(|\mathcal{P}_kG|) \cong \mathbb{Z}/k\mathbb{Z}$$

When $k = \pm 1$, $$|\mathcal{P}_kG| \cong \hat{G},$$

which is the topological group obtained by killing the third homotopy group of $G$.

When $G = \text{Spin}(n)$, $\hat{G}$ is called String($n$). When $k = \pm 1$, $|\mathcal{P}_kG| \cong \hat{G}$. 

The Lie 2-Algebra $\mathcal{P}_k g$

$\mathcal{P}_k G$ is a particularly nice kind of Lie 2-group: a *strict* one! Thus, its Lie 2-algebra is easy to compute. Moreover,

**Theorem.** $\mathcal{P}_k g \simeq g_k$
We know how to get Lie $n$-algebras from Lie algebra cohomology! We should:

- Classify their representations
- Find their corresponding Lie $n$-groups
- Understand their relation to higher braid theory