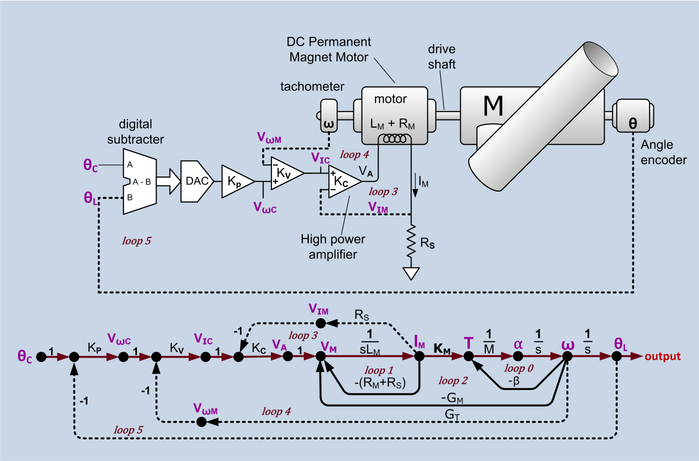
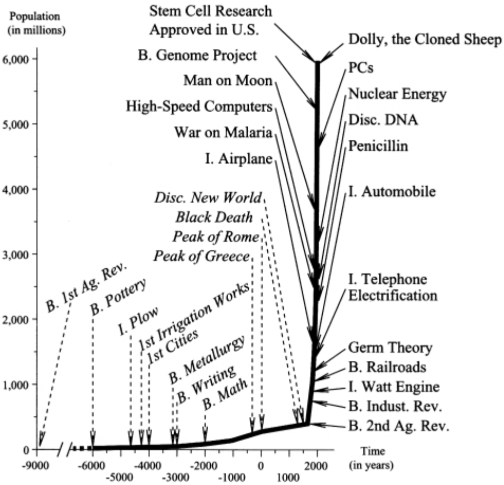
CATEGORIES IN CONTROL

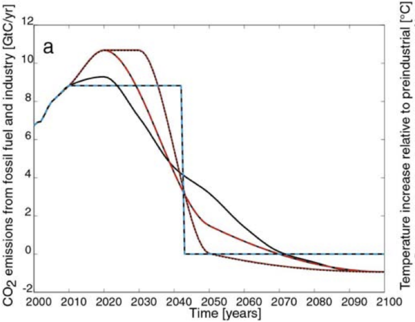


John Baez, Jason Erbele & Nick Woods
Higher-Dimensional Rewriting and Applications
Warsaw, 28 June 2015

We have left the Holocene and entered a new epoch, the Anthropocene, when the biosphere is rapidly changing due to human activities.



According to the 2014 IPCC report on climate change, to surely stay below 2 °C of warming, we need a *more than 100% reduction in carbon emissions*...

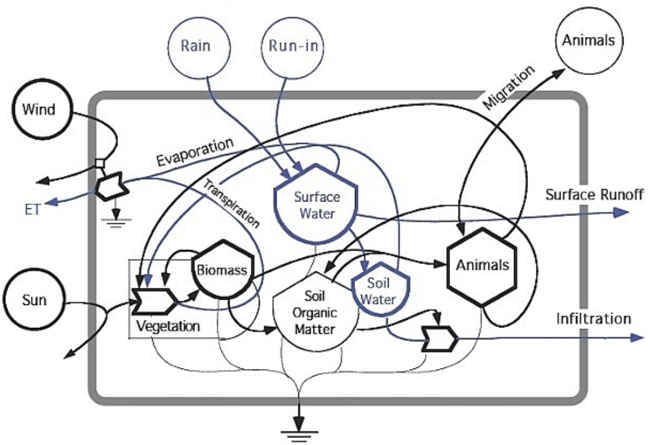


...unless we completely stop carbon emissions by 2040.

So, we can expect that in this century, scientists, engineers and mathematicians will be increasingly focused on *biology*, *ecology* and *complex networked systems* — just as the last century was dominated by physics.

What can category theorists contribute?

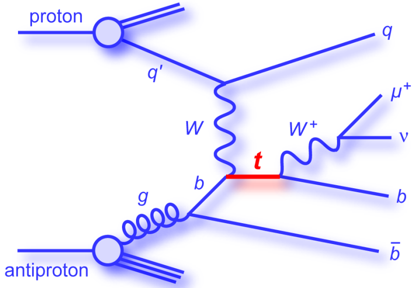
To understand ecosystems, ultimately will be to understand networks. — B. C. Patten and M. Witkamp



We need a good mathematical theory of networks.

The category with vector spaces as objects and linear maps as morphisms becomes symmetric monoidal with the usual $\otimes$.

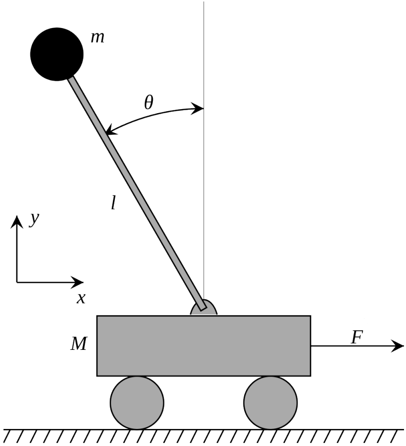
In quantum field theory, 'Feynman diagrams' are pictures of morphisms in this symmetric monoidal category:



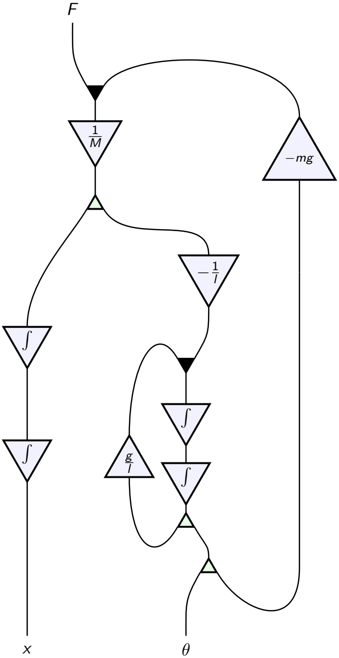
But the category of vector spaces also becomes symmetric monoidal with direct sum, $\oplus$, as its 'tensor product'. This is more important in electrical engineering and **control theory**: the art of getting systems to do what you want.

Control theorists use 'signal-flow diagrams' to describe how signals flow through a system and interact.

For example, an upside-down pendulum on a cart:



has the following signal-flow diagram...

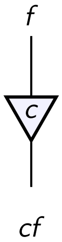


F
1/M
−mg
−1/l
∫
∫
∫
∫
g/l
x
θ

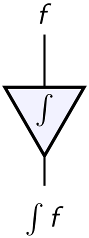
To formalize this, think of a signal as a smooth real-valued function of time:

$$f : \mathbb{R} \to \mathbb{R}$$

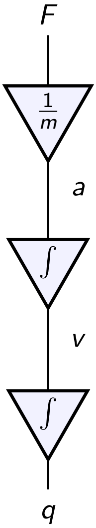
We can multiply a signal by a constant and get a new signal:



We can integrate a signal:



Here is the signal-flow diagram for the simplest machine in the world: a *rock!*



Integration introduces an ambiguity: the constant of integration. But electrical engineers often use Laplace transforms to write signals as linear combinations of exponentials

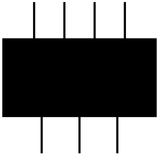
$$f(t) = e^{-st} \quad \text{for some } s > 0$$

Then they define

$$(\textstyle\int f)(t) = \frac{e^{-st}}{s}$$

This lets us think of integration as a special case of scalar multiplication! We extend our field of scalars from $\mathbb{R}$ to $\mathbb{R}(s)$, the field of rational real functions in one variable s.

Let us be general and work with an arbitrary field k. The simplest kind of signal-flow diagram with m input edges and n output edges:



stands for a linear map

$$F\colon k^m \to k^n$$

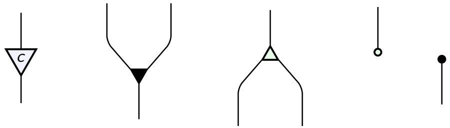
In other words: it's a string diagram for a morphism in $\mathrm{FinVect}_k$, the category of finite-dimensional vector spaces over k... *where we make this into a monoidal category using $\oplus$, not $\otimes$.*

Lemma (Jason Erbele)

The category $\mathbf{FinVect}_k$*, with*

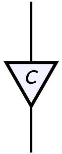
- *finite-dimensional vector spaces over* k *as objects,*
- *linear maps as morphisms,*

is symmetric monoidal with $\oplus$ *as its tensor product. It is generated as a symmetric monoidal category by one object,* k*, and these morphisms:*



where $c \in k$.

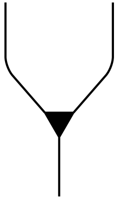
1. For each $c \in k$ we can multiply numbers by c:



This is a notation for the linear map

$$\begin{aligned} c\colon k &\to k \\ x &\mapsto cx \end{aligned}$$

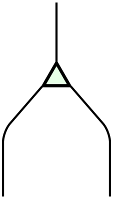
2. We can add numbers:



This is a notation for the linear map

$$\begin{array}{rccc} +: & k \oplus k & \to & k \\ & (x, y) & \mapsto & x + y \end{array}$$

3. We can **duplicate** a number:



This is a notation for the linear map

$$\begin{array}{rccc} \Delta\colon & k & \to & k \oplus k \\ & x & \mapsto & (x, x) \end{array}$$

4. We can **delete** a number:

This is a notation for the linear map

$$\begin{array}{rccc} ! \colon & k & \to & \{0\} \\ & x & \mapsto & 0 \end{array}$$

5. We have the number zero:

This is a notation for the linear map

$$\begin{array}{rccc} 0\colon & \{0\} & \to & k \\ & 0 & \mapsto & 0 \end{array}$$

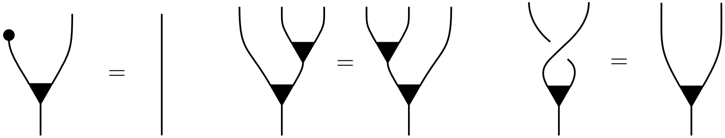
In fact we know what relations these generating morphisms obey:

Theorem (Erbele)

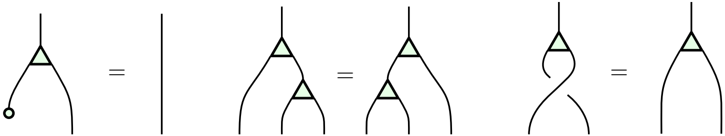
FinVect$_k$ *is the free symmetric monoidal category on a bicommutative bimonoid over* k*.*

The jargon here is a terse way to list the relations obeyed by scalar multiplication, addition, duplication, deletion and zero. In detail...

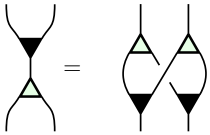
(1)–(3) Addition and zero make k into a commutative monoid:

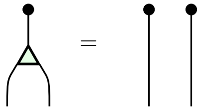


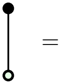
(4)–(6) Duplication and deletion make k into a cocommutative comonoid:

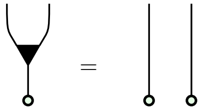


(7)–(10) The monoid and comonoid structures on k fit together to form a bimonoid:

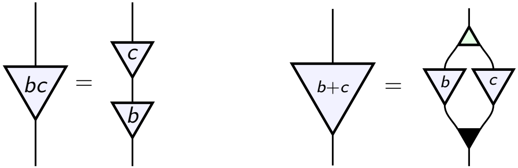




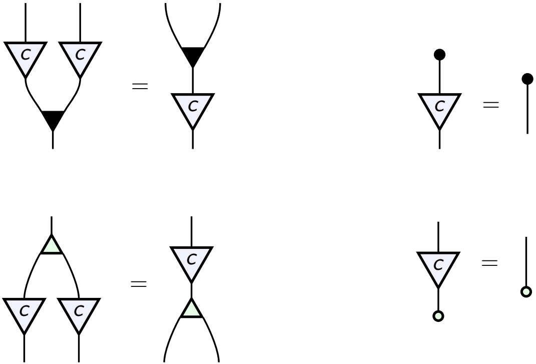




(11)–(14) The rig structure of k can be recovered from the generating morphisms:

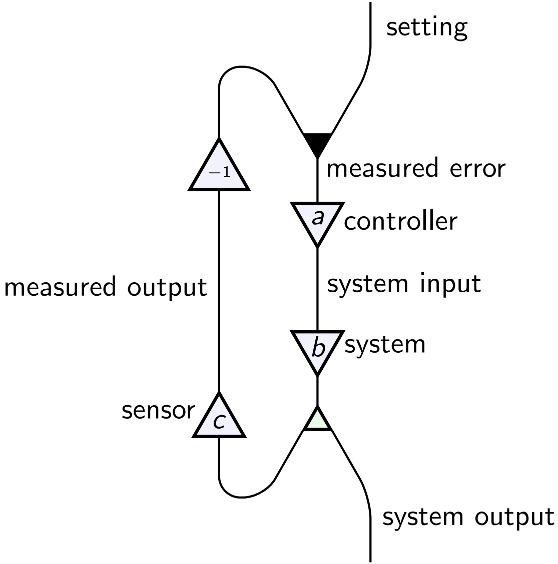


(15)–(18) Scalar multiplication by $c \in k$ commutes with the generating morphisms:



These are all the relations we need!

However, control theory also needs more general signal-flow diagrams, which have ‘feedback loops’:



Feedback is the most important concept in control theory: letting the output of a system affect its input. For this we should let wires 'bend back':

These aren't linear maps — they're linear *relations!*

A **linear relation** $F\colon U \rightsquigarrow V$ from a vector space U to a vector space V is a linear subspace $F \subseteq U \oplus V$.

We can compose linear relations $F\colon U \rightsquigarrow V$ and $G\colon V \rightsquigarrow W$ and get a linear relation $G \circ F\colon U \rightsquigarrow W$:

$$G \circ F = \{(u, w)\colon\ \exists v \in V \quad (u, v) \in F \text{ and } (v, w) \in G\}.$$

A linear map $\phi\colon U \to V$ gives a linear relation $F\colon U \rightsquigarrow V$, namely the graph of that map:

$$F = \{(u, \phi(u)) : u \in U\}$$

Composing linear maps becomes a special case of composing linear relations.

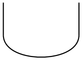
There is a category $\mathbf{FinRel}_k$ with finite-dimensional vector spaces over the field k as objects and linear relations as morphisms.

$\mathbf{FinRel}_k$ becomes symmetric monoidal using $\oplus$. It has $\mathbf{FinVect}_k$ as a symmetric monoidal subcategory.

Fully general signal-flow diagrams are pictures of morphisms in $\mathbf{FinRel}_k$, typically with $k = \mathbb{R}(s)$.

Erbele showed that besides the generators of $\mathbf{FinVect}_k$ we only need two more morphisms to generate $\mathbf{FinRel}_k$:

6. The **cup**:



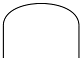
This is the linear relation

$$\cup \colon k \oplus k \rightsquigarrow \{0\}$$

given by

$$\cup \;=\; \{(x, x, 0) : x \in k\} \;\subseteq\; k \oplus k \oplus \{0\}$$

7. The **cap**:



This is the linear relation

$$\cap\colon \{0\} \rightsquigarrow k \oplus k$$

given by

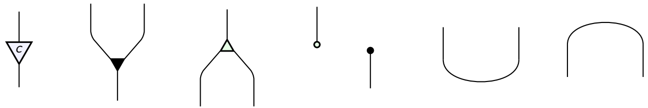
$$\cap \;=\; \{(0,x,x) : x \in k\} \;\subseteq\; \{0\} \oplus k \oplus k$$

Lemma (Erbele)

The category $\mathbf{FinRel}_k$, *with*

- *finite-dimensional vector spaces over* k *as objects,*
- *linear relations as morphisms,*

is symmetric monoidal with $\oplus$ *as its tensor product. It is generated as a symmetric monoidal category by one object,* k, *and these morphisms:*

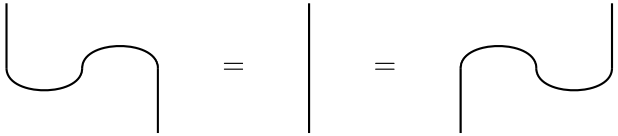


Theorem (Erbele, Bonchi–Sobociński–Zanasi)

FinRel$_k$ *is the free symmetric monoidal category on a pair of interacting bimonoids over* k.

Besides the relations we've seen so far, this statement summarizes the following extra relations:

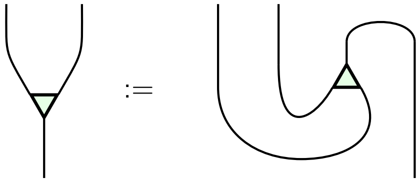
(19)–(20) $\cap$ and $\cup$ obey the zigzag relations:



It follows that $(\mathbf{FinRel}_k, \oplus)$ becomes a dagger-compact category, so we can 'turn around' any morphism $F\colon U \rightsquigarrow V$ and get its **adjoint** $F^\dagger\colon V \rightsquigarrow U$:

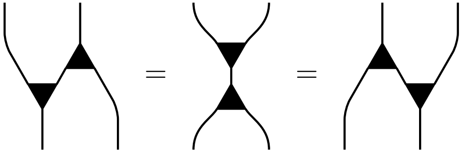
$$F^\dagger = \{(v,u) : (u,v) \in F\}$$

For example, turning around duplication $\Delta\colon k \to k \oplus k$ gives **coduplication**, $\Delta^\dagger\colon k \oplus k \rightsquigarrow k$:

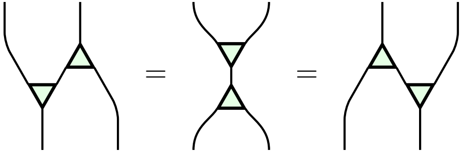


$$\Delta^\dagger = \{(x,x,x)\} \subseteq (k \oplus k) \oplus k$$

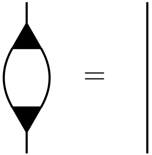
(21)–(22) $(k, +, 0, +^\dagger, 0^\dagger)$ is a Frobenius monoid:

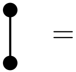


(23)–(24) $(k, \Delta^\dagger, !^\dagger, \Delta, !)$ is a Frobenius monoid:

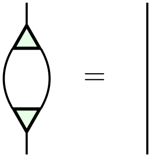


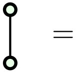
(25)–(26) The Frobenius monoid $(k, +, 0, +^\dagger, 0^\dagger)$ is extra-special:



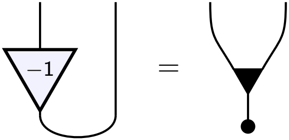


(27)–(28) The Frobenius monoid $(k, \Delta^\dagger, !^\dagger, \Delta, !)$ is extra-special:

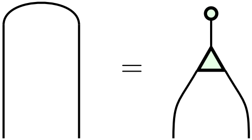




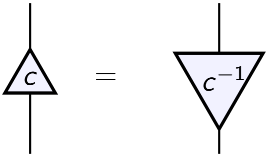
(29) $\cup$ with a factor of -1 inserted can be expressed in terms of $+$ and 0:



(30) $\cap$ can be expressed in terms of Δ and !:



(31) For any $c \in k$ with $c \neq 0$, scalar multiplication by c^{-1} is the adjoint of scalar multiplication by c:



A **PROP** is a symmetric monoidal category with natural numbers as objects, the tensor product on objects being addition.

The symmetric monoidal category $\mathbf{FinVect}_k$ is equivalent to the PROP $\mathbf{Mat}(k)$, where a morphism $f: m \to n$ is an $n \times m$ matrix with entries in k.

However, we can define $\mathbf{Mat}(k)$ whenever k is a rig. We have:

Theorem (Simon Wadsley and Nick Woods)

$\mathbf{Mat}(k)$ *is the PROP for bicommutative bimonoids over* k*.*

To understand this, note that for any bicommutative bimonoid A in a symmetric monoidal category $\mathbf{C}$, the bimonoid endomorphisms $f\colon A \to A$ can be added and composed, giving a rig $\mathrm{End}(A)$.

A bicommutative bimonoid **over $\boldsymbol{k}$** in $\mathbf{C}$ is one equipped with a rig homomorphism

$$\Phi_A\colon k \to \mathrm{End}(A)$$

Bicommutative bimonoids over k in $\mathbf{C}$ form a category where a morphism $f\colon A \to B$ is a bimonoid homomorphism such that for each $c \in k$ the square

$$\begin{tikzcd} A \arrow[r, "\Phi_A(c)"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "\Phi_B(c)"'] & B \end{tikzcd}$$

commutes.

Wadsley and Woods proved that this category is equivalent to the category of algebras of the PROP $\mathbf{Mat}(k)$ in $\mathbf{C}$.

Example: the commutative rig of natural numbers gives the PROP

$$\mathbf{Mat}(\mathbb{N}) \;\simeq\; \mathbf{FinSpan}$$

equivalent to the symmetric monoidal category of finite sets and spans, with disjoint union as tensor product.

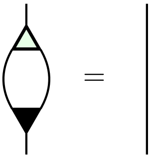
Steve Lack showed that this is the PROP for bicommutative bimonoids. But this also follows from the result of Wadsley and Woods.

Example: the commutative rig of booleans $\mathbb{B} = \{F, T\}$, with $\vee$ as addition and $\wedge$ as multiplication, gives the PROP

$$\mathbf{Mat}(\mathbb{B}) \simeq \mathbf{FinRel}$$

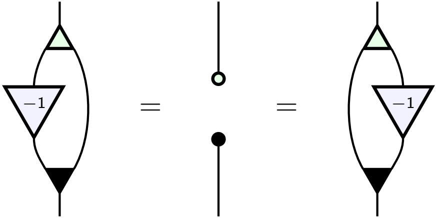
equivalent to the symmetric monoidal category of finite sets and relations, with disjoint union as tensor product.

Samuel Mimram showed that this is the PROP for **special** bicommutative bimonoids, meaning those where



Again, this follows from the general result of Wadsley and Woods.

Example: the commutative ring of integers $\mathbb{Z}$ gives the PROP $\mathbf{Mat}(\mathbb{Z})$. This is the PROP for bicommutative Hopf monoids. The key here is that scalar multiplication by -1 obeys the axioms for an antipode:



More generally, whenever k is a commutative ring, the presence of $-1 \in k$ guarantees that $\mathbf{Mat}(k)$ is the PROP for Hopf monoids over k.

So, there's no shortage of beautiful category theory and rewrite rules hiding in control theory.

Next: use them to help control theorists and save the world!