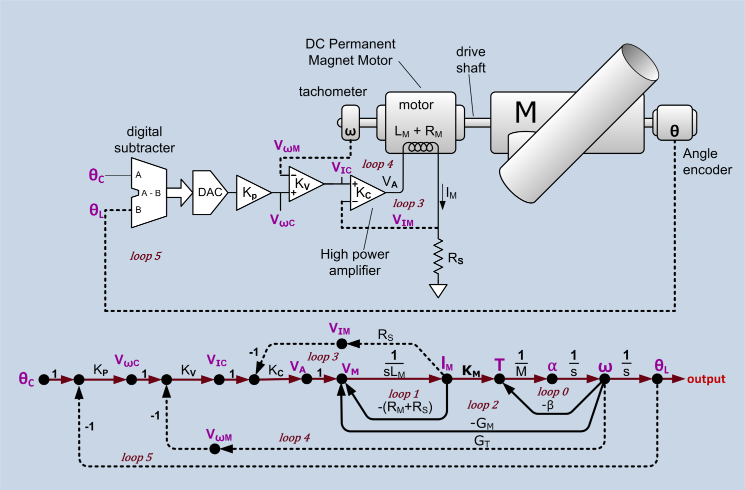
Network Theory I: Electrical Circuits and Signal-Flow Graphs

John Baez, Jason Erbele, Brendan Fong



The category with vector spaces as objects and linear maps as morphisms becomes symmetric monoidal with the usual $\otimes$.

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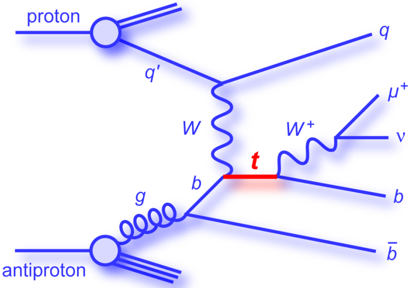
In quantum field theory, ‘Feynman diagrams’ are pictures of morphisms in this symmetric monoidal category:



But the category of vector spaces also becomes symmetric monoidal with direct sum, $\oplus$, as its 'tensor product'. Today we will explore this.

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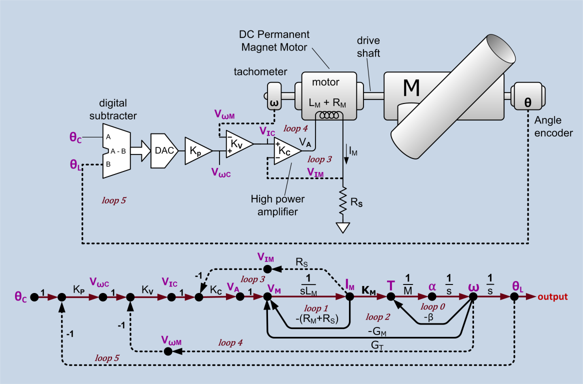
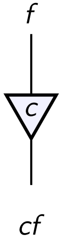
Control theorists use 'signal-flow graphs' to describe how signals flow through a system and interact:



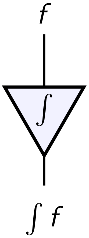
Think of a signal as a smooth real-valued function of time:

$$f : \mathbb{R} \to \mathbb{R}$$

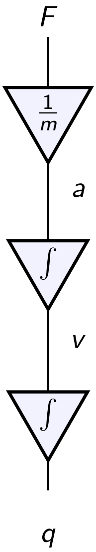
We can multiply a signal by a constant and get a new signal:



We can integrate a signal:



Here is what happens when you push on a mass m with a time-dependent force F:



Integration introduces an ambiguity: the constant of integration. But electrical engineers often use Laplace transforms to write signals as linear combinations of exponentials

$$f(t) = e^{-st} \quad \text{for some } s > 0$$

Then they define

$$(\textstyle\int f)(t) = \frac{e^{-st}}{s}$$

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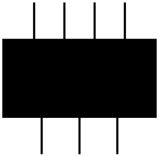
This lets us think of integration as a special case of scalar multiplication! We extend our field of scalars from $\mathbb{R}$ to $\mathbb{R}(s)$, the field of rational real functions in one variable s.

Let us be general and work with an arbitrary field k. For us, any signal-flow graph with m input edges and n output edges



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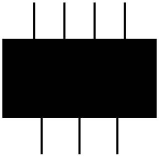

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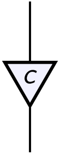
In other words: signal-flow graphs are pictures of morphisms in $\mathrm{FinVect}_k$, the category of finite-dimensional vector spaces over k... *where we make this into a monoidal category using* $\oplus$*, not* $\otimes$*.*

We build these pictures from a few simple 'generators'.

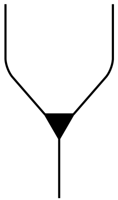
First, we have scalar multiplication:



This is a notation for the linear map

$$\begin{array}{ccc} k & \rightarrow & k \\ f & \mapsto & cf \end{array}$$

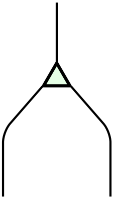
Second, we can add two signals:



This is a notation for

$$+\colon k \oplus k \to k$$

Third, we can 'duplicate' a signal:



This is a notation for the diagonal map

$$\begin{array}{rccc} \Delta\colon & k & \to & k \oplus k \\ & f & \mapsto & (f, f) \end{array}$$

Fourth, we can 'delete' a signal:

This is a notation for the linear map

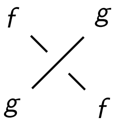
$$\begin{array}{rcl} k & \to & \{0\} \\ f & \mapsto & 0 \end{array}$$

Fifth, we have the zero signal:

This is a notation for the linear map

$$\begin{array}{ccc} \{0\} & \to & k \\ 0 & \mapsto & 0 \end{array}$$

Furthermore, ($\mathrm{FinVect}_k$, $\oplus$) is a *symmetric* monoidal category. This means we have a 'braiding': a way to switch two signals:

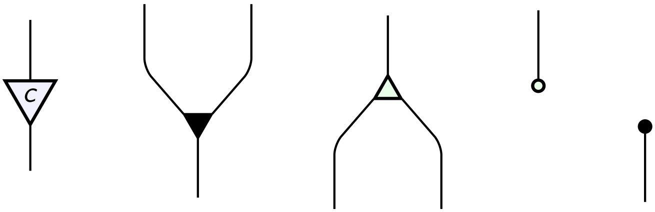


This is a notation for the linear map

$$\begin{array}{ccc} k \oplus k & \rightarrow & k \oplus k \\ (f, g) & \mapsto & (g, f) \end{array}$$

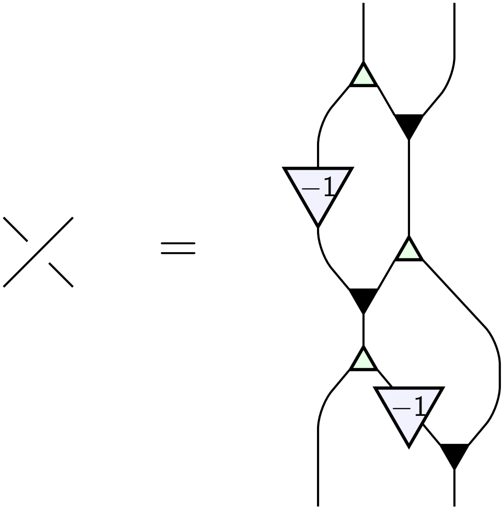
In a symmetric monoidal category, the braiding must obey a few axioms. I won't list them here, since they are easy to find.

From these ‘generators’:



together with the braiding, we can build complicated signal-flow graphs. In fact, we can describe *any* linear map $F\colon k^m \to k^n$ this way!

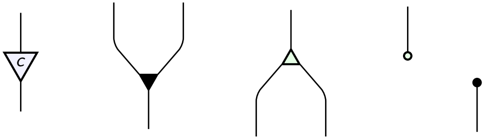
But these generators obey some unexpected relations:



Luckily, we can derive *all* the relations from some very nice ones!

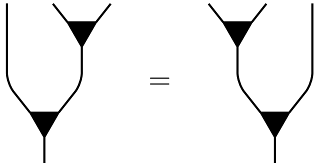
Theorem (Jason Erbele)

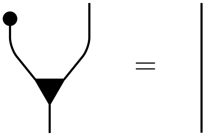
$FinVect_k$ is equivalent to the symmetric monoidal category generated by the object k and these morphisms:

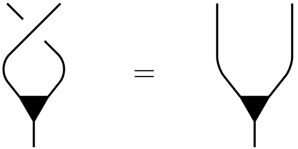


where $c \in k$, with the following relations.

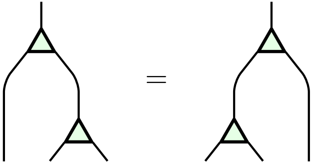
Addition and zero make k into a commutative monoid:

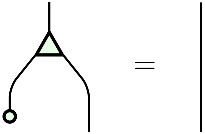


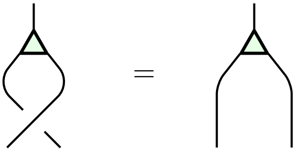




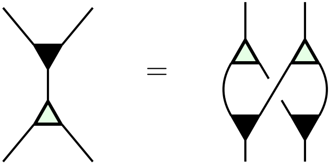
Duplication and deletion make k into a cocommutative comonoid:

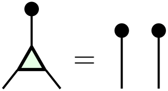


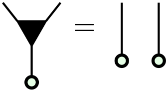


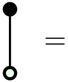


The monoid and comonoid operations are compatible, as in a bialgebra:

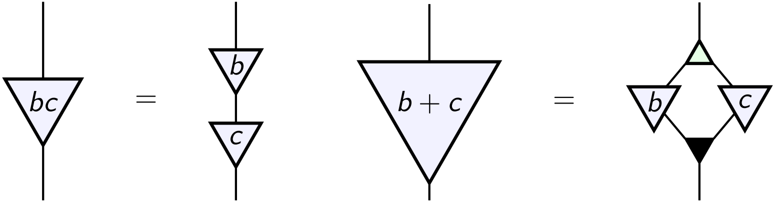


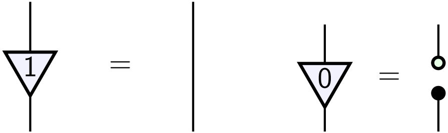




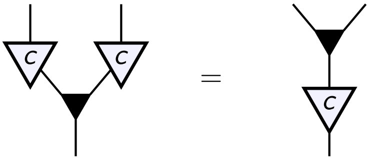


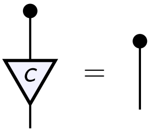
The ring structure of k can be recovered from the generators:



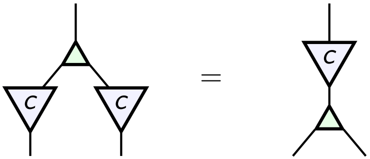


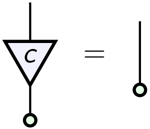
Scalar multiplication is linear (compatible with addition and zero):





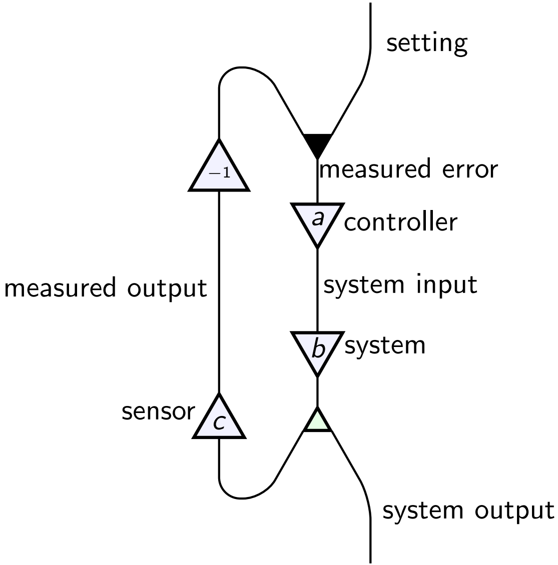
Scalar multiplication is 'colinear' (compatible with duplication and deletion):





Those are all the relations we need!

However, control theory also needs more general signal-flow graphs, which have 'feedback loops':



Feedback is the most important concept in control theory: letting the output of a system affect its input. For this we should let wires 'bend back':

These aren't linear functions — they're linear relations!

A **linear relation** $F\colon U \rightsquigarrow V$ from a vector space U to a vector space V is a linear subspace $F \subseteq U \oplus V$.

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We can compose linear relations $F\colon U \rightsquigarrow V$ and $G\colon V \rightsquigarrow W$ and get a linear relation $G \circ F\colon U \rightsquigarrow W$:

$$G \circ F = \{(u, w)\colon \ \exists v \in V \quad (u, v) \in F \text{ and } (v, w) \in G\}.$$

A linear map $\phi\colon U \to V$ gives a linear relation $F\colon U \rightsquigarrow V$, namely the graph of that map:

$$F = \{(u, \phi(u)) : u \in U\}$$

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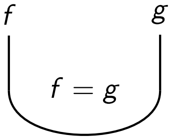
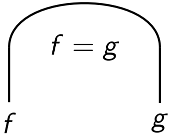
Fully general signal-flow graphs are pictures of morphisms in FinRel_k, typically with $k = \mathbb{R}(s)$.

Jason Erbele showed that besides the previous generators of $\mathrm{FinVect}_k$, we only need two more morphisms to generate all the morphisms in FinRel_k: the 'cup' and 'cap'.





These linear relations say that when a signal goes around a bend in a wire, the signal coming out equals the signal going in!

More formally, the cup is the linear relation

$$\cup \colon k \oplus k \rightsquigarrow \{0\}$$

given by:

$$\cup \;=\; \{(f, f, 0) : f \in k\} \;\subseteq\; k \oplus k \oplus \{0\}$$

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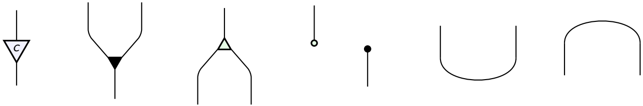
These make (FinRel_k, $\oplus$) into a 'dagger-compact category'.

Theorem (Jason Erbele)

FinRel_k is equivalent to the symmetric monoidal category generated by the object k and these morphisms:



where $c \in k$, and an explicit list of relations.

Instead of listing the relations, let me just sketch what comes next!

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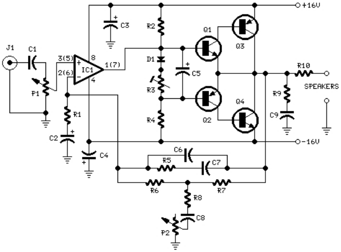
In both versions there's a general issue: engineers want to build devices that actually *implement* a given signal-flow graph. One way is to use electrical circuits. These are described using 'circuit diagrams':



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Thanks to work in progress by Brendan Fong, we know there is a *functor* from this bicategory to FinRel_k:

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where $k = \mathbb{R}(s)$.

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where $k = \mathbb{R}(s)$.

This functor says, for any circuit diagram, how the voltages and currents on the input wires are related to those on the ouput wires.

However, we do not get *arbitrary* linear relations this way. The space of voltages and currents on n wires:

$$k^n \oplus k^n$$

is a **symplectic** vector space, meaning that it's equipped with a skew-symmetric nondegenerate bilinear form:

$$\omega((V_1, I_1), (V_2, I_2)) = I_1 \cdot V_2 - I_2 \cdot V_1$$

called the **symplectic 2-form**.

This is similar to how in classical mechanics, the space of positions and momenta of a collection of particles is a symplectic 2-form on $\mathbb{R}^n \oplus \mathbb{R}^n$.

The linear relation

$$F \colon k^m \oplus k^m \rightsquigarrow k^n \oplus k^n$$

we get from a linear circuit is always a **Lagrangian relation**, meaning that

$$F \subseteq (k^m \oplus k^m) \oplus (k^n \oplus k^n)$$

is a **Lagrangian subspace**: a maximal subspace on which the symplectic 2-form vanishes.

Similarly, in classical mechanics, the inital and final positions/ momenta of a collection of particles lie in a Lagrangian submanifold of $\mathbb{R}^n \oplus \mathbb{R}^n$.

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- control theory
- electrical engineering
- category theory
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This should become even more interesting when we study nonlinear systems. And as we move from the networks important in *human-engineered* systems to those important in *biology* and *ecology*, the mathematics should become even more rich!