Higher Categories, Higher Gauge Theory – I

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Unni Namboodiri Lectures
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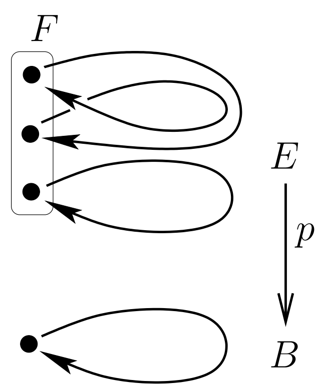
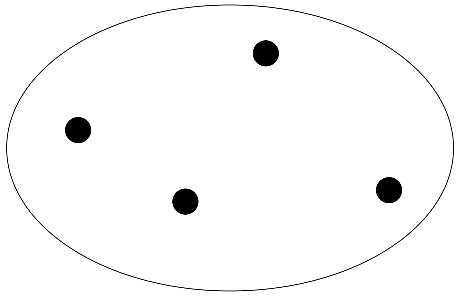


Notes and references at:

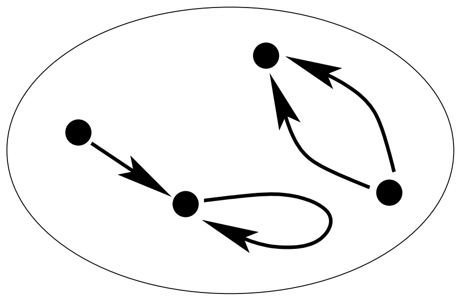
http://math.ucr.edu/home/baez/namboodiri/

The Big Picture

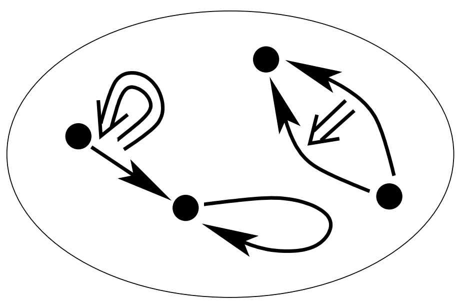
Mac Lane's work marked a revolution in how we think about mathematics. Instead of studying a set of things:



we can now start with a category of things and processes:

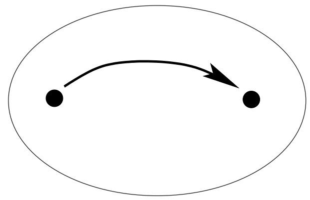


Doing this thoroughly forces us to go further, and study 2-categories of things, processes, and *processes between processes*:

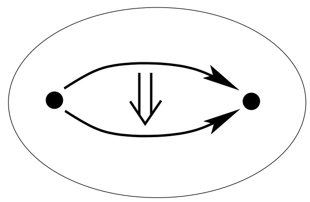


... and so on. This requires a theory of *n-categories*, which many mathematicians are struggling to build now.

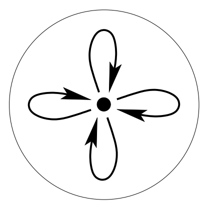
I'll illustrate these ideas with examples from *higher gauge theory*. This describes not only how particles transform as they move along paths in spacetime:



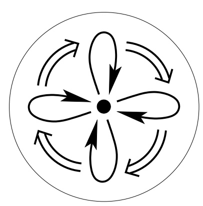
but also how strings transform as they trace out surfaces:



... and so on. Where ordinary gauge theory uses *groups*, which are special categories:



higher gauge theory uses *2-groups*:



which are special 2-categories. Where ordinary gauge theory uses *bundles*, higher gauge theory uses *2-bundles*. Everything gets 'categorified'!

But how did groups, bundles and gauge theory arise in the first place? Let's start at the beginning....

Galois Theory and the Erlangen Program

Around 1832, Galois discovered a basic principle:

> **We can classify the ways a little thing k can sit in a bigger thing K:**
>
> $$k \hookrightarrow K$$
>
> **by keeping track of the symmetries of K that map k to itself. These form a subgroup of the symmetries of K:**
>
> $$\mathrm{Gal}(K|k) \subseteq \mathrm{Aut}(K).$$

Galois applied this principle in a special case, but it's very general. In 1872 Klein announced his 'Erlangen program', which applies the principle to *geometry*.

For example, any projective plane $\mathbb{P}^2$ has a symmetry group $\mathrm{Aut}(\mathbb{P}^2)$ consisting of all transformations that carry lines to lines.

Each point in $\mathbb{P}^2$ is determined by the subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ fixing this point. Each line in $\mathbb{P}^2$ is determined by the subgroup preserving this line. Other subgroups correspond to other figures in the plane!

Algebras vs. Spaces

Galois applied his idea to situations where the 'things' in question were commutative algebras. In the mid-1800s, Dedekind, Kummer and Riemann realized that commutative algebra is like topology, only backwards! Any space X has a commutative algebra $\mathcal{O}(X)$ consisting of functions on it. Any map

$$f\colon X \to Y$$

gives a map

$$f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X).$$

If we're clever we can think of any commutative algebra as functions on some space — or 'scheme':

$$[\text{Affine Schemes}] = [\text{Commutative Rings}]^{\text{op}}.$$

Note how it's backwards: the *inclusion* of commutative rings

$$p^*\colon \mathbb{C}[z] \hookrightarrow \mathbb{C}[\sqrt{z}]$$

corresponds to the *branched cover* of the complex plane by the Riemann surface for $\sqrt{z}$:

$$\begin{array}{rcl} p\colon \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & z^2 \end{array}$$

So: classifying how a little commutative algebra can *sit inside* a big one amounts to classifying how a big space can *cover* a little one! Now the Galois group gets renamed the group of **deck transformations**: in the above example it's $\mathbb{Z}/2$:

$$\sqrt{z} \mapsto -\sqrt{z}.$$

The Fundamental Group

Around 1883, Poincaré discovered that any nice connected space B has a connected covering space that covers all others: its **universal cover**. This has the biggest deck transformation group of all: the **fundamental group** $\pi_1(B)$.

The idea behind Galois theory — turned backwards! — then says that:

Connected covering spaces of B are classified by subgroups $H \subseteq \pi_1(B)$.

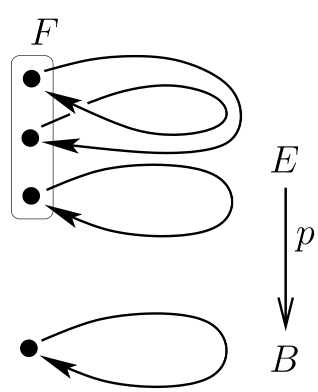
To remove the 'connectedness' assumption, say it like this instead:

Connected covering spaces of B with fiber F are classified by transitive actions of $\pi_1(B)$ on F.

Here $F = \pi_1(B)/H$. Now generalize:

Covering spaces of B with fiber F are classified by actions of $\pi_1(B)$ on F.

Here F is any set:



The Fundamental Groupoid

Having classified covering spaces of a nice *connected* space B, what if B is not connected? For this, replace $\pi_1(B)$ by $\Pi_1(B)$: the **fundamental groupoid** of B. This is the category where:

- objects are points of B: $\bullet x$
- morphisms are homotopy classes of paths in B:

$$x \bullet \xrightarrow{f} \bullet\, y$$

The basic principle of Galois theory then says this:

> **Covering spaces $F \hookrightarrow E \to B$ are classified by actions of $\Pi_1(B)$ on F: that is, functors**
>
> $$\Pi_1(B) \to \mathrm{Aut}(F).$$

Even better, we can let the fiber F be different over different components of the base B:

> **Covering spaces $E \to B$ are classified by functors**
>
> $$\Pi_1(B) \to \mathrm{Set}.$$

Eilenberg–Mac Lane Spaces

In 1945, Eilenberg and Mac Lane wrote their paper about categories *and* a paper showing any group G has a 'best' space with G as its fundamental group: the **Eilenberg-Mac Lane space** $K(G,1)$.

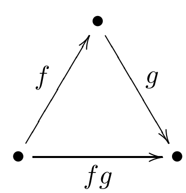
In fact, for any groupoid G we can build a space $K(G,1)$ by taking a vertex for each object of G:

$$\bullet\ x$$

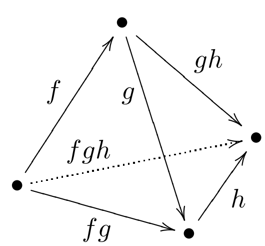
an edge for each morphism of G:

$$\bullet \xrightarrow{f} \bullet$$

a triangle for each composable pair of morphisms:



a tetrahedron for each composable triple:



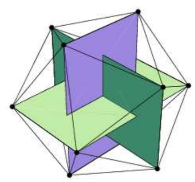
and so on! This space has G as its fundamental groupoid, and it's a **homotopy 1-type**: all its homotopy groups above the 1st vanish. These facts characterize it.

Using this idea, one can show a portion of topology is just groupoid theory:

Homotopy 1-types are the same as groupoids!

Klein's Favorite Example

Aut $\Bigg($  $\Bigg) \quad \cong \quad A_5$

Gal $\Bigg($  $\Big|$  $\Bigg) \quad \cong \quad A_4$

The symmetries of the icosahedron fixing a 'golden cross' form a subgroup $A_4 \hookrightarrow A_5$, so the set of golden crosses is $A_5/A_4 \cong 5$. We get a covering space:

$$\begin{array}{ccccc} F & \hookrightarrow & E & \longrightarrow & B \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ 5 & \hookrightarrow & K(A_4,1) & \longrightarrow & K(A_5,1) \end{array}$$

where

$$\begin{array}{rcl} B & = & K(S_5,1) \simeq \{\text{oriented 5-element subsets of } \mathbb{R}^\infty\} \\ E & = & K(S_4,1) \simeq \{\text{oriented 5-element subsets of } \mathbb{R}^\infty \text{ with chosen point}\} \end{array}$$

The group A_5 acts on the Riemann sphere, $\mathbb{C}\mathcal{P}^1$. The field of rational functions on $\mathbb{C}\mathcal{P}^1$ is $K = \mathbb{C}(z)$. The A_5-invariant rational functions form a subfield $k = \mathbb{C}(f)$, where f is Klein's 'icosahedral function'.

$$\mathrm{Gal}(K|k) \cong A_5,$$

and in his *Lectures on the Icosahedron*, Klein showed how the solution of $w = f(z)$ lets you solve the general quintic!

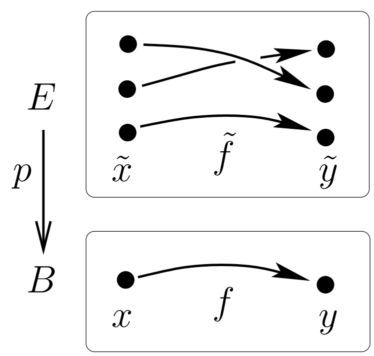
Galois Theory Revisited

Since the classification of covering spaces

$$E \to B$$

only involves the fundamental groupoid of B, we might as well assume B is a homotopy 1-type. Then E will be one too.

So, we might as well say E and B are *groupoids!* The analogue of a covering space for groupoids is a **discrete fibration**: a functor $p\colon E \to B$ such that for any morphism $f\colon x \to y$ in B and object $\tilde{x} \in E$ lifting x, there's a unique morphism $\tilde{f}\colon \tilde{x} \to \tilde{y}$ lifting f:



The basic principle of Galois theory then becomes:

> **Discrete fibrations** $E \to B$ **are classified by functors** $B \to \mathrm{Set}$**.**

This is true even when E and B are categories, though people use the term 'opfibrations'.

This — and much more — goes back to Grothendieck's 1971 book *Étale Coverings and the Fundamental Group* (SGA1).

Grothendieck's Dream

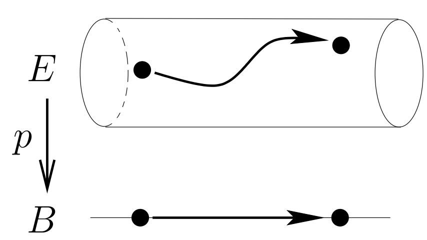
Say a space is a *homotopy n-type* if its homotopy groups above the nth all vanish. Since homotopy 1-types are 'the same' as groupoids, maybe

homotopy n-types are 'the same' as n-groupoids!

Grothendieck tackled this possibility in his 600-page letter to Quillen, *Pursuing Stacks*. It's certainly true if we use Kan's simplicial approach to n-groupoids — but we want it to emerge from a general theory of n-categories.

How does the basic principle of Galois theory generalize to this situation?

So far we have been studying *discrete* fibrations. For these the fiber is a mere *set*, and there is no choice about how to lift a path. The real fun starts when we let the fiber be a more general space:



Now we need a 'connection' to lift paths, leading to *gauge theory*.

Since spaces are like n-groupoids, we therefore expect something like a 'connection' to show up when we study fibrations of n-groupoids!

Nonabelian Cohomology

For n-groupoids, the basic principle of Galois theory should say something like this:

> **Fibrations $E \to B$ where E and B are n-groupoids are classified by n-functors $B \to n\mathrm{Gpd}$.**

Grothendieck proved this for $n = 1$; Hermida proved it for $n = 2$.

Let's see what it happens when $n = 1$. Suppose E, B are simply *groups*, and fix the fiber F, also a group:

> **Short exact sequences of groups**
>
> $$1 \to F \to E \to B \to 1$$
>
> **are classified by weak 2-functors**
>
> $$B \to \mathrm{AUT}(F).$$
>
> **where $\mathrm{AUT}(F)$ is the 'automorphism 2-group' of F.**

This is called **Schreier theory**, since a version of this result goes back to Schreier (1926). The classifications of abelian or central group extensions using Ext or H^2 are just watered-down versions of this!

AUT(F) is the **automorphism 2-group** of F, a 2-category with:

- F as its only object: $\bullet F$
- automorphisms of F as its morphisms:

$$F\bullet \xrightarrow{\ \alpha\ } \bullet\, F$$

- elements $g \in F$ with $g\alpha(f)g^{-1} = \beta(f)$ as its 2-morphisms:

$$F\bullet \underset{\beta}{\overset{\alpha}{\Downarrow g}} \bullet\, F$$

Given a short exact sequence of groups, we classify it by choosing a set-theoretic splitting... which is analogous to a *connection*:

$$1 \longrightarrow F \xrightarrow[i]{} E \underset{p}{\overset{s}{\rightleftarrows}} B \longrightarrow 1$$

This gives for any $b \in B$ an automorphism $\alpha(b)$ of F:

$$\alpha(b)(f) = s(b)fs(b)^{-1}$$

We do not have

$$\alpha(b)\,\alpha(b') = \alpha(bb')$$

but this holds *up to conjugation* by an element $\alpha(b, b') \in F$, so we get a *weak* 2-functor

$$\alpha \colon B \to \mathrm{AUT}(F).$$

Different splittings give equivalent 2-functors.

The set of equivalence clases of weak 2-functors $B \to \mathrm{Aut}(F)$ is the **nonabelian cohomology** $\mathcal{H}^2(B, F)$. This set is in one-to-one correspondence with the set of isomorphism classes of short exact sequences

$$1 \to F \to E \to B \to 1$$

What's Next?

• Tomorrow, Alissa Crans will explain how every simple Lie algebra $\mathfrak{g}$ has a 1-parameter deformation into a Lie 2-algebra $\mathfrak{g}_k$. For integer values of k, this Lie 2-algebra comes from a *Lie 2-group*. This Lie 2-group is built using a central extension of the loop group $C^\infty(S^1, G)$.

• On Sunday, Danny Stevenson will talk about the Lie algebra analogue of Schreier theory. He'll explain how a connection on a principal bundle is a splitting for some short exact sequence of Lie algebras. He'll then categorify this: a connection on a *categorified* principal bundle is a splitting for some short exact sequence of *Lie 2-algebras*. Curvature and the Bianchi identities then arise from nonabelian cohomology!

• On Monday I'll explain connections and 2-connections for *trivial* bundles and 2-bundles. These are just Lie-algebra-valued differential forms, so I won't even use the language of bundles.

• On Tuesday I'll explain connections and 2-connections for *nontrivial* bundles and 2-bundles.