

Representation Theory of Hopf Categories

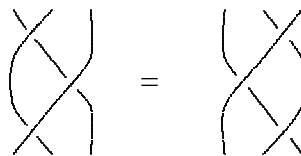
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Introduction

One feature that makes quantum groups and braided monoidal categories valuable for physicists is their supplying solutions of the Yang-Baxter equation, a symmetry equation that has many applications in different physical and mathematical subjects such as knot theory, 3-dimensional Topological Quantum Field Theory and the 2-dimensional lattice models of statistical mechanics [Dri, Fr, Tur, DW]. Roughly speaking the Yang-Baxter equation is an algebraic version of the following fundamental braid identity:

The diagram illustrates the Yang-Baxter equation using three strands. On the left, the strands are arranged such that the top and bottom strands cross each other twice, while the middle strand remains straight. On the right, the strands are arranged such that the middle strand crosses each of the other two strands once, while the top and bottom strands remain straight. An equals sign is placed between the two diagrams, indicating their equivalence.

The Yang-Baxter equation

Solutions for this equation are yielded by certain symmetry structures in monoidal categories, called braidings, or in Hopf algebras, where they are called quasitriangular structures. Via representation theory and Tannaka duality, braidings in the categories of representations of Hopf algebras correspond to the quasitriangular structures in the Hopf algebras. This interplay between the category and the Hopf algebra has been thoroughly examined during the last years [JS1, Sch2, Maj, Par3].

There are also higher dimensional versions of the Yang-Baxter equation, which have the same relevance to higher dimensional analogs of the applications mentioned above as 4-dimensional TQFTs, 3-dimensional lattice models or the theory of 2-knots (a 2-knot is an embedding of the 2-dimensional sphere into $\mathbb{R}^4$). Zamolodchikov [Za] introduced a class of such equations that he called tetrahedron equations, which are motivated by the theory of exactly solvable models and he also conjectured a solution for them. Though meanwhile some solutions are known, these complex equations are still not very well understood.

Recently, there has been an attempt in mathematics and mathematical physics to handle problems in higher dimensions by the new concept of *higher dimensional algebra*. In [BD] an overview over this new discipline and its connections to mathematical physics can be found. Let us review 2-categories, being the basic concept from higher dimensional algebra that is of most importance for this work. A 2-category is a category where for any pair of objects

A, B the ‘set’ of morphisms $\text{Hom}(A, B)$ is not merely a set, but a category. Objects of this category are called 1-morphisms and can be thought as morphisms $A \rightarrow B$, while morphisms in $\text{Hom}(A, B)$, are called 2-morphisms, and should be thought as morphisms of morphisms or as ‘homotopies’ $\alpha : f \Rightarrow g : A \rightarrow B$. Furthermore there are various obvious composition laws for 1- and 2-morphisms. This concept is familiar from homotopy theory, which gives rise to a first example of a 2-category. Topological spaces serve as objects, continuous maps as 1-morphisms and for two continuous maps f, g between spaces C and D , there may exist homotopies $\alpha : f \Rightarrow g$ which can be interpreted as 2-morphisms from f to g .

A monoidal 2-category then, is endowed with an appropriate multiplication, i.e. for any pair of objects A, B there exists a ‘tensor product’ $A \boxtimes B$, satisfying various coherence laws. In [KV] it has been outlined how symmetries in monoidal 2-categories can be used to construct solutions of the Zamolodchikov equation. Such *braided monoidal 2-categories* are highly nontrivial mathematical structures and examples are not close at hand. Several approaches towards braided monoidal 2-categories have been undertaken. The theory of 2-braids, 2-tangles and 2-knots furnishes geometric examples having a universal property. This connection is examined e.g. in [BL, CS]. The intimate relation between topological quantum field theories and higher dimensional category theory has been outlined in a number of papers [At, Fr, CY1, BD]. In [BN] semistrict braided 2-categories were accessed by a ‘center’ construction, starting from a given monoidal 2-category. Other examples are also known [DS, Yet].

However, a general framework that works out braided monoidal 2-categories as a sort of representation of some lower dimensional algebraic data is missing so far. Of course, even less is known about the inverse procedure of reconstructing this lower dimensional structure from a given monoidal 2-category.

In this paper we make a first step towards such a theory. We obtain monoidal 2-categories as representations of Hopf categories, a higher dimensional analog of the concept of a Hopf algebra. A first attempt towards Hopf categories has been made in [CF]. This paper also gave the first impetus for this work. A Hopf category, or rather a bimonoidal category, is a monoidal category $\mathcal{H}$ together with a second algebraic structure, shaped in such a way as to endow the 2-category of representations of $\mathcal{H}$ with a tensor product. The aim of this paper is to give a concise definition of Hopf categories and to examine their representation theories. We put special emphasis on the question when the resulting 2-category is supplied with a braiding. If a monoidal 2-category happens to be the category ${}_{\mathcal{H}}\mathfrak{M}$ of representations of a Hopf category $\mathcal{H}$, can we express all possible braidings purely in terms of the Hopf category? In other words, is there a notion of quasitriangularity for a Hopf category $\mathcal{H}$, wide enough to cover all possible braidings on ${}_{\mathcal{H}}\mathfrak{M}$?

It appears that to answer this question one has to take into account what is generally known as the *quantum hierarchy* of higher dimensional algebra (c.f. [BD]), or the *Main Principle in Category Theory* (c.f. [KV]). According to this principle it is *unnatural and undesirable to speak about equality of two objects in any category*. If we consider for example the category of vector spaces this principle says that it is not important (or even not possible) to know whether

two vector spaces are equal, but only whether they are isomorphic. The Cartesian product of vector spaces, say, is not associative, but only associative up to a canonical isomorphism. However, it appears that it is of importance to know the isomorphism, since to be an admissible morphism in the sense of a certain theory, it has to satisfy equations of its own. For two vector spaces X and Y , say, the linear map $\tau_{X,Y} : X \otimes Y \ni x \otimes y \mapsto -y \otimes x \in Y \otimes X$ is a canonical isomorphism of vector spaces, however it is not coherent. The fusion equation $(Y \otimes \tau_{X,Z}) \circ (\tau_{X,Y} \otimes Z) = \tau_{X,Y \otimes Z} : X \otimes Y \otimes Z \rightarrow Y \otimes Z \otimes X$ is not satisfied.

This principle becomes even more important if we apply it to higher category theory. By definition 2-categories are categories and composition of morphisms in categories is associative. However, morphisms in 2-categories are also objects in 1-categories and therefore associativity of composition should be replaced by associativity up to isomorphism. Bénabou [Ben] was the first to advocate this viewpoint. He developed a theory of weak 2-categories called *bicategories*, where he also replaced 2-functors (with their strict composition law $\mathcal{F}(f) \circ \mathcal{F}(g) = \mathcal{F}(f \circ g)$) by *pseudo functors* ($\mathcal{F}(f) \circ \mathcal{F}(g) \cong \mathcal{F}(f \circ g)$) and similarly for morphisms of 2-functors. Generally, coherence laws for morphisms in a 1-category become replaced in 2-categories by isomorphisms satisfying coherence laws of their own and the problem often consists in figuring out these laws.

Let us apply these considerations to the question of commutativity raised above. Commutativity for (usual) algebras is a mere property. In a monoidal category it is not sensible to require strict commutativity, as $A \otimes B = B \otimes A$. This is violated even for the cartesian product of sets. Rather we require the existence of an isomorphism $\tau : A \otimes B \cong B \otimes A$, fulfilling some axioms (the two celebrated fusion triangles which imply the Yang-Baxter equation). So, in a commutative monoidal category we are so sloppy to not require commutativity on the nose, but only up an isomorphism, but we are pedantic enough to want this isomorphism specified. Such an isomorphism we then call a braiding. Since there may be different braidings for a monoidal category, commutativity is now a set rather than a condition; namely the set of all possible braidings. In a monoidal 2-category $(\mathfrak{C}, \boxtimes)$ finally, we would not speak of equality of two braidings, but only of isomorphy. Therefore, commutativity is no longer a set, but a category. Likewise, it is not sensible to require the fusion triangles of a 1-braiding commuting on the nose. An object of this ‘braid-category’ is therefore a triple $(\mathcal{R}, \tilde{R}_1, \tilde{R}_2)$, consisting of a pseudo natural transformation $\mathcal{R}_{A,B} : A \boxtimes B \Rightarrow B \boxtimes A$ and isomorphisms of pseudo natural transformations called modifications $\tilde{R}_1$ and $\tilde{R}_2$ sitting inside the fusion triangles and fulfilling axioms of their own. Morphisms of braidings then are modifications $r : \mathcal{R} \Rightarrow \mathcal{R}'$ compatible with the additional structure.

Therefore the best we can hope for in order to answer our original question is a category equivalence between the category of braidings for the monoidal 2-category ${}_{\mathcal{H}}\mathfrak{M}$ and the to be defined category of quasitriangular structures for the Hopf category $\mathcal{H}$. An appropriate definition for the latter which allows to establish this equivalence is given in chapter 5.

What is the use of explaining the poorly understood braided monoidal 2-categories by new structures equally complicated? In the first place, this new structure is of one lower algebraic dimension and it should be easier to find examples for quasitriangular Hopf categories than

for braided monoidal 2-categories. An interesting class of examples can be constructed via a higher dimensional Drinfeld Double construction from any finite group. In this example, the 2-braidings are described by the second cohomology group of a certain cochain complex constructed from the group. To the best of the author's knowledge this is the first explicit algebraic construction of a nontrivial 2-braiding. Furthermore it is a commonplace in mathematics that examining the same thing from different points of view often yields a better understanding of the matter. In a joint work with John Baez [BN] the author showed by carrying out a general algebraic construction that the original definition of braided monoidal 2-categories due to Kapranov and Voevodsky had a gap. A new axiom had been added which by now is generally accepted as being indispensable.

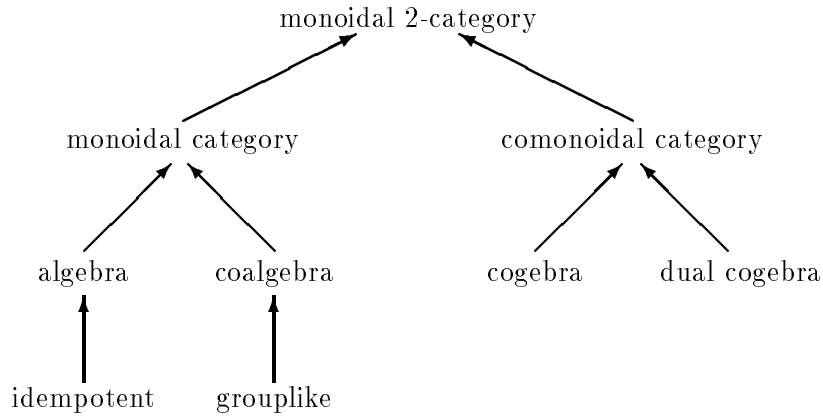
Recently, Street [DS] gave an interesting interpretation to this new axiom. It is well known that commutativity for an algebra is equivalent with the multiplication $\nabla : A \otimes A \rightarrow A$ being a morphism of algebras and with (A, ∇, η) being an algebra in the category of algebras $(A, \nabla, \eta) \in (\mathcal{C}\text{-Alg})\text{-Alg}$. In the same way, for a monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$ carrying a braid structure is equivalent with the functor $\boxtimes : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$ carrying the structure of a monoidal functor, or with $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ being a 2-algebra object in the monoidal 2-category of 2-algebra objects in $\mathbf{Cat}$. This equivalence was first proved in [JS1]. Obviously, the second condition implies the first, since being a morphism in $\mathfrak{C}\text{-Alg}$, $\boxtimes$ is a monoidal functor. But for the first condition to imply the second, we have to verify that the coherence morphisms α , λ and ρ are monoidal transformations, which as a matter of fact is true. Now we can examine the definitions of braided monoidal 2-categories given in [KV] and [BN] in the light of these considerations. It is a routine exercise to verify that the axioms in [KV] supply $\boxtimes : \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$ with the structure of a monoidal 2-functor (see [DS]). However, the coherence morphisms of $\boxtimes$ are not supplied with a monoidal structure. To ensure this we need the axiom introduced in [BN].

Also this work gives evidence for the singular character of the new axiom, which separates it from the remaining ones. In the group theoretic example mentioned before, we can interpret the three original axioms in terms of cocycles and coboundaries on the group, while the fourth is of a completely different type, and it is this axiom that makes the theory work so smoothly. As the theory will show, the pseudo natural transformation $\mathcal{R}$ and the modifications $\tilde{R}_1$ and $\tilde{R}_2$, constituting a 2-braiding are determined by three maps $\sigma, \alpha, \beta : G \times G \times G \rightarrow k^*$, where G denotes the finite group. The coherence axioms of pseudo natural transformations and modifications then translate into four lengthy and complicated conditions on these maps. The three original axioms of a 2-braiding due to Kapranov and Voevodsky say that the pair (α, β) is a 2-cocycle in the total complex of a double complex constructed from the group. The new axiom claims that σ is already determined by α and β and it turns out that the four lengthy conditions then are satisfied automatically. Therefore, any 2-braiding is completely determined by α and β . Since any braided monoidal 2-category gives rise to a solution of the Zamolodchikov equation, we can construct solutions out of any finite group, provided it has a nontrivial second cohomology group.

Going into this matter of higher dimensional algebra, one has to exploit the algebraic structures living in different algebraic dimensions that arise from the different levels of duality. Let us now give an overview over these structures. In a zero dimensional algebra, the structure morphism, the multiplication, becomes the identity. Hence, a zero dimensional algebra is an idempotent. To talk about an idempotent a only makes sense if one also gives the corresponding algebra A that a lives in. An idempotent can then be defined as a multiplicative mapping $a : k \rightarrow A$, where k is the ground field, or, in a sense to be made precise, the trivial algebra. The algebra itself lives in a category $\mathcal{A}$ that, to be able to define the notion algebra, has to be monoidal. An algebra is defined as a weak monoidal functor $A : \mathcal{I} \rightarrow \mathcal{A}$, where again $\mathcal{I}$ is the trivial monoidal category, depending on the theory one works in. Monoidal categories are algebra objects (also called 2-dimensional algebras) in monoidal 2-categories, which can be interpreted as algebra objects (3-dimensional algebras) in monoidal 3-categories. This view can be continued at will. A conceptual definition of an n -dimensional algebra can be formulated only in an $n + 1$ -dimensional algebra.

Making use of duality in the different levels of this framework we get new algebraic structures. Since there are no morphisms in 0-categories, the notion idempotent is selfdual. Dual to algebras are coalgebras, which are also defined as weak monoidal functors, but with coherence morphisms pointing in the opposite direction. A comultiplicative morphism $k \rightarrow C$ into a coalgebra C is usually called a grouplike element. In the same way we can dualize monoidal categories and define *comonoidal categories*. We shall call a weak comonoidal functor $\mathcal{I} \rightarrow \mathcal{C}$ into a comonoidal category $\mathcal{C}$ a *C-cogebra* or a *dual C-cogebra*, respectively, depending on the direction of the coherence morphisms.

We can arrange all algebras and their duals up to dimension 3 in the following tree.



Note that because of the absence of a trivial cogebra in a comonoidal category it is not possible to define anything like *coidempotents* in cogebars.

We will develop the appropriate representation theories in any dimension and examine what influence the presence of additional algebraic structure has for the particular representation

theory. The bimonoidal categories, mentioned before are monoidal categories equipped with a compatible comonoidal structure. It is well known that monoidal categories can be obtained as representations of algebras A that are equipped with a compatible coalgebra structure. It turns out that in the same way the presence of a cogeбра structure on an algebra A endows ${}_A\mathcal{M}$ with a comonoidal structure. Moreover, if A carries a coalgebra and a cogeбра structure and if these two are compatible (in which case we call A a *trialgebra*), then ${}_A\mathcal{M}$ is in fact a bimonoidal category.

It is well known that a braiding τ for a monoidal category is a monoidal transformation, if and only if it is a symmetry, a fact which causes many problems when one wants to deal with Hopf algebras in braided categories, since monoidality of τ is needed, symmetry however not wanted (c.f. [NS]). For cobraidings it is a much more natural condition to be monoidal. Therefore, working with cogebras instead of algebras could be a chance to overcome some difficulties. Yet, a comonoidal category is not as natural a mathematical structure as a monoidal category and there are not very many obvious examples. However, since cogebras are 2-dimensional grouplikes, they occur in all places where grouplikes occur in the 1-dimensional theory, a fact which indicates their importance and which should eventually lead to examples.

The outline of the paper is as follows. In chapter 1 we start from scratch. We define bicategories in their weakest versions and develop the appropriate versions of monoidal and braid structures for them. In our setup the braiding is not necessarily an equivalence, whence need arises for new axioms describing the behavior of the braiding on the unit objects. In the last section we introduce bicategories with only one object, which can be identified with monoidal categories. To be able to define Hopf categories one first has to fix a braided monoidal 2-category where everything takes place. This is done in chapter 2. In this paper we work constantly in the 2-category $\mathfrak{C}_k$ of k -linear additive small categories, where k is an algebraically closed field. We develop the monoidal structure and the braiding for this 2-category and develop the category of 2-vector spaces as defined in [KV] as a sub-2-category.

In the third chapter comonoidal categories as formal duals to monoidal categories and bimonoidal categories as a connection of monoidal and comonoidal categories are introduced. We prove some basic lemmas for bimonoidal categories. Furthermore we define Hopf categories as bimonoidal categories supplied with an antipode functor. This antipode is modeled in a way as to induce a rigid structure on the 2-category of representations of the Hopf category. In the next section we develop bigebras and trialgebras as the extra algebraic structure needed for an algebra A to equip the category of A -modules with a comonoidal, resp. a bimonoidal structure. These are in a sense negative results, since it turns out that in the category of vector spaces the only algebras admitting a bigebra structure are the commutative ones.

The next chapter develops the theory of representations of a monoidal category. Categories that are acted on by monoidal categories were first defined in [Par1]. Here we consider for a fixed monoidal category $\mathcal{C}$ the collection of all such categories, together with the actions and all functors and natural transformations compatible with these actions. This way we obtain the 2-category ${}_c\mathfrak{M}$ of representations of $\mathcal{C}$. Monoidal functors between monoidal categories

$\mathcal{C}$ and $\mathcal{D}$ then induce 2-functors between ${}_{\mathcal{D}}\mathfrak{M}$ and ${}_{\mathcal{C}}\mathfrak{M}$ and monoidal transformations induce 2-natural transformations. However this seems to be too restricted. According to the main principle we would rather be interested in the weakened versions, bicategories, pseudo-functors and pseudo-natural transformations. We do not address this problem as far as 0- and 1-cells are concerned. For 2-cells, however we show how this problem can be overcome, using a natural weakening of the notion of a monoidal transformation, which we call quasi monoidal transformation. Not only does it appear that these induce pseudo natural transformations as desired, it is even true that in a way all pseudo natural transformations are induced by quasi monoidal transformations. More exactly we prove that if $\mathfrak{F}, \mathfrak{G}$ are 2-functors induced by monoidal functors $\mathcal{F}, \mathcal{G}$, respectively, then the category of pseudo natural transformations $\mathfrak{F} \Rightarrow \mathfrak{G}$ is equivalent to the category of quasi monoidal transformations $\mathcal{F} \Rightarrow \mathcal{G}$. This result is the first step towards a reconstruction theory for 2-categories. It yields as a direct corollary that any monoidal category can be reconstructed from its 2-category of representations.

The last chapter then deals with representations of Hopf categories. A Hopf category $\mathcal{H}$ is supplied with a monoidal functor, the diagonal, which is responsible for the monoidal structure on ${}_{\mathcal{H}}\mathfrak{M}$. The same procedure as in the last chapter can then be applied to bimonoidal functors and bimonoidal transformations. They induce monoidal 2-functors and monoidal 2-natural transformations, respectively. We can also apply our weakening procedure by passing from monoidal transformations to quasi monoidal transformations. We will see that monoidal pseudo natural transformations are induced if and only if the corresponding quasi monoidal transformation carries a cogeбра structure.

Since we know already that any pseudo transformation is induced by a quasi monoidal transformation, to prove our main theorem mentioned before we only have to translate the axioms of a 2-braiding into axioms for the corresponding quasi monoidal transformation. We also show that, analogous to the 1-dimensional case, quasitriangular structures on finite Hopf categories $\mathcal{H}$ induce bimonoidal functors $\mathcal{H}^* \rightarrow \mathcal{H}^{\text{op}}$. In a last section we give the group theoretic example of the theory we mentioned already.

In the whole paper we use the following convention concerning axioms and proofs in 2-categories. These have the shape of convex polytopes and if possible we write them so. We then indicate by using bold lines which are the source and target 1-morphisms. We use dashed lines to draw the covered edges. If that 3-dimensional designing becomes too unclear we cut the polytopes along their common source and target 1-morphism and write both faces separately.

I want to express my deep thanks to Professor Dr. Bodo Pareigis for suggesting the subject of this work and for many helpful discussions. I owe special thanks to Dr. Peter Schauenburg for valuable discussions, encouragement and many helpful hints. I thank the working team, Frank Halanke, Harald Ludwig, Ho Hai Phung and Yorck Sommerhäuser for many discussions. I am deeply grateful to the Graduiertenkolleg *Mathematik im Bereich ihrer Wechselwirkung mit der Physik*, for the financial support.

Chapter 1

Bicategories and Braided Monoidal 2-Categories

In this chapter we collect the basic notions from the theory of 2-categories and monoidal 2-categories. We define a version of braided monoidal 2-category, suited for our applications in the later chapters. Additionally we examine the theory of bicategories with just one object, which can be identified with monoidal categories and are of special interest for us. We prove that braided 2-categories with one object are symmetric categories via this identification.

1.1 Bicategories

The notion *bicategory* goes back to Bénabou [Ben]. The striking difference to 2-categories consists in the fact that bicategories do not admit an underlying category, since the associativity law and the unitary law for the composition of 1-morphisms hold only up to an isomorphism.

1.1. Definition: *A bicategory $\mathfrak{A}$ consists of*

- (1) *a class $|\mathfrak{A}| = \text{Obj}(\mathfrak{A})$ of objects, also called 0-cells*
- (2) *for any pair $A, B \in |\mathfrak{A}|$ of objects, a small category $\mathfrak{A}(A, B) := \text{Mor}_{\mathfrak{A}}(A, B)$, whose objects $f : A \rightarrow B$ are called 1-morphisms or 1-cells. Morphisms $\gamma : f \Rightarrow g : A \rightarrow B$ in $\text{Mor}_{\mathfrak{A}}(A, B)$ are called 2-morphisms or 2-cells.*
- (3) *for any triple $A, B, C \in |\mathfrak{A}|$, a functor*

$$\begin{aligned} c_{A,B,C} : \text{Mor}_{\mathfrak{A}}(A, B) \times \text{Mor}_{\mathfrak{A}}(B, C) &\rightarrow \text{Mor}_{\mathfrak{A}}(A, C) \\ (f, g) &\mapsto g \circ f := c_{A,B,C}(f, g) \end{aligned}$$

- (4) *for any 0-cell $A \in |\mathfrak{A}|$, a 1-cell $\mathcal{I}_A := \text{Id}_A \in \text{Mor}_{\mathfrak{A}}(A, A)$*

(5) for each quadruple of objects $A, B, C, D \in \mathfrak{A}$ an isomorphism, natural in $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ called the associativity law

$$\alpha_{h,g,f} : h \circ (g \circ f) \Rightarrow (h \circ g) \circ f.$$

and for each pair of objects $A, B \in \mathfrak{A}$ two in $f : A \rightarrow B$ natural isomorphisms, called the left- and the right unitary law, respectively.

$$\lambda_f : \text{Id}_B \circ f \Rightarrow f \quad \rho_f : f \circ \text{Id}_A \Rightarrow f$$

such that the following diagrams commute for all $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$

$$\begin{array}{ccccc} k \circ (h \circ (g \circ f)) & \xrightarrow{1 \circ \alpha} & k \circ ((h \circ g) \circ f) & \xrightarrow{\alpha} & (k \circ (h \circ g)) \circ f \\ \alpha \downarrow & & & & \downarrow \alpha \circ 1 \\ (k \circ h) \circ (g \circ f) & \xrightarrow{\alpha} & & & ((k \circ h) \circ g) \circ f \end{array}$$

$$\begin{array}{ccc} g \circ (\text{Id}_B \circ f) & \xrightarrow{\alpha} & (g \circ \text{Id}_B) \circ f \\ 1 \circ \lambda \searrow & & \swarrow \rho \circ 1 \\ & g \circ f & \end{array}$$

A bicategory is called a 2-category, if the morphisms α, λ and ρ are the identical morphisms, respectively.

The composition $c_{A,B,C}$ is called *strong* and will be referred to by $f \circ g$. Strong composition of 2-morphisms is often called *horizontal composition*. The composition inside the Mor-categories is called *weak* or *vertical* composition and is denoted by $\alpha \cdot \beta$. If no confusion is possible, we refer to either composition just by juxtaposition.

2-categories admit a more conceptual definition. A 2-category is a category, enriched in the cartesian closed category Cat of all small categories. The paradigmatic example for a 2-category is Cat . A famous example for a bicategory that is not a 2-category is the following one that was invented by Bénabou.

1.2. Example: We define the bicategory Bim of bimodules. Objects of Bim are the unitary rings. For two rings A, B we define $\text{Bim}(A, B)$ as the category ${}_A\mathcal{M}_B$ of (A, B) -bimodules and for $M \in {}_A\mathcal{M}_B$ and $N \in {}_B\mathcal{M}_C$ the composition is defined as the tensor product $M \otimes_B N$ over B . The ring A viewed as (A, A) -bimodul is the unit in $\text{Bim}(A, A)$ and α, λ, ρ are the usual coherence morphisms. Obviously, the coherence laws for a bicategory are satisfied.

Since the notion *bifunctor* is occupied already for functors of two variables, we use the notion pseudofunctor for a homomorphism of bicategories.

1.3. Definition: A lax pseudofunctor $\mathcal{F} : \mathfrak{A} \rightarrow \mathfrak{B}$ between bicategories $\mathfrak{A}, \mathfrak{B}$ consists of

- (i) an object $\mathcal{F}(A) \in \mathfrak{B}$ for each object $A \in \mathfrak{A}$
- (ii) for each $A, B \in |\mathfrak{A}|$ a functor

$$\mathcal{F}_{A,B} : \text{Mor}_{\mathfrak{A}}(A, B) \rightarrow \text{Mor}_{\mathfrak{B}}(\mathcal{F}A, \mathcal{F}B)$$

- (iii) for each $A \xrightarrow{f} B \xrightarrow{g} C$ the natural morphisms

$$\xi : \mathcal{F}_{B,C}(g) \circ \mathcal{F}_{A,B}(f) \Rightarrow \mathcal{F}_{A,C}(g \circ f)$$

$$\zeta : \text{Id}_{\mathcal{F}(A)} \Rightarrow \mathcal{F}(\text{Id}_A)$$

such that for any $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ the following diagrams commute.

$$\begin{array}{ccccc} \mathcal{F}(h) \circ (\mathcal{F}(g) \circ \mathcal{F}(f)) & \xrightarrow{1 \circ \xi} & \mathcal{F}(h) \circ \mathcal{F}(g \circ f) & \xrightarrow{\xi} & \mathcal{F}(h \circ (g \circ f)) \\ \downarrow \alpha & & & & \downarrow \mathcal{F}(\alpha) \\ (\mathcal{F}(h) \circ \mathcal{F}(g)) \circ \mathcal{F}(f) & \xrightarrow{\xi \circ 1} & \mathcal{F}(h \circ g) \circ \mathcal{F}(f) & \xrightarrow{\xi} & \mathcal{F}((h \circ g) \circ f) \end{array}$$

$$\begin{array}{ccccc} \text{Id}_B \circ \mathcal{F}(f) & \xrightarrow{\zeta \circ 1} & \mathcal{F}(\text{Id}_B) \circ \mathcal{F}(f) & \xrightarrow{\xi} & \mathcal{F}(\text{Id}_B \circ f) \\ & \searrow \lambda & & \swarrow \mathcal{F}(\lambda) & \\ & & \mathcal{F}(f) & & \end{array}$$

$$\begin{array}{ccccc} \mathcal{F}(f) \circ \text{Id}_A & \xrightarrow{1 \circ \zeta} & \mathcal{F}(f) \circ \mathcal{F}(\text{Id}_A) & \xrightarrow{\xi} & \mathcal{F}(f \circ \text{Id}_A) \\ & \searrow \rho & & \swarrow \mathcal{F}(\rho) & \\ & & \mathcal{F}(f) & & \end{array}$$

A lax pseudofunctor is called a pseudofunctor, if all morphisms ξ and ζ are isomorphisms. A pseudofunctor is called a 2-functor, if all morphisms ξ and ζ are the identical morphisms.

Lax pseudofunctors with coherence morphisms leading in the opposite direction do not occur here. A lax pseudofunctor is called locally full (resp. faithful), if all functors $\mathcal{F}_{A,B}$ are full (resp. faithful). It is called a local equivalence, if all functors $\mathcal{F}_{A,B}$ are equivalencies. Note that 2-functors between 2-categories are Cat-functors in the sense of enriched category theory.

Lax pseudofunctors can be composed in a natural way. The coherence morphisms ξ, ζ then are defined as: $\xi_{\mathcal{G} \circ \mathcal{F}} : \mathcal{G}\mathcal{F}(g) \circ \mathcal{G}\mathcal{F}(f) \xrightarrow{\xi_{\mathcal{G}}} \mathcal{G}(\mathcal{F}(g) \circ \mathcal{F}(f)) \xrightarrow{\mathcal{G}(\xi_{\mathcal{F}})} \mathcal{G}\mathcal{F}(g \circ f)$ and $\zeta_{\mathcal{G}\mathcal{F}} : \text{Id}_{\mathcal{G}\mathcal{F}(A)} \xrightarrow{\zeta_{\mathcal{G}}} \mathcal{G}(\text{Id}_{\mathcal{F}(A)}) \xrightarrow{\mathcal{G}(\zeta_{\mathcal{F}})} \mathcal{G}\mathcal{F}(\text{Id}_A)$.

Next we define homomorphisms of pseudofunctors. Our terminology here is not consistent with the one in [Gray]. We pursue the policy that a *lax* notion has coherence morphisms, not necessarily invertible, while we use the prefix *pseudo* to indicate that all occurring coherence morphisms are isomorphisms.

1.4. Definition: Let $\mathcal{F}, \mathcal{G} : \mathfrak{A} \rightarrow \mathfrak{B}$ be lax pseudofunctors between bicategories. A lax natural transformation $\chi : \mathcal{F} \Rightarrow \mathcal{G}$ consists of

- (i) for each object $A \in \mathfrak{A}$ a 1-morphism $\chi_A : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$
- (ii) for each pair of objects $A, B \in \mathfrak{A}$ a transformation, natural in $f : A \rightarrow B$

$$\chi_f : \chi_B \circ \mathcal{F}_{A,B}(f) \Rightarrow \mathcal{G}_{A,B}(f) \circ \chi_A$$

such that for any $A \xrightarrow{f} B \xrightarrow{g} C$ the following diagrams commute.

$$\begin{array}{ccc}
 (\chi_C \circ \mathcal{F}(g)) \circ \mathcal{F}(f) & \xrightarrow{\chi \circ 1} & (\mathcal{F}(g) \circ \chi_A) \circ \mathcal{G}(f) \xrightarrow{\alpha^{-1}} \mathcal{G}(g) \circ (\chi_B \circ \mathcal{F}(f)) \xrightarrow{1 \circ \chi} \mathcal{G}(g) \circ (\mathcal{G}(f) \circ \chi_A) \\
 \downarrow \alpha^{-1} & & \downarrow \alpha \\
 \chi_A \circ (\mathcal{F}(g) \circ \mathcal{F}(f)) & & (\mathcal{G}(g) \circ \mathcal{G}(f)) \circ \chi_A \\
 \downarrow 1 \circ \xi_{\mathcal{G}} & & \downarrow \xi_{\mathcal{G}} \circ 1 \\
 \chi_C \circ \mathcal{F}(g \circ f) & \xrightarrow{\chi_{g \circ f}} & \mathcal{G}(g \circ f) \circ \chi_A
 \end{array}$$

$$\begin{array}{ccc}
 \chi_A \circ \mathcal{I}_{\mathcal{F}(A)} & \xrightarrow{\rho} & \chi_A \xrightarrow{\lambda^{-1}} \mathcal{I}_{\mathcal{G}(A)} \circ \chi_A \\
 \downarrow 1 \circ \zeta_{\mathcal{F}} & & \downarrow \zeta_{\mathcal{G}} \circ 1 \\
 \chi_A \circ \mathcal{F}(I_A) & \xrightarrow{\chi_{\text{id}}} & \mathcal{G}(I_A) \circ \chi_A
 \end{array}$$

If all 2-morphisms χ_f are isomorphisms χ is called a pseudo natural transformation. If all 2-morphisms χ_f are identities χ is called a 2-natural transformation.

For lax transformations a weak composition can be defined. If $\chi : \mathcal{F} \Rightarrow \mathcal{G}$ and $\psi : \mathcal{G} \Rightarrow \mathcal{H}$ are lax transformations, then the weak composition $\psi \cdot \chi : \mathcal{F} \Rightarrow \mathcal{H}$ is defined by $(\chi \cdot \psi)_A := \chi_A \circ \psi_A$ and $(\chi \cdot \psi)_f := (\psi_A \circ \chi_A) \circ \mathcal{F}(f) \Rightarrow \mathcal{H}(f) \circ (\psi_A \circ \chi_A)$ by an appropriate composition of α, ψ_f, χ_f and α^{-1} , respectively.

Strong composition of lax natural transformations is more subtle an affair. Let $\mathcal{F}, \mathcal{F}' : \mathfrak{A} \rightarrow \mathfrak{B}$ and $\mathcal{G}, \mathcal{G}' : \mathfrak{B} \rightarrow \mathfrak{C}$ be lax pseudofunctors and $\chi : \mathcal{F} \Rightarrow \mathcal{F}'$ and $\psi : \mathcal{G} \Rightarrow \mathcal{G}'$ be lax natural transformations. It is easy to see that $\psi \circ \mathcal{F} : \mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F} : \mathfrak{A} \rightarrow \mathfrak{C}$ is again a lax natural transformation. But we only can supply $\mathcal{G} \circ \chi : \mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G} \circ \mathcal{F}' : \mathfrak{A} \rightarrow \mathfrak{C}$ with the structure of a lax natural transformation, in case $\mathcal{G}$ is a pseudofunctor, since we have to define

$$(\mathcal{G} \circ \chi)_f := \mathcal{G}(\chi)_f : \mathcal{G}(\chi) \circ \mathcal{G}(\mathcal{F}(f)) \xrightarrow{\xi_{\mathcal{G}}} \mathcal{G}(\chi \circ \mathcal{F}(f)) \xrightarrow{\mathcal{G}(\chi_f)} \mathcal{G}(\mathcal{F}'(f) \circ \chi) \xrightarrow{\xi_{\mathcal{G}}^{-1}} \mathcal{G}(\mathcal{F}'(f)) \circ \mathcal{G}(\chi).$$

If all occurring functors are pseudofunctors, then we can define strong composition in two ways. Namely $\beta \circ \alpha := (\beta \circ \mathcal{F}') \cdot (\mathcal{G} \circ \alpha)$ and $\beta \circ \alpha := (\mathcal{G}' \circ \alpha) \cdot (\beta \circ \mathcal{F})$. In case β is pseudo natural these two compositions are isomorphic, via the modification β_{α} .

There is no canonical direction for the natural transformation χ_f . Following [Gray], we use the notion *colax natural transformation* resp. *copseudo natural transformation*, if χ leads in

the opposite direction. In the sequel we will mostly be concerned with 2-categories. Because of the strict associativity law in 2-categories we can make unlimited use of pasting-diagrams. Therefore we give an equivalent objectwise definition for lax natural transformations.

1.5. Definition: Let $\mathfrak{A}, \mathfrak{B}$ be 2-categories and $\mathcal{F}, \mathcal{G} : \mathfrak{A} \rightarrow \mathfrak{B}$ lax pseudofunctors. A lax natural transformation $\chi : \mathcal{F} \Rightarrow \mathcal{G} : \mathfrak{A} \rightarrow \mathfrak{B}$ consists of a family of 1-morphisms $\chi_A : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$ in $\mathfrak{B}$ and a family of 2-morphisms $\chi_f : \chi_B \circ \mathcal{F}(f) \Rightarrow \mathcal{G}(f) \circ \chi_A$,

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\chi_A} & \mathcal{G}(A) \\ \mathcal{F}(f) \downarrow & \uparrow \chi_f & \downarrow \mathcal{G}(f) \\ \mathcal{F}(B) & \xrightarrow{\chi_B} & \mathcal{G}(B) \end{array}$$

such that for any 2-morphism $\alpha : f \Rightarrow g : A \rightarrow B$ the following diagram, which expresses naturality of χ_f commutes.

$$(1) \quad \begin{array}{ccccc} \mathcal{F}(A) & \xrightarrow{\chi_A} & & \mathcal{G}(A) & \\ \uparrow \chi_g & & & & \uparrow \chi_f \\ \mathcal{F}(f) & \xRightarrow{\mathcal{F}(\alpha)} & \mathcal{F}(g) & \xRightarrow{\mathcal{G}(\alpha)} & \mathcal{G}(g) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(B) & \xrightarrow{\chi_B} & & \mathcal{G}(B) & \end{array}$$

The following two diagrams are the coherence diagrams for associativity and unit, respectively.

$$(2) \quad \begin{array}{ccccc} \mathcal{F}(A) & \xrightarrow{\chi_A} & \mathcal{G}(A) & & \\ \downarrow \xi_{\mathcal{F}} & \searrow \mathcal{F}(f) & \downarrow \xi_{\mathcal{G}} & \searrow \mathcal{G}(f) & \\ \mathcal{F}(B) & \xrightarrow{\chi_B} & \mathcal{G}(B) & & \\ \downarrow \mathcal{F}(gf) & \searrow \mathcal{F}(g) & \downarrow \mathcal{G}(g) & \searrow \mathcal{G}(g) & \\ \mathcal{F}(C) & \xrightarrow{\chi_C} & \mathcal{G}(C) & & \end{array}$$

$$(3) \quad \begin{array}{ccccc} \mathcal{F}(A) & \xrightarrow{\chi_A} & \mathcal{G}(A) & & \\ \uparrow \chi_{\text{Id}_A} & & & & \uparrow \text{Id} \\ \mathcal{F}(\text{Id}_A) & \xRightarrow{\zeta_{\mathcal{F}}} & \mathcal{F}(\text{Id}_A) & \xRightarrow{\zeta_{\mathcal{G}}} & \mathcal{G}(\text{Id}_A) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(A) & \xrightarrow{\chi_A} & \mathcal{G}(A) & & \end{array}$$

In bicategory theory there also exist natural 2-morphisms between natural 1-morphisms, which are called *modifications*.

1.6. Definition: Let $\chi, \psi : \mathcal{F} \Rightarrow \mathcal{G} : \mathfrak{A} \rightarrow \mathfrak{B}$ be two lax natural transformations. A modification $\mathfrak{a} : \chi \Rightarrow \psi$ consists of a family of 2-morphisms $\mathfrak{a}_A : \chi_A \Rightarrow \psi_A$ where $A \in \mathfrak{A}$, such that for any 1-morphism $f : A \rightarrow B \in \mathfrak{A}$ holds

$$(\mathcal{G}(f) \circ \mathfrak{a}_A) \cdot \chi_f = \psi_f \cdot (\mathfrak{a}_B \circ \mathcal{F}(f)).$$

This can be expressed equivalently by the commutativity of the following diagram.

$$\begin{array}{ccc}
 & \xrightarrow{\chi_A} & \\
 \mathcal{F}(A) & \Downarrow \mathfrak{a}_A & \mathcal{G}(A) \\
 & \xleftarrow{\psi_A} & \\
 \uparrow \psi_f & & \downarrow \mathcal{G}(f) \\
 \mathcal{F}(f) & & \\
 & \xrightarrow{\chi_B} & \\
 \mathcal{F}(B) & \Downarrow \mathfrak{a}_B & \mathcal{G}(B) \\
 & \xleftarrow{\psi_B} & \\
 \uparrow \chi_f & &
 \end{array}$$

We define modifications between colax natural transformations by reversing the direction of the vertical 2-morphisms in the diagram. This means we have the equation $(\mathfrak{a}_B \circ \mathcal{F}(f)) \cdot \chi_f = \psi_f \cdot (\mathcal{G}(f) \circ \mathfrak{a}_A)$.

The following facts on bicategories are well known [Gray]. Given two bicategories $\mathfrak{A}, \mathfrak{B}$ we define a new bicategory $\text{Pseud}(\mathfrak{A}, \mathfrak{B})$, consisting of pseudofunctors $\mathcal{F} : \mathfrak{A} \rightarrow \mathfrak{B}$, lax natural transformations $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$ between pseudofunctors and modifications $\mathfrak{a} : \alpha \Rightarrow \beta$. The composition in $\text{Pseud}(\mathfrak{A}, \mathfrak{B})$ is the weak composition of lax natural transformations. Since this composition is defined objectwise in $\mathfrak{B}$, the bicategory $\text{Pseud}(\mathfrak{A}, \mathfrak{B})$ is a 2-category if $\mathfrak{B}$ is. It follows that Bicat , the collection of all bicategories, pseudofunctors, lax natural transformations and modifications, is not a 3-category (3-categories are enriched in 2-Cat), but rather a strict tricategory. (*Strict* since composition of pseudofunctors is of course associative.) Tricategories have been defined in [GPS]. We will not go into the theory of tricategories here.

If $\mathcal{F} : \mathfrak{A}' \rightarrow \mathfrak{A}$ is a pseudofunctor, then

$$\begin{aligned}
 \mathcal{F}^* := \text{Pseud}(\mathcal{F}, \mathfrak{B}) : \text{Pseud}(\mathfrak{A}, \mathfrak{B}) &\rightarrow \text{Pseud}(\mathfrak{A}', \mathfrak{B}) \\
 \mathcal{H} &\mapsto \mathcal{H} \circ \mathcal{F} \\
 (\alpha, \alpha_-) : \mathcal{H} \Rightarrow \mathcal{H}' &\mapsto \alpha \circ \mathcal{F} := (\alpha_{\mathcal{F}}, \alpha_{\mathcal{F}(-)}) : \mathcal{H}\mathcal{F} \Rightarrow \mathcal{H}'\mathcal{F} \\
 \mathfrak{a} : \alpha \Rightarrow \beta &\mapsto \mathfrak{a}_{\mathcal{F}}
 \end{aligned}$$

is a 2-functor. Being a composition of two pseudofunctors, $\mathcal{H} \circ \mathcal{F}$ is a pseudofunctor as well. We already indicated in the remark following definition 1.4 that $\alpha \circ \mathcal{F}$ is in fact lax natural.

Obviously, for any pair of composable modifications $\mathfrak{a}, \mathfrak{b} \in \text{Pseud}(\mathfrak{A}, \mathfrak{B})$, $\mathfrak{a}_{\mathcal{F}}$ and $\mathfrak{b}_{\mathcal{F}}$ are modifications as well and $(\mathfrak{a} \cdot \mathfrak{b})_{\mathcal{F}} = \mathfrak{a}_{\mathcal{F}} \cdot \mathfrak{b}_{\mathcal{F}}$. Note that for composable lax natural transformations α and β holds

$$\mathcal{F}^*(\alpha \circ \beta) = (\alpha \cdot \beta)_{\mathcal{F}(-)} = \alpha_{\mathcal{F}(-)} \circ \beta_{\mathcal{F}(-)} = \mathcal{F}^*(\alpha) \circ \mathcal{F}^*(\beta),$$

whence $\text{Pseud}(\mathcal{F}, \mathfrak{B})$ is always a 2-functor, even if $\mathcal{F}$ is only a pseudofunctor.

Now, let $\chi : \mathcal{F} \Rightarrow \mathcal{G} : \mathfrak{A}' \rightarrow \mathfrak{A}$ be a lax natural transformation. Then, $\text{Pseud}(\chi, \mathfrak{B}) : \text{Pseud}(\mathcal{F}, \mathfrak{B}) \Rightarrow \text{Pseud}(\mathcal{G}, \mathfrak{B}) : \text{Pseud}(\mathfrak{A}', \mathfrak{B}) \rightarrow \text{Pseud}(\mathfrak{A}, \mathfrak{B})$ is a colax natural transformation. For any object $\mathcal{H} \in \text{Pseud}(\mathfrak{A}', \mathfrak{B})$ we have a 1-morphism $\text{Pseud}(\chi, \mathfrak{B})_{\mathcal{H}} := \mathcal{H}(\chi) : \mathcal{H} \circ \mathcal{F} \Rightarrow \mathcal{H} \circ \mathcal{G} \in \text{Pseud}(\mathfrak{A}, \mathfrak{B})$. This is in fact a lax natural transformation as indicated in the remark following definition 1.4. For any 1-morphism, (i.e. a lax natural transformation $\alpha : \mathcal{H} \Rightarrow \mathcal{H}' \in \text{Pseud}(\mathfrak{A}, \mathfrak{B})$) we define a 2-morphism (i.e. a modification) $\text{Pseud}(\chi, \mathfrak{B})_{\alpha} := \alpha_{\chi(-)} : \mathcal{G}^*(\alpha) \circ \text{Pseud}(\chi, \mathfrak{B})_{\mathcal{H}} = \alpha_{\mathcal{G}} \circ \mathcal{H}(\chi) \Rightarrow \mathcal{H}'(\chi) \circ \alpha_{\mathcal{F}} = \text{Pseud}(\chi, \mathfrak{B})_{\mathcal{H}'} \circ \mathcal{F}^*(\alpha) \in \text{Pseud}(\mathfrak{A}', \mathfrak{B})$ by

$$\begin{array}{ccc} \mathcal{H}(\mathcal{F}(X)) & \xrightarrow{\alpha_{\mathcal{F}(X)}} & \mathcal{H}'(\mathcal{F}(X)) \\ \mathcal{H}(\chi_X) \downarrow & \uparrow \alpha_{\chi(X)} & \downarrow \mathcal{H}'(\chi_X) \\ \mathcal{H}(\mathcal{G}(X)) & \xrightarrow{\alpha_{\mathcal{G}(X)}} & \mathcal{H}'(\mathcal{G}(X)) \end{array}$$

We do not give the lengthy proof that this defines in fact a colax natural transformation (note that it is really *colax*). We shall prove a special case of this later on. In case we only handle with pseudo natural transformations we can get rid of the prefix *co*-, by defining the modification $\text{Pseud}(\chi, \mathfrak{B})$ as $\alpha_{\chi(X)}^{-1} : \mathcal{H}'(\chi) \circ \alpha_{\mathcal{F}} \Rightarrow \alpha_{\mathcal{G}} \circ \mathcal{H}(\chi)$ and hence get the 2-morphisms pointing in the right direction.

Note that even in the case when χ is a 2-natural transformation, $\text{Pseud}(\chi, \mathfrak{B})$ is in general only colax natural.

For bicategories there are three versions of duality defined, see [Ben].

- By $\mathfrak{A}^{\text{op}}$ we denote the *dual* bicategory, defined by $\text{Mor}_{\mathfrak{A}^{\text{op}}}(A, B) = \text{Mor}_{\mathfrak{A}}(B, A)$, $\mathcal{I}^{\text{op}} = \mathcal{I}$ and

$$\begin{aligned} c_{A|B,C}^{\text{op}} = c_{C,B,A} & : \text{Mor}_{\mathfrak{A}^{\text{op}}}(A, B) \times \text{Mor}_{\mathfrak{A}^{\text{op}}}(B, C) \cong \text{Mor}_{\mathfrak{A}}(C, B) \times \text{Mor}_{\mathfrak{A}}(B, A) \\ & \ni (g, f) \mapsto f \circ g \in \text{Mor}_{\mathfrak{A}}(C, A) \cong \text{Mor}_{\mathfrak{A}^{\text{op}}}(A, C) \end{aligned}$$

Moreover $\alpha_{h,g,f}^{\text{op}} = \alpha_{f,g,h}^{-1}$, $\lambda_f^{\text{op}} = \rho_f$ and $\rho_f^{\text{op}} = \lambda_f$.

- By $\mathfrak{A}^{\text{co}}$ we denote the *conjugated* bicategory, defined by $\text{Mor}_{\mathfrak{A}^{\text{co}}}(A, B) = \text{Mor}_{\mathfrak{A}}(A, B)^{\text{op}}$, $\mathcal{I}^{\text{co}} = \mathcal{I}$ and $c_{(A,B,C)}^{\text{co}} = c_{(C,B,A)}^{\text{op}}$. Moreover $\alpha_{f,g,h}^{\text{co}} = \alpha_{f,g,h}^{-1}$, $\lambda_f^{\text{co}} = \lambda_f^{-1}$ and $\rho_f^{\text{co}} = \rho_f^{-1}$.
- The *symmetric* bicategory arises as a connection of both and it holds $\mathfrak{A}^{\text{coop}} \simeq \mathfrak{A}^{\text{opco}}$.

1.7. Example: Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. We define a 2-category $\mathcal{C}\text{-Alg}$. The class of objects of $\mathcal{C}\text{-Alg}$ consists of the algebras in $\mathcal{C}$. The 1-morphisms of $\mathcal{C}\text{-Alg}$ are defined to be the algebra morphisms. If $f, g : (A, \nabla_A, \eta_A) \rightarrow (B, \nabla_B, \eta_B)$ are algebra morphisms in $\mathcal{C}$, a 2-cell $R : f \Rightarrow g$ is defined to be a morphism $R : I \rightarrow B$, such that $\nabla_B \circ (R \otimes f) = \nabla_B \circ (g \otimes R) : A \rightarrow B$. We write $g * R := \nabla \circ (g \otimes R)$ and $R * f := \nabla \circ (R \otimes f)$. Then, $\mathcal{C}\text{-Alg}(A, B)$ is a category for any $A, B \in \mathcal{C}\text{-Alg}$, where the weak composition of 2-cells $R : f \Rightarrow g$ and $T : g \Rightarrow h$ is defined by $T \cdot R := T \otimes R : I \cong I \otimes I \rightarrow B \otimes B \xrightarrow{\nabla_B} B$. This composition is known as *convolution*. The strong composition of 1-morphisms is the composition in $\mathcal{C}$. For $R : f \Rightarrow g : A \rightarrow B$ and $T : f' \Rightarrow g' : B \rightarrow C$, we define $f' \circ R : I \xrightarrow{R} B \xrightarrow{f'} C$ and $T \circ f : I \xrightarrow{T} C$. It is readily verified that this defines a 2-category.

There is a 2-functor $*$: $\mathcal{C}\text{-Alg} \rightarrow \text{Cat}$, which maps a $\mathcal{C}$ -algebra A to the category of A -modules and an algebra morphism $f : A \rightarrow B$ to the functor

$$\begin{aligned} {}_f\mathcal{M} : {}_B\mathcal{M} &\rightarrow {}_A\mathcal{M} \\ (M, B \otimes M \xrightarrow{\mu} M) &\mapsto (M, A \otimes M \xrightarrow{f \otimes M} B \otimes M \xrightarrow{\nabla} M) \\ h &\mapsto h \end{aligned}$$

If $R : f \Rightarrow g$ is a 2-morphism then the family of morphisms, defined for any $M \in {}_B\mathcal{M}$ by $\tilde{R}_M : M \xrightarrow{R \otimes M} B \otimes M \xrightarrow{\mu} M$ is a natural transformation ${}_f\mathcal{M} \Rightarrow {}_g\mathcal{M}$. Naturality is plain, since for each B -module morphism $h : M \rightarrow N$ we have

$${}_g\mathcal{M}(h) \circ \tilde{R}_M = h \circ \mu_{B,M} \circ (R \otimes M) = \mu_{B,N} \circ (B \otimes h) \circ (R \otimes M) = \mu_{B,N} \circ (R \otimes N) \circ h = \tilde{R}_N \circ {}_f\mathcal{M}(h)$$

Moreover $\tilde{R}$ is an A -module morphism, since the following diagram commutes.

$$\begin{array}{ccccc} A \otimes M & \xrightarrow{A \otimes R \otimes M} & A \otimes B \otimes M & \xrightarrow{A \otimes \mu} & A \otimes M \\ f \otimes M \downarrow & & g \otimes B \otimes M \downarrow & & \downarrow g \otimes M \\ B \otimes M & & B \otimes B \otimes M & \xrightarrow{B \otimes \mu} & B \otimes M \\ \mu \downarrow & & \nabla \otimes M \downarrow & & \downarrow \mu \\ M & \xrightarrow{R \otimes M} & B \otimes M & \xrightarrow{\mu} & M \end{array}$$

The two right inner squares are obvious. The left inner part, together with the morphism $\mu : B \otimes M \rightarrow M$, commutes since $f * R = R * g$.

It is well known that in case A is a $\mathcal{C}$ -bialgebra, the algebra morphisms $\Delta : A \rightarrow A \otimes A$ and $\varepsilon : A \rightarrow I$ induce a monoidal structure on ${}_A\mathcal{M}$. Bialgebra morphisms $f : A \rightarrow B$ then induce monoidal functors ${}_f\mathcal{M} : {}_A\mathcal{M} \rightarrow {}_B\mathcal{M}$ and a 2-cell $R : f \Rightarrow g : A \rightarrow B$ induces a monoidal transformation, if $R : I \rightarrow B$ is a coalgebra morphism (i.e. R is grouplike and $\varepsilon(R) = 1$).

1.2 Semistrict Braided Monoidal 2-Categories

Using the notion of a tricategory, general monoidal bicategories were defined in [GPS]. Here we do not give the general definition, but make do with a strictified version, the so called semistrict monoidal 2-categories. In [GPS] it was proved that any monoidal bicategory is equivalent to a semistrict monoidal 2-categories. Following [DS], we will refer to semistrict monoidal 2-categories by the shorter name *Gray monoid*.

1.8. Definition: A semistrict monoidal 2-category consists of a 2-category $\mathfrak{C}$ together with the following data:

1. An object $I \in \mathfrak{C}$.
2. For any object $A \in \mathfrak{C}$ two 2-functors $\mathcal{L}_A := A \boxtimes - : \mathfrak{C} \rightarrow \mathfrak{C}$ and $\mathcal{R}_A := - \boxtimes A : \mathfrak{C} \rightarrow \mathfrak{C}$, such that $\mathcal{L}_A(B) = \mathcal{R}_B(A)$ for any pair of objects $A, B \in \mathfrak{C}$.
3. For any two 1-morphisms $f : A \rightarrow A'$ and $g : B \rightarrow B'$ a 2-isomorphism

$$\begin{array}{ccc} A \boxtimes B & \xrightarrow{A \boxtimes g} & A \boxtimes B' \\ f \boxtimes B \downarrow & \Downarrow \boxtimes_{f,g} & \downarrow f \boxtimes B' \\ A' \boxtimes B & \xrightarrow{A' \boxtimes g} & A' \boxtimes B' \end{array}$$

Moreover, these data must satisfy the following conditions.

- (i) The following holds as an equation of 2-functors: $- \boxtimes I = I \boxtimes - = \text{Id} : \mathfrak{C} \rightarrow \mathfrak{C}$
- (ii) We have the following equalities of 2-functors: $A \boxtimes (B \boxtimes -) = (A \boxtimes B) \boxtimes -$, $A \boxtimes (- \boxtimes B) = (A \boxtimes -) \boxtimes B$ and $- \boxtimes (A \boxtimes B) = (- \boxtimes A) \boxtimes B$.
- (iii) For any 1-morphisms $f : A \rightarrow A'$, $g : B \rightarrow B'$ and $h : C \rightarrow C'$ in $\mathfrak{C}$ we have $\boxtimes_{A \boxtimes g, h} = A \boxtimes \boxtimes_{g, h}$, $\boxtimes_{f \boxtimes B, h} = \boxtimes_{f, B \boxtimes h}$ and $\boxtimes_{f, g \boxtimes C} = \boxtimes_{f, g} \boxtimes C$.
- (iv) For any 1-morphism $f : A \rightarrow A'$, any 1-morphisms $g, g' : B \rightarrow B'$, and any 2-morphism $\beta : g \Rightarrow g'$ the following diagram commutes:

$$\begin{array}{ccc} A \boxtimes B & \xrightarrow{\quad \Downarrow A \boxtimes \beta \quad} & A \boxtimes B' \\ \downarrow \boxtimes_{f, g'} & \nearrow & \downarrow \boxtimes_{f, g} \\ A' \boxtimes B & \xrightarrow{\quad \Downarrow A' \boxtimes \beta \quad} & A' \boxtimes B' \end{array}$$

- (v) For any 1-morphism $g : B \rightarrow B'$, any 1-morphisms $f, f' : A \rightarrow A'$, and any 2-morphism $\alpha : f \Rightarrow f'$, the following diagram commutes:

$$\begin{array}{ccc}
 A \boxtimes B & \xrightarrow{\quad \Downarrow \alpha \boxtimes B \quad} & A' \boxtimes B \\
 \uparrow \boxtimes f', g & \searrow & \downarrow \\
 A \boxtimes B' & \xrightarrow{\quad \Downarrow \alpha \boxtimes B' \quad} & A' \boxtimes B' \\
 & \nwarrow & \uparrow \boxtimes f, g
 \end{array}$$

- (vi) For any 1-morphisms $f : A \rightarrow A'$, $g : B \rightarrow B'$ and $g' : B' \rightarrow B''$ the 2-isomorphism $\boxtimes_{f, gg'}$ coincides with the pasting of $\boxtimes_{f, g}$ and $\boxtimes_{f, g'}$ as in the following diagram.

$$\begin{array}{ccccc}
 A \boxtimes B & \longrightarrow & A \boxtimes B' & \longrightarrow & A \boxtimes B'' \\
 \downarrow & & \downarrow \boxtimes_{f, g} & & \downarrow \boxtimes_{f, g'} \\
 A' \boxtimes B & \longrightarrow & A' \boxtimes B' & \longrightarrow & A' \boxtimes B''
 \end{array}$$

- (vii) For any 1-morphisms $f : A \rightarrow A'$, $f' : A' \rightarrow A''$ and $g : B \rightarrow B'$ the 2-isomorphism $\boxtimes_{ff', g}$ coincides with the pasting of $\boxtimes_{f, g}$ and $\boxtimes_{f', g}$ as in the following diagram.

$$\begin{array}{ccccc}
 A \boxtimes B & \longrightarrow & A' \boxtimes B & \longrightarrow & A'' \boxtimes B \\
 \downarrow & & \uparrow \boxtimes_{f, g} & & \downarrow \boxtimes_{f', g} \\
 A \boxtimes B' & \longrightarrow & A' \boxtimes B' & \longrightarrow & A'' \boxtimes B'
 \end{array}$$

Note that there are two possibilities to define the tensor product $f \boxtimes g$ of two 1-morphisms f and g . We define for $f : A \rightarrow A'$, $g : B \rightarrow B'$

$$f \boxtimes g := (f \boxtimes B') \circ (A \boxtimes g) : A \boxtimes B \rightarrow A' \boxtimes B'$$

as the source of the 2-morphism $\boxtimes_{f, g}$ and similar for 2-morphisms.

We call the morphism $\boxtimes_{-, -}$ the *Gray morphism* of the semistrict monoidal 2-category. Note that we can replace the clauses (iv) and (vi), by requiring that for any 1-morphisms $f : A \rightarrow A'$

$$(f \boxtimes -, \boxtimes_{f, -}) : A \boxtimes - \Rightarrow A' \boxtimes -$$

be a pseudo natural transformation. We can replace the clauses (v) and (vii), by requiring that for any 1-morphisms $g : B \rightarrow B'$

$$(- \boxtimes g, \boxtimes_{-,g}) : - \boxtimes B \Rightarrow - \boxtimes B'$$

be a copseudo natural transformation. Indeed, the conditions (vi) and (vii) and the invertibility of the 2-morphism $\boxtimes_{f,g}$ imply that $\boxtimes_{1_A, g} = 1_g$ and $\boxtimes_{f, 1_B} = 1_f$, for any $f : A \rightarrow A'$ and any $g : B \rightarrow B'$. Thus the coherence axiom for the unit of a pseudo natural transformation is satisfied automatically. Moreover, it follows that $\alpha \boxtimes -$ (resp. $- \boxtimes \beta$) is a modification for any 2-morphism $\alpha : f \Rightarrow f' : A \rightarrow A'$, (resp. $\beta : g \Rightarrow g' : B \rightarrow B'$).

The definition of Gray monoids can be stated more elegantly using the notion of a *cubical functor* that was introduced in [GPS].

1.9. Definition: *A semistrict monoidal 2-category consists of a 2-category $\mathfrak{C}$, together with*

1. *an object $\mathcal{I} \in \mathfrak{C}$, the unit or neutral object*
2. *a cubical functor $\boxtimes : \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$, the tensor product*

such that:

- $\boxtimes \circ (1 \times \boxtimes) = \boxtimes \circ (\boxtimes \times 1)$ *holds as an equation of cubical functors*
- $\mathcal{I} \boxtimes - = - \boxtimes \mathcal{I} = \text{Id} : \mathfrak{C} \rightarrow \mathfrak{C}$

It is easy to verify that both definitions are in fact equivalent.

We write $\mathfrak{C}^{\text{sym}} := (\mathfrak{C}, \boxtimes^{\text{sym}}, \mathcal{I})$ for the Gray monoid obtained from $(\mathfrak{C}, \boxtimes, \mathcal{I})$, by reversing the tensor product $A \boxtimes^{\text{sym}} B := B \boxtimes A$. Note that the Gray morphism then reads $\boxtimes_{f,g}^{\text{sym}} := \boxtimes_{g,f}^{-1}$. We also need the monoidal versions of pseudofunctors, lax natural transformations and modifications. But to avoid an overloading of this chapter with definitions, we shift the definitions of these entities to the appendix.

Now we want to equip the Gray monoids with the appropriate notion of braiding. Braided Gray monoids were first defined in [KV] and modified in [BN], where a new axiom has been added. In the definition we state here, we do not assume the braiding to be an equivalence (which does not hold in our applications). Rather we include new data and axioms for the behavior of the braiding on the unit object, which then become necessary.

1.10. Definition:. *Let $(\mathfrak{C}, \boxtimes, \mathcal{I})$ be a Gray monoid. A 2-braid structure for $\mathfrak{C}$ consists of*

1. *A pseudo natural morphism $\mathcal{R} : \boxtimes \Rightarrow \boxtimes^{\text{sym}}$*
2. *Two invertible modifications $\tilde{R}_1(-|-,-)$ and $\tilde{R}_2(-,-|-)$, giving for any objects $A, B, C \in \mathfrak{C}$ the following 2-isomorphisms*

$$\begin{array}{ccc}
A \boxtimes B \boxtimes C & \xrightarrow{\mathcal{R}_{A,B \boxtimes C}} & B \boxtimes C \boxtimes A \\
\mathcal{R}_{A,B} \boxtimes C \searrow \uparrow \tilde{R}_{1(A|B,C)} & & \nearrow B \boxtimes \mathcal{R}_{A,C} \\
& B \boxtimes A \boxtimes C &
\end{array}
\qquad
\begin{array}{ccc}
A \boxtimes B \boxtimes C & \xrightarrow{\mathcal{R}_{A \boxtimes B,C}} & C \boxtimes A \boxtimes B \\
A \boxtimes \mathcal{R}_{B,C} \searrow \uparrow \tilde{R}_{2(A,B|C)} & & \nearrow \mathcal{R}_{A,C} \boxtimes B \\
& A \boxtimes C \boxtimes B &
\end{array}$$

3. Two invertible modifications $\mathfrak{r}_1$ and $\mathfrak{r}_2$, giving for any object $A \in \mathfrak{C}$ the following 2-isomorphisms

$$\begin{array}{ccc}
& \text{Id} & \\
& \curvearrowright & \\
A \boxtimes I & \xrightarrow{\mathcal{R}_{A,I}} & I \boxtimes A \\
& \Downarrow \mathfrak{r}_{1A} &
\end{array}
\qquad
\begin{array}{ccc}
& \text{Id} & \\
& \curvearrowright & \\
I \boxtimes A & \xrightarrow{\mathcal{R}_{I,A}} & A \boxtimes I \\
& \Downarrow \mathfrak{r}_{2A} &
\end{array}$$

These data must satisfy the following relations. For any objects $A, B, C, D \in \mathfrak{C}$ the following diagrams commute:

$$((\bullet \boxtimes \bullet \boxtimes \bullet) \boxtimes \bullet)$$

$$\begin{array}{ccccc}
& & DABC & & \\
& \nearrow & \uparrow 1. & \nwarrow 2. & \\
& & \Leftarrow & & \\
ABCD & \xleftarrow{4.} & & \xrightarrow{3.} & ADBC \\
& \searrow & \uparrow 3. & \nearrow & \\
& & ABDC & &
\end{array}$$

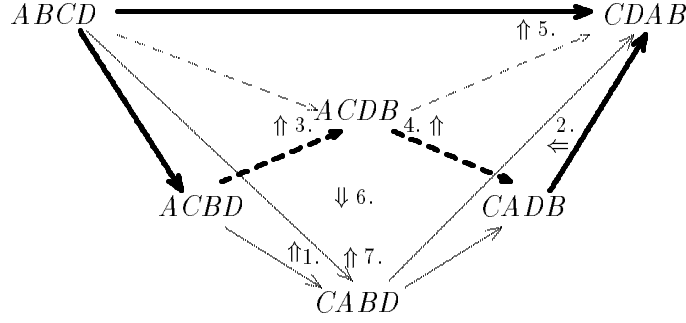
$$\begin{array}{ll}
1. \tilde{R}_{2(A \boxtimes B, C|D)} & 2. = \tilde{R}_{2(A, B|D)} \boxtimes C \\
3. = A \boxtimes \tilde{R}_{2(B, C|D)} & 4. = \tilde{R}_{2(A, B \boxtimes C|D)}
\end{array}$$

$$(\bullet \boxtimes (\bullet \boxtimes \bullet \boxtimes \bullet))$$

$$\begin{array}{ccccc}
& & BCDA & & \\
& \nearrow & \uparrow 1. & \nwarrow 2. & \\
& & \Leftarrow & & \\
ABCD & \xleftarrow{4.} & & \xrightarrow{3.} & BCAD \\
& \searrow & \uparrow 3. & \nearrow & \\
& & BACD & &
\end{array}$$

$$\begin{aligned}
1. &= \tilde{R}_{1(A|B,C \boxtimes D)} & 2. &= B \boxtimes \tilde{R}_{1(A|C,D)} \\
3. &= \tilde{R}_{1(A|B,C)} \boxtimes D & 4. &= \tilde{R}_{1(A|B \boxtimes C,D)}
\end{aligned}$$

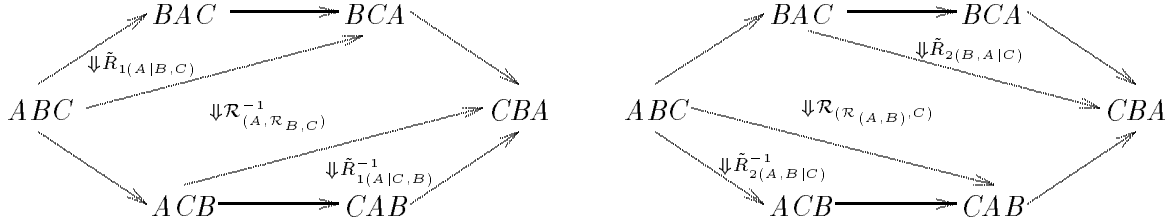
$$((\bullet \boxtimes \bullet) \boxtimes (\bullet \boxtimes \bullet))$$



$$\begin{aligned}
1. &= \tilde{R}_{2(A,B|C)} \boxtimes D & 2. &= C \boxtimes \tilde{R}_{2(A,B|D)} & 3. &= A \boxtimes \tilde{R}_{1(B|C,D)} \\
4. &= \tilde{R}_{1(A|C,D)} \boxtimes B & 5. &= \tilde{R}_{2(A,B|C \boxtimes D)} & 6. &= \boxtimes (\mathcal{R}_{A,C}, \mathcal{R}_{B,D}) \\
7. &= \tilde{R}_{1(A \boxtimes B|C,D)}
\end{aligned}$$

$$(S^+ = S^-)$$

For any objects $A, B, C \in \mathcal{C}$, the following 2-isomorphisms corresponding to the two proofs of the Yang-Baxter hexagon in a braided monoidal category have to agree:

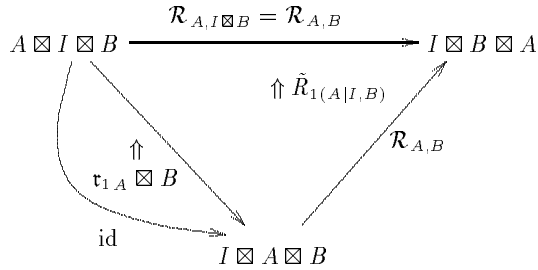


We refer to these 2-morphisms as $S_{A,B,C}^+$ and $S_{A,B,C}^-$, respectively.

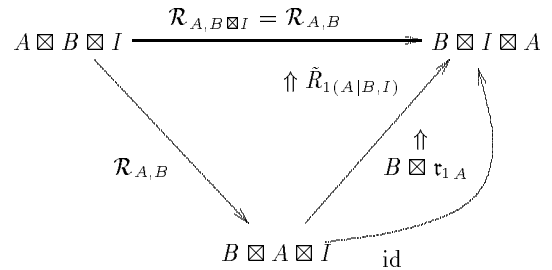
The following seven axioms are the new axioms for the unit object.

For any objects $A, B \in \mathcal{C}$, the following four compositions agree with the identity 2-morphism $\text{id} : \mathcal{R}_{A,B} \Rightarrow \mathcal{R}_{A,B}$.

$$(\bullet \boxtimes (I \boxtimes \bullet))$$



$$(\bullet \boxtimes (\bullet \boxtimes I))$$



$((\bullet \boxtimes I) \boxtimes \bullet)$

$$\begin{array}{ccc}
 A \boxtimes I \boxtimes B & \xrightarrow{\mathcal{R}_{A \boxtimes I, B} = \mathcal{R}_{A, B}} & B \boxtimes A \boxtimes I \\
 \downarrow \scriptstyle A \boxtimes \tau_{2B} & \nearrow \scriptstyle \mathcal{R}_{A, B} & \\
 A \boxtimes B \boxtimes I & &
 \end{array}
 \quad \begin{array}{c} \uparrow \scriptstyle \tilde{R}_{2(A, I|B)} \end{array}$$

 $((I \boxtimes \bullet) \boxtimes \bullet)$

$$\begin{array}{ccc}
 I \boxtimes A \boxtimes B & \xrightarrow{\mathcal{R}_{I \boxtimes A, B} = \mathcal{R}_{A, B}} & B \boxtimes I \boxtimes A \\
 \downarrow \scriptstyle \mathcal{R}_{A, B} & \nearrow \scriptstyle \tau_{2B} \boxtimes A & \\
 I \boxtimes B \boxtimes A & &
 \end{array}
 \quad \begin{array}{c} \uparrow \scriptstyle \tilde{R}_{2(I, A|B)} \end{array}$$

For any objects $A, B \in \mathfrak{C}$, the following two diagrams commute. Here we identify along the identity 1-morphisms.

 $((\bullet \boxtimes \bullet) \boxtimes I)$

$$\begin{array}{ccc}
 A \boxtimes B \boxtimes I & \xrightarrow{\mathcal{R}_{A \boxtimes B, I}} & I \boxtimes A \boxtimes B \\
 \downarrow \scriptstyle A \boxtimes \tau_{1B} & \nearrow \scriptstyle \tau_{1A} \boxtimes B & \\
 A \boxtimes I \boxtimes B & &
 \end{array}
 \quad \begin{array}{c} \uparrow \scriptstyle \tilde{R}_{2(A, B|I)} \end{array}$$

 $(I \boxtimes (\bullet \boxtimes \bullet))$

$$\begin{array}{ccc}
 I \boxtimes A \boxtimes B & \xrightarrow{\mathcal{R}_{I, A \boxtimes B}} & A \boxtimes B \boxtimes I \\
 \downarrow \scriptstyle \tau_{2A} \boxtimes B & \nearrow \scriptstyle A \boxtimes \tau_{2B} & \\
 A \boxtimes I \boxtimes B & &
 \end{array}
 \quad \begin{array}{c} \uparrow \scriptstyle \tilde{R}_{1(I|A, B)} \end{array}$$

Last we require $\tau_{1I} = \tau_{2I}$

 $(I \boxtimes I)$

$$\begin{array}{ccc}
 I \boxtimes I & \xrightarrow{\mathcal{R}_{I, I}} & I \boxtimes I \\
 \downarrow \scriptstyle \tau_{1I} & \nearrow \scriptstyle \tau_{2I} & \\
 I \boxtimes I & &
 \end{array}$$

In [BN] we gave an unraveled version of this definition, though without the unit axioms. We used the hieroglyphical notion introduced in [KV] for the operations and axioms that emerge from this unpacking. In the sequel we make freely use of these hieroglyphs. The new data τ_1 and τ_2 being modifications give rise to two new hieroglyphs in the unpacked version of the definition, namely:

For any 1-morphism $f : A \rightarrow B$, the following prisms commute:

$$\begin{array}{ccc}
(\rightarrow \boxtimes I) & & (I \boxtimes \rightarrow) \\
\begin{array}{c}
\begin{array}{ccc}
& \text{id}_A & \\
A \boxtimes I & \xrightarrow{\quad} & I \boxtimes A \\
\downarrow \tau_{1A} & & \downarrow \tau_{1A} \\
\uparrow \mathcal{R}_{f,I} & \mathcal{R}_{A,I} & \\
f \boxtimes I & \xrightarrow{\quad} & I \boxtimes f \\
\downarrow \tau_{1B} & & \downarrow \tau_{1B} \\
B \boxtimes I & \xrightarrow{\quad} & I \boxtimes B \\
\uparrow \text{id}_B & \mathcal{R}_{B,I} & \\
\end{array}
\end{array}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{ccc}
& \text{id}_A & \\
I \boxtimes A & \xrightarrow{\quad} & A \boxtimes I \\
\downarrow \tau_{2A} & & \downarrow \tau_{2A} \\
\uparrow \mathcal{R}_{I,f} & \mathcal{R}_{I,A} & \\
I \boxtimes f & \xrightarrow{\quad} & f \boxtimes I \\
\downarrow \tau_{2B} & & \downarrow \tau_{2B} \\
I \boxtimes B & \xrightarrow{\quad} & B \boxtimes I \\
\uparrow \text{id}_B & \mathcal{R}_{I,B} & \\
\end{array}
\end{array}
\end{array}$$

The axioms concerning the unit objects are new. They are shaped in a way to fit the requirements of a 2-quasitriangular structure in chapter 5.

1.11. Lemma: *Let $(\mathcal{R}, \tilde{R}_1(-|-,-), \tilde{R}_2(-,-|-), \tau_1, \tau_2)$ be a 2-braid structure for a Gray monoid $(\mathfrak{C}, \boxtimes, I)$. We have:*

1. $\tilde{R}_{1(I|I,I)} = \tilde{R}_{2(I,I|I)}$
2. $\mathcal{R}_{f,I} = \mathcal{R}_{I,f}$ for any 1-morphism $f : I \rightarrow I$

PROOF:

If we specialize $A = B = I$ in the axioms $((\bullet \boxtimes \bullet) \boxtimes I)$ and $(I \boxtimes (\bullet \boxtimes \bullet))$, then from $(I \boxtimes I)$ follows that $\tilde{R}_{1(I|I,I)} = \tilde{R}_{2(I,I|I)}$. Also, if we specialize $A = I$ in $(\rightarrow \boxtimes I)$ and $(I \boxtimes \rightarrow)$, then $(I \boxtimes I)$ ensures that $\mathcal{R}_{f,I} = \mathcal{R}_{I,f}$ for any 1-morphism $f : I \rightarrow I$. $\square$

As noted before, $\text{End}(X)$ is a monoidal category for any Gray monoid $\mathfrak{C}$ and any $X \in \mathfrak{C}$. Now, given a 2-braid structure $(\mathcal{R}, \tilde{R}_1, \tilde{R}_2, \tau_1, \tau_2)$ it is an interesting observation that $(\mathcal{R}_{X,I}, \tilde{R}_1, \tau_1)$ is an algebra in $\text{End}(X)$. The associativity follows from $((\bullet \boxtimes \bullet \boxtimes \bullet) \boxtimes \bullet)$, the unitary laws from $(\bullet \boxtimes (I \boxtimes \bullet))$ and $(\bullet \boxtimes (\bullet \boxtimes I))$, respectively. Since $\mathcal{R}$ is pseudo natural, $\mathcal{R}_{X,I}$ is an object in the center $\mathcal{Z}(\text{End}(X))$ of the monoidal category $\text{End}(X)$. Since $\tilde{R}_1$ is a modification, $\mathcal{R}_{X,I}$ is even as an algebra in $\mathcal{Z}(\text{End}(X))$.

1.12. Definition: *A semistrict braided monoidal 2-category consists of a Gray monoid, together with a 2-braid structure.*

The definitions of the appropriate notions of morphisms and 2-morphisms of semistrict braided monoidal 2-categories can be found in the appendix. As pointed out in [DS], a 2-braiding supplies the functor $\boxtimes$ with a monoidal structure in the sense of A.1.

As noted in the introduction, the 2-braid structures for a given Gray monoid form a category. Now we define the morphisms of this category.

1.13. Definition: Let $(\mathcal{R}, \tilde{R}_1(-|-,-), \tilde{R}_2(-|-,-), \mathfrak{r}_1, \mathfrak{r}_2)$ and $(\mathcal{T}, \tilde{T}_1(-|-,-), \tilde{T}_2(-|-,-), \mathfrak{t}_1, \mathfrak{t}_2)$ be two 2-braid structures for a Gray monoid $(\mathfrak{C}, \boxtimes, \mathcal{I})$. A modification $\mathfrak{a} : \mathcal{R} \Rightarrow \mathcal{T}$ is called a morphism of 2-braid structures, if:

1. for any object $A \in \mathfrak{C}$ holds: $\mathfrak{a}_{A,I} \cdot \mathfrak{r}_{1A} = \mathfrak{t}_1$
2. for any object $A \in \mathfrak{C}$ holds: $\mathfrak{a}_{I,A} \cdot \mathfrak{r}_{2A} = \mathfrak{t}_2$
3. for any objects $A, B, C \in \mathfrak{C}$ the following equations hold

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccc}
 & \xrightarrow{\quad T_{A,B \boxtimes C} \quad} & \\
 \uparrow \mathfrak{a}_{A,B \boxtimes C} & & \\
 ABC & \xrightarrow{\quad \mathcal{R}_{A,B \boxtimes C} \quad} & BCA \\
 \downarrow \mathcal{R}_{A,B} \boxtimes C & & \uparrow B \boxtimes \mathcal{R}_{A,C} \\
 & BAC &
 \end{array} \\
 \\
 \begin{array}{ccc}
 & \xrightarrow{\quad T_{A \boxtimes B,C} \quad} & \\
 \uparrow \mathfrak{a}_{A \boxtimes B,C} & & \\
 ABC & \xrightarrow{\quad \mathcal{R}_{A \boxtimes B,C} \quad} & CAB \\
 \downarrow A \boxtimes \mathcal{R}_{B,C} & & \uparrow \mathcal{R}_{A,C} \boxtimes B \\
 & ACB &
 \end{array}
 \end{array}
 =
 \begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{\quad T_{A,B \boxtimes C} \quad} & \\
 \uparrow \tilde{T}_1(A|B,C) & & \\
 ABC & \xrightarrow{\quad \mathcal{R}_{A,B \boxtimes C} \quad} & BCA \\
 \downarrow \mathfrak{a}_{A,B} \boxtimes C & & \uparrow B \boxtimes \mathfrak{a}_{A,C} \\
 & BAC &
 \end{array} \\
 \\
 \begin{array}{ccc}
 & \xrightarrow{\quad T_{A \boxtimes B,C} \quad} & \\
 \uparrow \tilde{T}_2(A,B|C) & & \\
 ABC & \xrightarrow{\quad \mathcal{R}_{A \boxtimes B,C} \quad} & CAB \\
 \downarrow A \boxtimes \mathfrak{a}_{B,C} & & \uparrow \mathfrak{a}_{A,C} \boxtimes B \\
 & ACB &
 \end{array}
 \end{array}
 \end{array}$$

The category of 2-braid structures for a Gray monoid $(\mathfrak{C}, \boxtimes, \mathcal{I})$ is called $2\text{-Braid}(\mathfrak{C})$. Note that $\mathfrak{a} : \mathcal{R} \Rightarrow \mathcal{T}$ is a morphism of 2-braid structures, if and only if $(\text{id}, \mathfrak{a}) : (\boxtimes, \mathcal{R}) \Rightarrow (\boxtimes, \mathcal{T}) : \mathfrak{C} \boxtimes \mathfrak{C} \rightarrow \mathfrak{C}$ is a monoidal lax natural transformation (c.f. definition A.2).

According to the quantum hierarchy and unlike in the case of 1-categories, being a symmetry is a new datum for a 2-braid structure rather than a condition. We call this new datum an *involution* for the braiding. The following definition is the same as in [DS].

1.14. Definition: Let $(\mathfrak{C}, \boxtimes, \mathcal{I})$ be a Gray monoid and $(\mathcal{R}, \tilde{R}_1, \tilde{R}_2, \mathfrak{r}_1, \mathfrak{r}_2)$ a 2-braid structure. An involution for $\mathcal{R}$ is an invertible modification given for any pair of objects $X, Y \in \mathfrak{C}$ by isomorphisms $\mathfrak{I} : \text{Id}_{A \boxtimes B} \Rightarrow \mathcal{R}_{A,B} \circ \mathcal{R}_{B,A}$, such that the following equations hold.

$$\begin{array}{ccc}
\begin{array}{ccc}
A \boxtimes B \boxtimes C & \xrightarrow{1} & A \boxtimes B \boxtimes C \\
\downarrow 1 \boxtimes \mathcal{R}_{B,C} & \searrow \mathcal{R}_{A \boxtimes B, C} & \uparrow \mathcal{R}_{C, A \boxtimes B} \\
A \boxtimes C \boxtimes B & \xrightarrow{\mathcal{R}_{A,C} \boxtimes B} & C \boxtimes A \boxtimes B
\end{array} & = & \begin{array}{ccc}
A \boxtimes B \boxtimes C & \xrightarrow{1} & A \boxtimes B \boxtimes C \\
\downarrow 1 \boxtimes \mathcal{J}_{B,C} & \searrow \mathcal{J}_{A,C} \boxtimes B & \uparrow \mathcal{R}_{C, A \boxtimes B} \\
A \boxtimes C \boxtimes B & \xrightarrow{\mathcal{R}_{A,C} \boxtimes B} & C \boxtimes A \boxtimes B
\end{array} \\
\begin{array}{ccc}
A \boxtimes B \boxtimes C & \xrightarrow{1} & A \boxtimes B \boxtimes C \\
\downarrow \mathcal{R}_{A, B \boxtimes C} & \searrow \mathcal{R}_{B \boxtimes C, A} & \uparrow \mathcal{R}_{B, A} \boxtimes C \\
B \boxtimes C \boxtimes A & \xrightarrow{B \boxtimes \mathcal{R}_{C, A}} & B \boxtimes A \boxtimes C
\end{array} & = & \begin{array}{ccc}
A \boxtimes B \boxtimes C & \xrightarrow{1} & A \boxtimes B \boxtimes C \\
\downarrow \mathcal{J}_{A, B \boxtimes C} & \searrow \mathcal{J}_{A, B} \boxtimes C & \uparrow \mathcal{R}_{B, A} \boxtimes C \\
B \boxtimes C \boxtimes A & \xrightarrow{B \boxtimes \mathcal{R}_{C, A}} & B \boxtimes A \boxtimes C
\end{array}
\end{array}$$

An involutor is called a *strong involutor* or a *symmetrizer*, if the following pasting of 2-morphisms is the identity morphism on $\mathcal{R}_{X,Y}$.

$$\begin{array}{ccccc}
X \boxtimes Y & \xrightarrow{1} & X \boxtimes Y & \xrightarrow{\mathcal{R}_{X,Y}} & Y \boxtimes X \\
& \searrow \mathcal{R}_{X,Y} & \downarrow \mathcal{J}_{X,Y} & \downarrow \mathcal{J}_{Y,X}^{-1} & \nearrow 1 \\
& & Y \boxtimes X & & \\
& & \uparrow \mathcal{R}_{Y,X} & &
\end{array}$$

It can be proved that for any involutor holds $\mathcal{J}_{X,I} = \mathfrak{r}_{1X}$ and $\mathcal{J}_{I,X} = \mathfrak{r}_{2X}$. Only 1-morphisms of braided Gray monoids can be sensitive to involtors. We state the definition of an *involutory braided pseudofunctor* in the appendix.

1.15. Definition: If $(\mathcal{R}, \mathcal{J}_{\mathcal{R}})$ and $(\mathcal{T}, \mathcal{J}_{\mathcal{T}})$ are two involutory 2-braid structures for a Gray monoid $(\mathfrak{C}, \boxtimes)$ and $\mathfrak{a} : \mathcal{R} \Rightarrow \mathcal{T}$ is a morphism of 2-braidings, then $\mathfrak{a}$ is called *involutory* in case, the following equation holds.

$$\begin{array}{ccc}
A \boxtimes B & \xrightarrow{\text{Id}} & A \boxtimes B \\
\searrow \mathcal{R}_{A,B} & \downarrow \mathcal{J}_{\mathcal{R}} & \nearrow \mathcal{R}_{B,A} \\
& B \boxtimes A &
\end{array}
=
\begin{array}{ccc}
A \boxtimes B & \xrightarrow{\text{Id}} & A \boxtimes B \\
\searrow \mathcal{R}_{A,B} & \downarrow \mathcal{J}_{\mathcal{T}} & \nearrow \mathcal{R}_{B,A} \\
& B \boxtimes A &
\end{array}$$

1.3 One Object Bicategories are Monoidal Categories

The following construction is essential for the sequel. We can identify monoidal categories with bicategories with just one object. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathcal{I}_{\mathcal{C}}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho})$ be a monoidal category. We define a bicategory $\mathfrak{U}(\mathcal{C})$. Therefore, we choose a one object set $\{*\}$, say and define the following data of a bicategory.

$$\begin{aligned} \text{Mor}_{\mathfrak{U}(\mathcal{C})}(*, *) &:= \mathcal{C} \\ c_{*,*,*} &:= \otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \\ \mathcal{I}_{\mathfrak{U}(\mathcal{C})} &:= \mathcal{I}_{\mathcal{C}} \end{aligned}$$

and for any $A, B, C \in \mathcal{C}$ we define $\alpha_{A,B,C} := \tilde{\alpha}_{A,B,C}$, $\lambda_A := \tilde{\lambda}_A$ and $\rho_A := \tilde{\rho}_A$. Obviously, these data satisfy the axioms of a bicategory. As noted before, for any bicategory $\mathfrak{C}$ and any $X \in |\mathfrak{C}|$, the category $\text{Mor}_{\mathfrak{C}}(X, X)$ is equipped with a monoidal structure. Especially, to any bicategory $\mathfrak{C}$ with just one object $\{*\}$, there belongs a monoidal category $\mathfrak{C}^S := \text{Mor}_{\mathfrak{C}}(*, *)$. If the coherence morphisms α, λ and ρ are identities, and hence $\mathfrak{C}$ is a 2-category, then $\mathfrak{C}^S$ is a strict monoidal category. Combining these observations, we can identify monoidal categories with bicategories consisting of one single object.

If $(\mathcal{F}, \mathcal{F}_{*,*}, \xi, \zeta) : \mathfrak{C} \rightarrow \mathfrak{D}$ is a pseudofunctor between bicategories with one object, then $(\mathcal{F}_{*,*}, \xi, \zeta) : \text{Mor}_{\mathfrak{C}}(*, *) \rightarrow \text{Mor}_{\mathfrak{D}}(*, *)$ is a monoidal functor. On the other hand, a monoidal functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ induces a pseudofunctor $\mathcal{F} : \mathfrak{U}(\mathcal{C}) \rightarrow \mathfrak{U}(\mathcal{D})$. Therefore, pseudofunctors correspond to monoidal functors and, obviously, 2-functors to strict monoidal functors.

One might think that due to this identification lax natural transformations between pseudofunctors correspond to monoidal transformations. This is not the case. Rather we define *lax quasi monoidal transformations* between monoidal functors as lax transformations between the corresponding pseudofunctors. It is easy to see that this is equivalent to the following definition. (Do not confuse this notion with the monoidal lax transformations, defined in the appendix.)

1.16. Definition: Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \otimes, I, \alpha, \lambda, \rho)$ be two monoidal categories and $(\mathcal{F}, \xi_{\mathcal{F}}, \zeta_{\mathcal{F}}), (\mathcal{G}, \xi_{\mathcal{G}}, \zeta_{\mathcal{G}}) : \mathcal{C} \rightarrow \mathcal{D}$ be monoidal functors. A lax quasi monoidal transformation $(X, \chi) : (\mathcal{F}, \xi_{\mathcal{F}}, \zeta_{\mathcal{F}}) \Rightarrow (\mathcal{G}, \xi_{\mathcal{G}}, \zeta_{\mathcal{G}})$ is a pair (X, χ) consisting of an object $X \in \mathcal{D}$ and a natural transformation $\chi : X \otimes \mathcal{F} \Rightarrow \mathcal{G} \otimes X$, such that the following diagrams commute.

$$\begin{array}{ccc} X \otimes \mathcal{F}(C) \otimes \mathcal{F}(D) & \xrightarrow{\chi \otimes 1} & \mathcal{G}(C) \otimes X \otimes \mathcal{F}(D) \xrightarrow{1 \otimes \chi} \mathcal{G}(C) \otimes \mathcal{G}(D) \otimes X \\ \downarrow 1 \otimes \xi_{\mathcal{F}} & & \downarrow \xi_{\mathcal{G}} \otimes 1 \\ X \otimes \mathcal{F}(C \otimes D) & \xrightarrow{\chi} & \mathcal{G}(C \otimes D) \otimes X \\ & & \downarrow \\ & & X \otimes I \xrightarrow{\rho} X \xrightarrow{\lambda^{-1}} I \otimes X \\ & & \downarrow 1 \otimes \zeta_{\mathcal{F}} \quad \downarrow \zeta_{\mathcal{G}} \otimes 1 \\ & & X \otimes \mathcal{F}(I) \xrightarrow{\chi} \mathcal{G}(I) \otimes X \end{array}$$

A lax quasi monoidal transformation is called quasi monoidal if χ is an isomorphism.

Now we can define $\otimes$ -modifications between lax quasi monoidal transformations as modifications between the corresponding lax transformations.

1.17. Definition: Let $(X, \chi), (Y, \psi) : \mathcal{F} \Rightarrow \mathcal{G}$ be lax quasi monoidal transformations between monoidal functors. A $\otimes$ -modification $f : (X, \chi) \Rightarrow (Y, \psi)$ is a morphism $f : X \rightarrow Y$ in $\mathcal{D}$, such that the following diagram commutes.

$$\begin{array}{ccc} X \otimes \mathcal{F}(C) & \xrightarrow{\chi} & \mathcal{G}(C) \otimes X \\ f \otimes 1 \downarrow & & \downarrow 1 \otimes f \\ Y \otimes \mathcal{F}(C) & \xrightarrow{\psi} & \mathcal{G}(C) \otimes Y \end{array}$$

Lax quasi monoidal transformations $\mathcal{F} \Rightarrow \mathcal{G}$ together with their $\otimes$ -modifications constitute a category $\text{QMon}(\mathcal{F}, \mathcal{G})$.

1.18. Remark: Note the following special cases of this definition.

- (i) If $X = I$, then $\chi : \mathcal{F} \cong I \otimes \mathcal{F} \xrightarrow{\chi} \mathcal{G} \otimes I \cong \mathcal{G}$ is a monoidal transformation. If $\chi, \psi : \mathcal{G} \Rightarrow \mathcal{F}$ are monoidal transformations, a $\otimes$ -modification f is an endomorphism of the neutral object I such that

$$f \circ \chi = \psi \circ f$$

for the action induced by f on the objects $\mathcal{F}(C)$ and $\mathcal{G}(C)$, respectively.

- (ii) If $\mathcal{F} = \mathcal{G} = \text{Id} : \mathcal{C} \rightarrow \mathcal{C}$, then a quasi monoidal transformation (X, χ) is an object in $\mathcal{Z}(\mathcal{C})$, the center of the monoidal category $\mathcal{C}$, as defined in [JS2]. A $\otimes$ -modification is then a morphism in the center of $\mathcal{C}$, whence we have established a category equivalence $\mathcal{Z}(\mathcal{C}) \simeq \text{QMon}(\text{Id}, \text{Id})$ that is monoidal, as can easily be checked.

- (iii) 2-natural transformations $\mathcal{F} \Rightarrow \mathcal{G}$ correspond to objects $X \in \mathcal{D}$ with $X \otimes \mathcal{F} = \mathcal{G} \otimes X$ as an equation of functors.

Note that a lax quasi monoidal transformation $(X, \chi) : \mathcal{F} \Rightarrow \mathcal{G}$ can also be understood as a colax natural transformation $\mathcal{G} \Rightarrow \mathcal{F}$. Via our identification we recover Mon as a subcategory of Bicat .

Next we examine, what happens with the monoidal category $\mathfrak{C}^S$, if the given one object 2-category $\mathfrak{C}$ is equipped with a Gray monoid structure. Of course, the functors $- \boxtimes *$ and $* \boxtimes -$ are the identity functors and the unique object serves as the unit object. But now, for any pair of 1-morphisms $f, g \in \mathfrak{C}$, and thus for any pair of objects $A, B \in \mathfrak{C}^S$, we have a 2-isomorphism

$$\boxtimes_{f,g} : f \circ g = (f \boxtimes *) \circ (* \boxtimes g) \Rightarrow (* \boxtimes g) \circ (f \boxtimes *) = g \circ f.$$

Hence, this 2-isomorphism corresponds to an isomorphism $\sigma_{A,B} : A \otimes B \rightarrow B \otimes A$ in $\mathfrak{C}^S$. By clauses (iv) and (v) of definition 1.8, σ is natural in both its arguments. The clauses (vi) and (vii) correspond to the two coherence axioms of a braiding, respectively. Thus we have proved the following theorem.

1.19. Theorem: *Via the described identification, Gray monoids with one object correspond to braided strict monoidal categories.*

Of course this theorem holds analogously for arbitrary monoidal bicategories. But since we have not defined this more general notion we do not want to use it here. As the last step in this quantum hierarchy we show that if a one object Gray monoid becomes equipped with a 2-braid structure, the effect for the corresponding 1-category is that the braiding becomes a symmetry. As far as known to the author, this result has not been proved yet.

1.20. Theorem: *Let $(\mathfrak{C}, \boxtimes)$ be a one object Gray monoid. There exists a 2-braid structure $\mathcal{R}$ for $\mathfrak{C}$, if and only if the braided monoidal category $(\mathfrak{C}^S, \otimes, \sigma)$ is symmetric.*

PROOF:

It is easily verified that in case $\mathfrak{C}^S$ is symmetric, the unit object $I \in \mathfrak{C}^S$, serves as a 2-braid structure for $\mathfrak{C}$. For the proof of the reverse statement we use the axioms corresponding to the hieroglyphs $(\rightarrow \boxtimes \rightarrow)$ which is diagram 1.5(1) applied to the 2-morphism $\sigma_{A,B}$, $(\rightarrow \rightarrow \boxtimes \bullet)$ which is an application of diagram 1.5(2) and $(\Downarrow \boxtimes \bullet)$ which is an application of diagram 1.5(1) (c.f. [BN]).

Let $(\mathfrak{C}, \boxtimes)$ be a one object Gray monoid that is equipped with a 2-braid structure $\mathcal{R}$ and let $\mathfrak{C}^S$ be the corresponding braided monoidal 1-category. Let A, B be arbitrary objects in $\mathfrak{C}^S$ (or equivalently 1-morphisms in $\mathfrak{C}$). The 2-braiding $\mathcal{R}$ of $\mathfrak{C}$ gives rise to a distinguished object in the braided monoidal category $(\mathfrak{C}^S, \otimes, \sigma)$, called R and the pseudo natural structure of $\mathcal{R}$ gives rise to two natural isomorphisms $\mathcal{R}_{A,*} : A \otimes R \rightarrow R \otimes A$ and $\mathcal{R}_{*,A} : A \otimes R \rightarrow R \otimes A$. By lemma 1.11 these isomorphisms are equal. In $\mathfrak{C}^S$ the axiom $(\rightarrow \boxtimes \rightarrow)$ can be stated in the following form.

$$(1) \quad \begin{array}{ccccc} A \otimes B \otimes R & \xrightarrow{A \otimes R_{B,*}} & A \otimes R \otimes B & \xrightarrow{R_{*,A} \otimes B} & R \otimes A \otimes B \\ \sigma_{A,B} \otimes R \downarrow & & & & \downarrow R \otimes \sigma_{B,A}^{-1} \\ B \otimes A \otimes R & \xrightarrow{B \otimes R_{*,A}} & B \otimes R \otimes A & \xrightarrow{R_{B,*} \otimes A} & R \otimes B \otimes A \end{array}$$

Recall that $\mathfrak{C}^{\text{sym}}$ is a Gray monoid with Gray morphism $\boxtimes_{A,B}^{\text{sym}} = \boxtimes_{B,A}^{-1}$, whence we have an occurrence of σ^{-1} here and in axiom $(\rightarrow \boxtimes \rightarrow)$. Since we know $R_{*,A} = R_{A,*}$ for any $A \in \mathfrak{C}^S$ we can apply axiom $(\rightarrow \rightarrow \boxtimes \bullet)$, which reads:

$$\begin{array}{ccc}
A \otimes B \otimes R & \xrightarrow{R_{A \otimes B, *}} & R \otimes B \otimes A \\
& \searrow & \nearrow \\
A \otimes R_{B, *} & & R_{A, *} \otimes B \\
& \searrow & \nearrow \\
& A \otimes R \otimes B &
\end{array}$$

Thus we can rewrite (1) as

$$(2) \quad \begin{array}{ccc}
A \otimes B \otimes R & \xrightarrow{R_{A \otimes B, *}} & R \otimes A \otimes B \\
\sigma_{A, B} \otimes R \downarrow & & \uparrow R \otimes \sigma_{B, A} \\
B \otimes A \otimes R & \xrightarrow{R_{B \otimes A, *}} & R \otimes B \otimes A
\end{array}$$

By definition of a pseudo natural transformation $R_{A, *}$ is natural in A (a fact, we expressed by $(\Downarrow \boxtimes \bullet)$ in the hieroglyphical notation). Applying this to the morphism $\sigma_{B, A}$ in (2), we get

$$\begin{array}{ccc}
A \otimes B \otimes R & \xrightarrow{R_{A \otimes B, *}} & R \otimes A \otimes B \\
\sigma_{A, B} \otimes R \downarrow & & \uparrow R_{A \otimes B, *} \\
B \otimes A \otimes R & \xrightarrow{\sigma_{B, A} \otimes R} & A \otimes B \otimes R
\end{array}$$

which yields the result, since $R_{A \otimes B, *}$ is invertible. □

Chapter 2

2-Dimensional Linear Algebra

In this chapter we develop a symmetric Gray monoid, we are working within constantly from now, as our base 2-category. In a way this 2-category is to take the place, the category of vector spaces has, if one deals with ordinary Hopf algebras. We fix once and for all an algebraically closed field k . We denote by vec , (resp. Vec) the category of finite dimensional k -vector spaces (resp. the category of all k -vector spaces). From now on, $\mathfrak{C}_k$ always denotes the 2-category of small k -linear additive categories, with the k -linear functors as 1-morphisms. By k -linear we mean that the categories are enriched in vec . This 2-category admits a Gray monoid structure and a strongly involutory 2-braiding that we develop as a main goal of this chapter. In [KV] the monoidal 2-category 2-Vec of 2-vector spaces was introduced, as the 2-dimensional analog of the category of finite vector spaces. We recover the 2-vector spaces as a sub 2-category of $\mathfrak{C}_k$. Since a good deal of the theory also holds in $\mathfrak{C}_k$, we start our considerations in this more general setup and withdraw to 2-vector spaces only when necessary.

We fix some more notations. Let $k\text{-lin}$ be the 2-category of k -linear small categories and Ab the 2-category of abelian k -linear small categories. Then, Ab is a full subcategory of $\mathfrak{C}_k$ and $\mathfrak{C}_k$ a full subcategory of $k\text{-lin}$.

$$\text{Ab} \xrightarrow{\mathcal{V}_1} \mathfrak{C}_k \xrightarrow{\mathcal{V}_2} k\text{-lin}$$

As 2-morphisms in $\mathfrak{C}_k$, we may choose arbitrary natural transformations, since:

2.1. Lemma: *Any natural transformation between additive functors is additive.*

The area (I) of the diagram in the proof makes clear what is meant by additivity of a natural transformation.

PROOF:

Let $\alpha : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ be a natural transformation between additive functors. Let $X \in \{A, B\}$ and let $i_X : X \hookrightarrow A \oplus B$ be the universal morphism for a chosen biproduct of A and B . The following diagram commutes and therefore, because of the universal property of $\mathcal{F}(i)$, also (I) commutes.

$$\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{\alpha_X} & \mathcal{G}(X) \\
\downarrow \mathcal{F}(i) & & \downarrow \mathcal{G}(i) \\
\mathcal{F}(A \oplus B) & \xrightarrow{\alpha_{A \oplus B}} & \mathcal{G}(A \oplus B) \\
\downarrow \exists_1 \xi_{\mathcal{F}} & \text{(I)} & \downarrow \exists_1 \xi_{\mathcal{G}} \\
\mathcal{F}(A) \oplus \mathcal{F}(B) & \xrightarrow{\alpha_A \oplus \alpha_B} & \mathcal{G}(A) \oplus \mathcal{G}(B)
\end{array}$$

$\swarrow i$ $\searrow$

Note that the uniquely determined morphisms ξ_F occurring in the diagram constitute natural isomorphisms. $\square$

If $\mathcal{C} \in \mathfrak{C}_k$ and $\mathcal{C}' \subset \mathcal{C}$ a full subcategory, we denote by $\mathfrak{A}(\mathcal{C}')$ the full k -linear additive replete subcategory of $\mathcal{C}$, generated by $\mathcal{C}'$.

2.2. Proposition: *Let $\mathcal{C}, \mathcal{D} \in \mathfrak{C}_k$ and $\mathcal{C} \supset \mathcal{C}' \in k\text{-lin}$, such that $\mathfrak{A}(\mathcal{C}') = \mathcal{C}$.*

1. *Any k -linear functor $\mathcal{F} : \mathcal{C}' \rightarrow \mathcal{D}$ has an extension $\mathcal{F}' : \mathcal{C} \rightarrow \mathcal{D}$ that is unique up to isomorphism.*
2. *Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ be k -linear functors. Any natural transformation $\alpha : \mathcal{F}|_{\mathcal{C}'} \Rightarrow \mathcal{G}|_{\mathcal{C}'} : \mathcal{C}' \rightarrow \mathcal{D}$ has a unique extension $\alpha' : \mathcal{F} \Rightarrow \mathcal{G}$. Additionally, α' is an isomorphism if α is.*

PROOF:

For any $X, Y \in \mathcal{C}$ we choose $X_1, \dots, X_n \in \mathcal{C}'$ and $Y_1, \dots, Y_m \in \mathcal{C}'$ such that there are isomorphisms $i_X : X \cong X_1 \oplus \dots \oplus X_n$ and $i_Y : Y \cong Y_1 \oplus \dots \oplus Y_m$. For any $f : X \rightarrow Y$ there exists a unique $m \times n$ matrix of morphisms $(f_{i,j} : X_i \rightarrow Y_j)$, such that $f = i_Y^{-1} \circ (f_{i,j}) \circ i_X$. Then, i_X is natural since $i_Y \circ f = i_Y \circ i_Y^{-1} \circ (f_{i,j}) \circ i_X = (f_{i,j}) \circ i_X$.

To prove part 1., we define $\mathcal{F}'(X) := \mathcal{F}(X_1) \oplus \dots \oplus \mathcal{F}(X_n)$ for an arbitrarily chosen biproduct of $\mathcal{F}(X_1), \dots, \mathcal{F}(X_n)$ and $\mathcal{F}'(f) := (\mathcal{F}(f_{i,j}))$ with respect to this biproduct. This obviously defines a k -linear functor. If $\mathcal{G}$ is another extension of $\mathcal{F}$, then by part 2., the identity on $\mathcal{F}$ extends to a unique natural isomorphism $\mathcal{F}' \Rightarrow \mathcal{G}$.

Now we prove part 2. For $A_1, \dots, A_n \in \mathcal{C}'$ and a k -linear functor $\mathcal{F}$ let $\xi_{\mathcal{F}} : \mathcal{F}(A_1 \oplus \dots \oplus A_n) \rightarrow \mathcal{F}(A_1) \oplus \dots \oplus \mathcal{F}(A_n)$ be the unique isomorphism. Using lemma 2.1, we define for any $X \in \mathcal{C}$

$$\alpha'_X := \mathcal{G}(i_Y)^{-1} \circ \xi_{\mathcal{G}}^{-1} \circ (\alpha_{X_1} \oplus \dots \oplus \alpha_{X_n}) \circ \xi_{\mathcal{F}} \circ \mathcal{F}(i_X) : \mathcal{F}(X) \rightarrow \mathcal{G}(X).$$

Then, α' is natural as a composition of natural morphisms. Now let $\beta : \mathcal{F} \Rightarrow \mathcal{G}$ be another natural transformation with $\beta|_{\mathcal{C}'} = \alpha$. Then again with lemma 2.1:

$$\begin{aligned}
\alpha'_X &= \mathcal{G}(i_X^{-1}) \circ \xi_{\mathcal{G}}^{-1} \circ (\alpha_{A_1} \oplus \dots \oplus \alpha_{A_n}) \circ \xi_{\mathcal{F}} \circ \mathcal{F}(i_X) \\
&= \mathcal{G}(i_X^{-1}) \circ \xi_{\mathcal{G}}^{-1} \circ (\beta_{A_1} \oplus \dots \oplus \beta_{A_n}) \circ \xi_{\mathcal{F}} \circ \mathcal{F}(i_X) = \mathcal{G}(i_X^{-1}) \circ \beta_{A_1 \oplus \dots \oplus A_n} \circ \mathcal{F}(i_X) \\
&= \beta_X
\end{aligned}$$

This proves uniqueness of α' . By lemma 2.1, $\oplus_i \alpha_{A_i}$ is invertible, if $\alpha_{A_1}, \dots, \alpha_{A_n}$ are and hence α' is an isomorphism if α is. $\square$

2.3. Theorem: *The category $\mathfrak{C}_k$ is a pseudo reflective subcategory of $k\text{-lin}$. This means, there is a pseudofunctor $\mathfrak{A} : k\text{-lin} \rightarrow \mathfrak{C}_k$ and a pseudo natural category equivalence $\mathfrak{C}_k(\mathfrak{A}(\mathcal{A}), \mathcal{B}) \simeq k\text{-lin}(\mathcal{A}, \mathcal{V}_2(\mathcal{B}))$.*

PROOF:

First we define for any k -linear category $\mathcal{A}$, a k -linear additive category $\mathfrak{A}(\mathcal{A})$ and show that it has an appropriate universal property. For $\mathcal{A} \in k\text{-lin}$ we define a category $\mathfrak{A}(\mathcal{A})$ whose objects are n -tupel of objects and whose morphisms are matrices of morphisms from $\mathcal{A}$.

- $\text{Obj}(\mathfrak{A}(\mathcal{A})) := \{(A_1, \dots, A_n) | n \in \mathbb{N} \text{ and } A_i \in \mathcal{A}\}$
- $\text{Mor}_{\mathfrak{A}(\mathcal{A})}((A_1, \dots, A_n), (B_1, \dots, B_m)) := \{(f_{i,j})_{j \in \{1, \dots, n\}, i \in \{1, \dots, m\}} \mid f_{i,j} : A_j \rightarrow B_i\}$

It is easy to see that this in fact defines a k -linear additive category, if we take as biproduct: $(A_1, \dots, A_n) \oplus (B_1, \dots, B_m) := (A_1, \dots, A_n, B_1, \dots, B_m)$. The embedding $\mathcal{J}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathfrak{A}(\mathcal{A})$ is a k -linear functor. The notation $\mathfrak{A}(\mathcal{A})$ is justified, since the full repletive subcategory of $\mathfrak{A}(\mathcal{A})$, generated by $\mathcal{J}_{\mathcal{A}}(\mathcal{A})$ is in fact $\mathfrak{A}(\mathcal{A})$. If $\mathcal{X}$ is any k -linear additive category and $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{X}$ is a k -linear functor, then by proposition 2.2 there is a k -linear additive functor $\tilde{\mathcal{F}} : \mathfrak{A}(\mathcal{A}) \rightarrow \mathcal{X}$ with $\mathcal{F} = \tilde{\mathcal{F}} \circ \mathcal{J}$ that is unique up to isomorphism. If $\alpha : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{A} \rightarrow \mathcal{X}$ is a natural transformation, then, also by proposition 2.2, there is a unique natural transformation $\tilde{\alpha} : \tilde{\mathcal{F}} \Rightarrow \tilde{\mathcal{G}} : \mathfrak{A}(\mathcal{A}) \rightarrow \mathcal{X}$.

To make a pseudo functor out of $\mathfrak{A}$, we have to define a family of functors $\mathcal{F}_{\mathcal{A}, \mathcal{B}} : k\text{-lin}(\mathcal{A}, \mathcal{B}) \rightarrow \mathfrak{C}_k(\mathfrak{A}(\mathcal{A}), \mathfrak{A}(\mathcal{B}))$. For $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \in k\text{-lin}$ we define $\mathfrak{A}(\mathcal{F})$ as a representative of the equivalence class of $\tilde{\mathcal{J}}_{\mathcal{D}} \circ \mathcal{F}$. Any natural transformation $\alpha : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D} \in k\text{-lin}$ can be uniquely continued to $\mathfrak{A}(\mathcal{F}) \Rightarrow \mathfrak{A}(\mathcal{G})$. As coherence morphisms $\xi_{\mathfrak{A}} : \mathfrak{A}(\mathcal{F}) \circ \mathfrak{A}(\mathcal{G}) \Rightarrow \mathfrak{A}(\mathcal{F} \circ \mathcal{G})$ and $\zeta_{\mathfrak{A}} : \text{Id}_{\mathfrak{A}} \Rightarrow \mathfrak{A}(\text{Id})$ we take for any $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{F} : \mathcal{B} \rightarrow \mathcal{C}$ the unique extensions of $\xi_{\mathfrak{A}}|_{\mathcal{A}} := \text{id} : \mathcal{F} \circ \mathcal{G} \Rightarrow \mathcal{F} \circ \mathcal{G}$ and $\zeta_{\mathfrak{A}}|_{\mathcal{A}} := \text{id} : \text{Id}_{\mathcal{A}} \Rightarrow \text{Id}_{\mathcal{A}}$.

Thus we have defined the pseudofunctor we were looking for and the universal property gives the category equivalence. It is easy to show the pseudo naturality of this equivalence. $\square$

We show that the category $k\text{-lin}$ admits a 2-tensor product.

2.4. Proposition: *Let $\mathcal{C}, \mathcal{D} \in k\text{-lin}$ be k -linear categories. There is a k -linear category $\mathcal{C} \triangle \mathcal{D}$ and a bilinear functor $\triangle : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \triangle \mathcal{D}$, such that for any bilinear functor $\mathcal{F} : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ there exists a unique k -linear functor $\tilde{\mathcal{F}} : \mathcal{C} \triangle \mathcal{D} \rightarrow \mathcal{E}$ with $\tilde{\mathcal{F}} \circ \triangle = \mathcal{F}$. The category $\mathcal{C} \triangle \mathcal{D}$ is called the tensor product of $\mathcal{C}$ and $\mathcal{D}$.*

PROOF:

We define a category $\mathcal{C} \triangle \mathcal{D}$.

- $\text{Obj}(\mathcal{C} \triangle \mathcal{D}) := \text{Obj}(\mathcal{C} \times \mathcal{D})$. We write $C \triangle D$ for an object $(C, D) \in \text{Obj}(\mathcal{C} \triangle \mathcal{D})$.

- $\text{Mor}(C \triangle D, C' \triangle D') := \text{Mor}(C, C') \otimes_k \text{Mor}(D, D')$.

This category is k -linear and $\triangle$ defines a bilinear functor. The universal property follows from the universal property of $\otimes_k$. $\square$

The tensor product defined this way satisfies a 2-universal property. For any natural transformation between bilinear functors $\mu_X : \mathcal{F}(X) \Rightarrow \mathcal{G}(X)$ there is a unique natural transformation $\tilde{\mu}_X : \tilde{\mathcal{F}}(X) \Rightarrow \tilde{\mathcal{G}}(X)$, such that $\mu = \tilde{\mu} \circ \triangle$. In other words, there is a k -linear isomorphism of categories $\text{Bilin}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong k\text{-lin}(\mathcal{C} \triangle \mathcal{D}, \mathcal{E})$. This holds, since a family of morphisms $\mu_X : \mathcal{F}(X) \Rightarrow \mathcal{G}(X)$ is a natural transformation between bilinear functors, if and only if the same family is a natural transformation $\mu : \tilde{\mathcal{F}} \Rightarrow \tilde{\mathcal{G}}$ between the corresponding k -linear functors. The last point is clear because of the universal property of $\otimes_k$.

2.5. Theorem: *Let $\mathcal{C}, \mathcal{D} \in \mathfrak{C}_k$ be k -linear additive categories. There is a k -linear additive category $\mathcal{C} \boxtimes \mathcal{D}$ and a bilinear functor $\boxtimes : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \boxtimes \mathcal{D}$, such that for all bilinear functors $\mathcal{F} : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ there is a k -linear functor $\tilde{\mathcal{F}} : \mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{E}$ unique up to isomorphism satisfying $\tilde{\mathcal{F}} \circ \boxtimes = \mathcal{F}$. The category $\mathcal{C} \boxtimes \mathcal{D}$ is called the tensor product of $\mathcal{C}$ and $\mathcal{D}$.*

PROOF:

We define $\mathcal{C} \boxtimes \mathcal{D} := \mathfrak{A}(\mathcal{V}_2(\mathcal{C}) \triangle \mathcal{V}_2(\mathcal{D}))$. This means:

- $\text{Obj}(\mathcal{C} \boxtimes \mathcal{D}) := \{(A_1, \dots, A_n) | n \in \mathbb{N} \text{ and } A_i \in \mathcal{C} \triangle \mathcal{D}\}$
- $\text{Mor}((A_1, \dots, A_n), (B_1, \dots, B_m)) := \{(f_{i,j})_{j \in \{1, \dots, n\}, i \in \{1, \dots, m\}} \mid f_{i,j} : A_j \Rightarrow B_i\}$

The universal property follows from the ones of $\triangle$ and $\mathfrak{A}$, respectively. $\square$

The tensor product defined this way satisfies a biuniversal property. For any natural transformation between bilinear functors $\mu_X : \mathcal{F}(X) \Rightarrow \mathcal{G}(X)$ there is a unique natural transformation $\tilde{\mu}_X : \tilde{\mathcal{F}}(X) \Rightarrow \tilde{\mathcal{G}}(X)$, such that $\mu = \tilde{\mu} \circ \boxtimes$. This means there is a k -linear category equivalence $\text{Bilin}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathfrak{C}_k(\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E})$. This follows from the corresponding observation for $k\text{-lin}$ and theorem 2.3. Therefore, the tensor product for $\mathfrak{C}_k$ is a bilimes rather than a 2-limes, since the representation of a bilinear functor will not be unique in general.

Let us summarize in a diagram what we have defined so far.

$$\begin{array}{ccccc}
 \mathcal{C} \times \mathcal{D} & \xrightarrow{\triangle} & \mathcal{C} \triangle \mathcal{D} & \xrightarrow{\mathfrak{A}} & \mathcal{C} \boxtimes \mathcal{D} \\
 & \searrow & \downarrow & & \swarrow \\
 & \forall \mathcal{F} \text{ bilinear} & \exists_1 \tilde{\mathcal{F}} \text{ k-linear} & & \exists_1 \text{ up to iso } \hat{\mathcal{F}} \text{ k-linear} \\
 & & \downarrow & & \\
 & & \mathcal{E} & &
 \end{array}$$

Note that the tensor product for categories we defined also makes sense if we worked with categories, enriched in monoidal categories other than vec . Especially the restriction to finite vector spaces is not necessary.

Because of proposition 2.2 and theorem 2.3 and the definition of $\boxtimes$, all functors $\mathcal{F} : \mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{E}$ in $\mathfrak{C}_k$ are defined up to isomorphism by their values on $\mathcal{C} \triangle \mathcal{D}$. Likewise, the equality of two transformations $\alpha, \beta : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{E}$ has only to be checked on objects the decomposable tensors of $\mathcal{C} \triangle \mathcal{D}$.

2.6. Lemma: *For any $\mathcal{C} \in \mathfrak{C}_k$ there exists a pseudo natural equivalence $\hat{\otimes} : \text{vec} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$.*

PROOF:

It is easy to verify that the bilinear mapping $\hat{\otimes} : \text{vec} \times \mathcal{C} \rightarrow \mathcal{C}$ defined in [Sch1] lemma 2.2.2, has the universal property of the tensor product. The equivalence reads:

$$\begin{aligned} \text{vec} \boxtimes \mathcal{C} &\rightarrow \mathcal{C} \\ k^n \boxtimes X &\mapsto X^n \\ k \boxtimes Y &\leftarrow Y \\ 1 \boxtimes f &\leftarrow f \end{aligned}$$

□

Any k -linear functor is a vec -functor and any k -linear transformation is a vec -transformation. This is proved in [Par3]. Thus, $\hat{\otimes}$ induces a vec -category structure on any $\mathcal{C} \in \mathfrak{C}_k$. Additionally, $V \hat{\otimes} C \boxtimes D \rightarrow C \boxtimes V \hat{\otimes} D \in \mathcal{C} \boxtimes \mathcal{D}$ is an isomorphism natural in V, C, D .

2.7. Theorem: *The 2-categories $\mathfrak{C}_k$ and $k\text{-lin}$ admit the structure of a strongly involutory monoidal 2-category.*

The cautious formulation in the theorem is due to the fact that the tensor product just defined is not strictly associative and thus does not induce a Gray monoid structure. Since any monoidal 2-category is monoidally equivalent to a Gray monoid, we will assume $\boxtimes$ as strict.

PROOF:

We only show the claim for $\mathfrak{C}_k$. Note that $k\text{-lin}$ is symmetric monoidal, even as a 1-category. By its universal property, $\boxtimes$ is defined uniquely up to equivalence (resp. up to isomorphism, resp. uniquely) on the objects (resp. on the 1-morphisms, resp. on the 2-morphisms) of $\mathfrak{C}_k$. By lemma 2.6, the k -linear additive category vec serves as a unit object for $\boxtimes$. We can make the selections on objects and 1-morphisms in such a way that $\boxtimes$ really becomes a strictly associative and unitary 2-functor.

We have to determine the Gray isomorphism $\boxtimes_{-, -}$ in $\mathfrak{C}_k$. By proposition 2.2, it is enough to define $\boxtimes_{-, -}$ on the decomposable tensors $C \boxtimes D \in \mathcal{C} \boxtimes \mathcal{D}$. By our choice of $\boxtimes$ on decomposable tensors it holds for any $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}'$ and $\mathcal{G} : \mathcal{D} \rightarrow \mathcal{D}'$ and any $C \boxtimes D \in \mathcal{C} \boxtimes \mathcal{D}$:

$$(\mathcal{F} \boxtimes 1)(1 \boxtimes \mathcal{G})(C \boxtimes D) = \mathcal{F}(C) \boxtimes \mathcal{G}(D) = (1 \boxtimes \mathcal{G})(\mathcal{F} \boxtimes 1)(C \boxtimes D)$$

Therefore, we may choose $\boxtimes_{\mathcal{F}, \mathcal{G}}$ as the uniquely determined natural isomorphism that is the identity on decomposable functors. The coherence of $\boxtimes_{\cdot, \cdot}$ is obvious, since the identity satisfies all coherence laws, whatsoever. Of course $\boxtimes_{\mathcal{F}, \mathcal{G}}$ is not altogether the identity.

We also determine the braiding and the involutor. The pseudo natural morphism is chosen as the family of functors, defined uniquely up to isomorphism by:

$$\begin{aligned} \mathcal{R}_{\mathcal{C}, \mathcal{D}} : \mathcal{C} \boxtimes \mathcal{D} &\rightarrow \mathcal{C} \boxtimes \mathcal{D} \\ C \boxtimes D &\mapsto D \boxtimes C \\ f \boxtimes g &\mapsto g \boxtimes f \end{aligned}$$

The corresponding family of 2-morphisms are defined uniquely by: $R_{\mathcal{F}, \mathcal{D}}(C, D) := \text{Id}_{D, \mathcal{F}(C)} : D \boxtimes \mathcal{F}(C) \rightarrow D \boxtimes \mathcal{F}(C)$ and $R_{\mathcal{C}, \mathcal{G}}(C, D) := \text{Id}_{\mathcal{G}(D), C} : \mathcal{G}(D) \boxtimes C \rightarrow \mathcal{G}(D) \boxtimes C$. This defines the pseudo natural transformation up to an isomorphism. Note that $\mathcal{R}$ is even an equivalence. We define the corresponding modifications $\tilde{R}_{1(-|\cdot, -)}, \tilde{R}_{2(-, -|\cdot)}, \mathfrak{r}_1$ and $\mathfrak{r}_2$ on decomposable tensors as identities, as well. The axioms of a braided monoidal 2-category are satisfied. Also the involutor $\mathcal{J} : \mathcal{R} \circ \mathcal{R} \Rightarrow \text{Id}$ is defined to be the identity on the decomposable tensors. Plainly, all axioms of a strong involutor are satisfied, then. $\square$

This tensor product admits an inner hom-functor, namely:

2.8. Theorem: *For k -linear additive categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ there is a k -linear category equivalence*

$$\mathfrak{C}_k(\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}) \simeq \mathfrak{C}_k(\mathcal{C}, \mathfrak{C}_k(\mathcal{D}, \mathcal{E}))$$

PROOF:

Obviously, the mapping

$$\begin{aligned} \sim : \text{Bilin}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\rightarrow \mathfrak{C}_k(\mathcal{C}, \mathfrak{C}_k(\mathcal{D}, \mathcal{E})) \\ \mathcal{F} &\mapsto \tilde{\mathcal{F}} : \mathcal{C} \ni X \mapsto \mathcal{F}(X, -) \in \mathfrak{C}_k(\mathcal{D}, \mathcal{E}) \end{aligned}$$

is a bijection on the class of objects. $\sim$ becomes a full and faithful functor, if we define for any natural transformation $\alpha : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ and any $X \in \mathcal{C}$

$$\tilde{\alpha}_X := f(X, -) : \mathcal{F}(X, -) \Rightarrow \mathcal{G}(X, -),$$

Thus, $\sim : \text{Bilin}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathfrak{C}_k(\mathcal{C}, \mathfrak{C}_k(\mathcal{D}, \mathcal{E}))$ is an isomorphism of categories, which is obviously k -linear. The universal property of $\boxtimes$ ensures the existence of the k -linear category equivalence we were looking for. $\square$

For $\mathcal{C}, \mathcal{D} \in \mathfrak{C}_k$, we denote by $[\mathcal{C}, \mathcal{D}]$ the category of all k -linear exact functors $\mathcal{C} \rightarrow \mathcal{D}$ and by $k\text{-lin}_r(\mathcal{C}, \text{vec})$ the category of representable functors. We define the 2-functors $*$: $k\text{-lin} \rightarrow \text{Ab}$ by $\mathcal{C}^* := k\text{-lin}(\mathcal{C}, \text{vec})$, $\circ$: $k\text{-lin} \rightarrow \text{Ab}$ by $\mathcal{C}^\circ := k\text{-lin}_r(\mathcal{C}, \text{vec})$ and $[-, \text{vec}]$: $k\text{-lin} \rightarrow \text{Ab}$ by $[\mathcal{C}, \text{vec}]$.

2.9. Lemma: *For any k -linear categories $\mathcal{C}$ and $\mathcal{D}$ we have:*

$$\begin{aligned}\mathfrak{A}(\mathcal{C} \triangle \mathcal{D}) &\simeq \mathfrak{A}(\mathcal{C}) \boxtimes \mathfrak{A}(\mathcal{D}) \\ \mathcal{C}^* &\simeq \mathfrak{A}(\mathcal{C})^*\end{aligned}$$

PROOF:

The first equation follows with $\mathfrak{A}(\mathcal{C}) \boxtimes \mathfrak{A}(\mathcal{D}) \cong \mathfrak{A}(\mathfrak{A}(\mathcal{C}) \triangle \mathfrak{A}(\mathcal{D}))$ from:

$$\begin{aligned}k\text{-lin}(\mathfrak{A}(\mathcal{C}) \triangle \mathfrak{A}(\mathcal{D}), \mathcal{E}) &\cong \text{bilin}(\mathfrak{A}(\mathcal{C}) \times \mathfrak{A}(\mathcal{D}), \mathcal{E}) \simeq \text{bilin}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \\ &\cong k\text{-lin}(\mathcal{C} \triangle \mathcal{D}, \mathcal{E})\end{aligned}$$

The second equation is clear, since the universal property of $\mathfrak{A}$ implies $k\text{-lin}(\mathcal{C}, \text{vec}) \simeq k\text{-lin}(\mathfrak{A}(\mathcal{C}), \text{vec})$. $\square$

2.10. Example: Let $A \in k\text{-Alg}$ be a finite k -algebra and let $\mathcal{M}_A^f$ be the category of finite flat A -modules, then $[_A\mathcal{M}, \text{vec}] \simeq \mathcal{M}_A^f$.

Indeed, for any $X \in \mathcal{M}_A^f$ the functor $X \otimes_A - \in [_A\mathcal{M}, \text{vec}]$ is exact and k -linear. For $\mathcal{F} : [_A\mathcal{M} \rightarrow \text{vec}]$ any exact k -linear functor, $\mathcal{F}(A)$ is a flat A -module and $\mathcal{F} \cong \mathcal{F}(A) \otimes_A -$, since, as can easily be shown, $\mathcal{F}(X)$ fulfills the universal property of the tensor product.

2.11. Example: Let (A, μ_A, η_A) and (B, μ_B, η_B) be two semisimple finite k -algebras, resp. $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be two finite cosemisimple k -coalgebras. Then

$$[_A\mathcal{M} \boxtimes _B\mathcal{M} \simeq _{A \otimes B}\mathcal{M} \quad \text{resp.} \quad \mathcal{M}^C \boxtimes \mathcal{M}^D \simeq \mathcal{M}^{C \otimes D}$$

PROOF:

For a k -algebra A let $\mathcal{E}(A)$ be the set of equivalence classes of the simple A -modules. Since k is algebraically closed and hence the endomorphism spaces of simple representations are 1-dimensional, the category of representations of a semisimple algebra A is generated up to equivalence by the set $\mathcal{E}(A)$. By definition of the tensor product $[_A\mathcal{M} \boxtimes _B\mathcal{M}$ is determined up to equivalence by $\mathcal{E}(A) \times \mathcal{E}(B)$. Now, by [Bou] chapter 8, page 94, there is a bijection $\mathcal{E}(A) \times \mathcal{E}(B) \cong \mathcal{E}(A \otimes B)$. Hence the functor $\otimes_k : [_A\mathcal{M} \boxtimes _B\mathcal{M} \rightarrow _{A \otimes B}\mathcal{M}$, $M \boxtimes N \mapsto M \otimes_k N$ is an equivalence of categories. $\square$

For k -linear additive categories holds $\mathcal{C}^\circ \boxtimes \mathcal{D}^\circ \simeq (\mathcal{C} \boxtimes \mathcal{D})^\circ$. However, for k -linear categories we do not have as a law: $\mathcal{C}^* \boxtimes \mathcal{D}^* \simeq (\mathcal{C} \triangle \mathcal{D})^*$ and thus with lemma 2.9 for additive categories we do not have in general $(\mathcal{C} \boxtimes \mathcal{D})^* \simeq \mathcal{C}^* \boxtimes \mathcal{D}^*$. For example the tensor product of two abelian categories has not to be abelian again. Consider two k -algebras A and B . We can identify A and B with two one element k -linear categories $\mathcal{C}(A)$ and $\mathcal{C}(B)$, respectively. Then we have $\mathcal{C}(A) \triangle \mathcal{C}(B) \cong \mathcal{C}(A \otimes B)$ and $\mathcal{C}(A)^* \simeq [_A\mathcal{M}$. There is a full and faithful functor $[_A\mathcal{M} \boxtimes _B\mathcal{M} \rightarrow _{A \otimes B}\mathcal{M}$, $M \boxtimes N \mapsto M \otimes N$, that is not surjective in general, though. If A and B are not semisimple then not every $A \otimes B$ -module is of the form $\sum M_i \otimes N_i$, but rather a subquotient thereof. Now we introduce a distinguished class of k -linear additive categories, whose ‘dual spaces’ behave rather good-natured.

2.12. Definition: A 2-vector space is a k -linear additive category $(\mathcal{C}, \oplus)$ that admits a subset $B \subset \mathcal{C}$ with:

- Any object $X \in B$ is simple.
- For any $X \in B$ we have $\dim_k(\text{Hom}(X, X)) = 1$
- For any object $A \in \mathcal{C}$ there is a unique finite subset $B' \subset B$, such that $A \cong \bigoplus_{X_i \in B'} X_i^{n_i}$.

Any set $B \subset \mathcal{C}$ with these properties is called a basis of $\mathcal{C}$. A 2-vector space is called finite, if it admits a finite basis. The 2-category of 2-vector spaces (resp. of finite dimensional 2-vector spaces) is denoted in the sequel with 2-Vec (resp. with 2-vec).

Note that the categories ${}_A\mathcal{M}$ of representations of semisimple algebras, considered in example 2.11 always comprise 2-vector spaces.

2.13. Lemma: Any finite 2-vector space is an abelian category and it holds:

- Any 2-vector space $\mathcal{V} \in 2\text{-vec}$ is defined by its basis up to equivalence.
- Any k -linear functor $\mathcal{F} : \mathcal{V} \rightarrow \mathcal{C}$ with source a 2-vector space and range an arbitrary k -linear additive category is defined by its values on a basis B up to isomorphism and any mapping $B \rightarrow \mathcal{C}$ defines such a functor.
- Any natural transformation $\alpha : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{V} \rightarrow \mathcal{C}$ between k -linear functors is defined uniquely by its values on a basis and any family of morphisms $(\mathcal{F}(B_i) \rightarrow \mathcal{G}(B_i))_{B_i \in B}$ defines a natural transformation.

PROOF:

Let $\mathcal{V}$ be a 2-vector space with basis $B := \{X_1, \dots, X_n\}$. With the notation of proposition 2.2 we have $\mathcal{V} = \mathfrak{A}(B)$. Using lemma 2.6 we can write for any $X, Y \in \mathcal{V}$, $X \cong V_1 \hat{\otimes} X_1 \oplus \dots \oplus V_n \hat{\otimes} X_n$ and $Y \cong W_1 \hat{\otimes} X_1 \oplus \dots \oplus W_n \hat{\otimes} X_n$, with appropriate $V_i, W_j \in \text{vec}$. Then, any morphism $f : X \rightarrow Y$ can be written as

$$f_1 \hat{\otimes} X_1 \oplus \dots \oplus f_n \hat{\otimes} X_n : (X \cong) V_1 \hat{\otimes} X_1 \oplus \dots \oplus V_n \hat{\otimes} X_n \rightarrow W_1 \hat{\otimes} X_1 \oplus \dots \oplus W_n \hat{\otimes} X_n (\cong Y)$$

with appropriate $f_i : V_i \rightarrow W_i$. Especially $\mathcal{C} \simeq \text{vec}^n$ and is thus an abelian category. Any mapping $B \rightarrow \mathcal{C}$ is a k -linear functor and hence by proposition 2.2 defines a k -linear functor $\mathcal{V} \rightarrow \mathcal{C}$, up to isomorphism. Because of the absence of nontrivial morphisms between different elements of the basis, any family of morphisms $\alpha_{X_i} : \mathcal{F}(X_i) \rightarrow \mathcal{G}(X_i)$ defines a natural transformation $\mathcal{F}|_B \Rightarrow \mathcal{G}|_B$. Hence, by proposition 2.2, any natural transformation $\alpha : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{V} \rightarrow \mathcal{C}$ is determined uniquely by its values $\alpha_{X_i} : \mathcal{F}(X_i) \rightarrow \mathcal{G}(X_i)$. $\square$

In [KV] a strictified version 2-vec_c of 2-vec was defined. The set of objects of 2-vec_c is the set $\mathbb{N}$ of natural numbers. The set of 1-morphisms $f : n \rightarrow m$ is the set ${}^n\mathbb{N}^m$ of $n \times m$ matrices

of natural numbers. Composition of 1-morphisms is defined by matrix multiplication. A 2-morphism $\alpha : (n_{i,j}) \Rightarrow (n'_{i,j}) : n \rightarrow m$ is a $n \times m$ matrix $(p_{i,j})$ where $p_{i,j} : \mathbb{R}^{n_{i,j}} \rightarrow \mathbb{R}^{n'_{i,j}}$ are k -linear mappings. By lemma 2.13 we have immediately that 2-vec is biequivalent to 2-vec_c .

2.14. Lemma: *Let $\mathcal{V}, \mathcal{W} \in 2\text{-vec}$ and $\mathcal{V} \simeq \text{vec}^n, \mathcal{W} \simeq \text{vec}^m$. Then $\mathcal{V} \boxtimes \mathcal{W} \simeq \text{vec}^{nm}$.*

PROOF:

The claim follows with lemma 2.6 and the equation $X \boxtimes (Y \oplus Z) \cong X \boxtimes Y \oplus X \boxtimes Z$ for any $X \in \mathcal{V}$ and any $Y, Z \in \mathcal{W}$. $\square$

The following theorem summarizes the essential advantages of 2-vector spaces.

2.15. Theorem: *Let $\mathcal{V}, \mathcal{W} \in 2\text{-vec}$.*

1. *Any k -linear functor $\mathcal{F} : \mathcal{V} \rightarrow \mathcal{W}$ is exact.*
2. *Any k -linear functor $\mathcal{F} : \mathcal{V} \rightarrow \text{vec}$ is representable.*

PROOF:

Both statements were proved by Yetter [Yet]. Part (1) follows from the obvious fact that for 2-vector spaces all short exact sequences split. Part (2) is a k -linear version of Freyd's representability theorem. $\square$

2.16. Corollary: *Let $\mathcal{V}$ be a finite dimensional 2-vector space. There are natural equivalences $\mathcal{C} \simeq \mathcal{C}^{\text{op}} \simeq \mathcal{C}^{\circ} \simeq \mathcal{C}^* \simeq [\mathcal{C}, \text{vec}]$.*

PROOF:

The first equivalence is induced by the functor, which is defined being the identity on the basis, the remaining one are consequences of the last theorem. $\square$

The following result characterizes the 2-vector spaces as k -linear additive categories that admit duals. This theorem is due to Peter Schauenburg.

2.17. Theorem: *Let $\mathcal{C}$ be a k -linear additive category. If there exists a k -linear additive category $\mathcal{C}^*$ and k -linear additive functors $\text{ev} : \mathcal{C}^* \boxtimes \mathcal{C} \rightarrow \text{vec}$ and $\text{db} : \text{vec} \rightarrow \mathcal{C} \boxtimes \mathcal{C}^*$, such that*

- (i) $(\mathcal{C} \xrightarrow{\text{db} \boxtimes 1} \mathcal{C} \boxtimes \mathcal{C}^* \boxtimes \mathcal{C} \xrightarrow{1 \boxtimes \text{ev}} \mathcal{C}) \cong \text{id}_{\mathcal{C}}$
- (ii) $(\mathcal{C}^* \xrightarrow{1 \boxtimes \text{db}} \mathcal{C}^* \boxtimes \mathcal{C} \boxtimes \mathcal{C}^* \xrightarrow{\text{ev} \boxtimes 1} \mathcal{C}^*) \cong \text{id}_{\mathcal{C}^*}$

then $\mathcal{C}$ is a finite 2-vector space. The finite 2-vector space $\mathcal{C}^$ is called a left dual for $\mathcal{C}$.*

PROOF:

For arbitrary $X^* \in \mathcal{C}^*$ and $X \in \mathcal{C}$, we define $X^*(X) := \text{ev}(X^* \boxtimes X)$, and we choose appropriate $\mathcal{C} \ni X_i \neq 0$ and $\mathcal{C}^* \ni X^i \neq 0$, such that $\text{db}(k) = \bigoplus_i X_i \boxtimes X^i$.

Because of axiom (i) we have $X \cong \bigoplus_i X^i(X) \hat{\otimes} X_i$ for any $X \in \mathcal{C}$. Thus, $X^j(X_k) \cong \bigoplus_i X^j(X_i) \hat{\otimes} X^i(X_k)$. The quadratic matrix $M = \dim_k(X^i(X_j))$ is hence idempotent and has only nonnegative integer entries. Since all $X_j \neq 0$ the matrix M contains neither a zero line nor a zero column. By lemma 2.18 we have $M = \text{id}$ and hence $X^i(X_j) \cong \delta_{i,j} k$. Now we conclude $\dim_k(\text{Hom}(X_i, X_j)) = \delta_{i,j}$, since for any $f : X_i \rightarrow X_j$ the following diagram commutes by assumption.

$$\begin{array}{ccc} X_i & \xrightarrow{\cong} & \bigoplus_k X_k \boxtimes X^k(X_i) \\ f \downarrow & & \downarrow \bigoplus_k X_k \boxtimes X^k(f) \\ X_j & \xrightarrow{\cong} & \bigoplus_k X_k \boxtimes X^k(X_j) \end{array}$$

It remains to show that all X_i are simple. Let $0 \neq U \xhookrightarrow{f} X_j$ be a subobject of X_j . Then, we have $U \cong \bigoplus_i X_i \boxtimes X^i(U) \xrightarrow{g} \bigoplus_i X_i \boxtimes X^i(X_j) \cong X_j$ with the monomorphism $g = \bigoplus_i X_i \boxtimes X^i(f)$. Hence for all i , the morphism $X_i \boxtimes X^i(f) : X_i \boxtimes X^i(U) \rightarrow X_i \boxtimes X^i(X_j)$ is mono. This holds, since for $(f_i : X_i \rightarrow Y_i)_{i \in \{1, \dots, n\}}$ an arbitrary family of morphisms in a k -linear, additive category, $\bigoplus f_i : \bigoplus X_i \rightarrow \bigoplus Y_i$ is a monomorphism, if and only if f_i is for all i . We conclude $X^i(U) = 0$ for $i \neq j$ and therefore $U \cong X_j \hat{\otimes} X^j(U)$. Since U is a subobject of X_j , we have $U \cong X_j$. $\square$

2.18. Lemma: *Any idempotent matrix with nonnegative integer entries is either the unit matrix, or has a zero line or a zero column.*

PROOF:

Let $M = (a_{i,j})$ be an idempotent matrix with nonnegative integer entries. Being a projection, M has a positive trace, unless $M = 0$. Therefore, there is an entry in the diagonal of M , $a_{i,i} = 1$, say. If in the i th line there is another entry $a_{i,j} \neq 0$, then the j th line is zero, since we have $a_{i,k} = \sum_m a_{i,m} a_{m,k} \geq a_{i,i} a_{i,k} + a_{i,j} a_{j,k} \geq a_{i,k}$, if $a_{j,k} \neq 0$. Analogously, if in the i th column there is another entry $a_{k,i} \neq 0$, then the k th column is zero. If both is not the case, we can cancel the i th line and column and get an idempotent matrix of a smaller degree. $\square$

The notion $\mathcal{C}^*$ for the left dual of $\mathcal{C}$ was justified, since we have:

2.19. Theorem: *For $\mathcal{V}$ any finite dimensional 2-vector space, $\mathcal{V}^* := k\text{-lin}(\mathcal{V}, \text{vec})$ is a left dual for $\mathcal{V}$.*

PROOF:

Choose a basis $B = \{X_1, \dots, X_n\}$ of $\mathcal{V}$ and the dual basis $B = \{X^1, \dots, X^n\}$ of $\mathcal{V}^*$. $\mathcal{V}^*$ is a left dual for $\mathcal{V}$, with $\text{db} : \text{vec} \rightarrow \mathcal{V} \boxtimes \mathcal{V}^*, k \mapsto \bigoplus_i X_i \boxtimes X^i$. and $\text{ev} : \mathcal{V}^* \boxtimes \mathcal{V} \rightarrow \text{vec}, F \boxtimes X \mapsto F(X)$.

We only show $(\mathcal{V} \xrightarrow{\text{db} \boxtimes 1} \mathcal{V} \boxtimes \mathcal{V}^* \boxtimes \mathcal{V} \xrightarrow{1 \boxtimes \text{ev}} \mathcal{V}) \cong \text{id}_{\mathcal{V}}$. Let $X \in \mathcal{V}$ be an object and $X \cong \bigoplus V_i \hat{\otimes} X_i$ be a basis representation of X .

$$X \mapsto \bigoplus X_i \boxtimes X^i \boxtimes X \mapsto \bigoplus X_i \boxtimes X^i(X) \cong \bigoplus X_i \boxtimes X^i(\bigoplus V_j \hat{\otimes} X_j) \cong \bigoplus V_i \hat{\otimes} X_i \cong X$$

This assignment is obviously natural in X . $\square$

2.20. Example: Let A be a semisimple finite k -algebra. We already know that ${}_A\mathcal{M}$ is a finite 2-vector space. We can identify ${}_A\mathcal{M}^* \simeq \mathcal{M}_A$, with $\text{db} : \text{vec} \rightarrow {}_A\mathcal{M} \boxtimes \mathcal{M}_A, k \mapsto \bigoplus X_i \boxtimes X^i$, where X_i and X^i are representatives of the simple left and right A -modules, respectively. The evaluation is then defined by $\text{ev} : {}_A\mathcal{M}^* \boxtimes {}_A\mathcal{M} \rightarrow \text{vec}, M \boxtimes N \mapsto M \otimes_A N$.

Quasitriangular Bialgebras

As a last point of this chapter we show how the tensor product of categories can be used, to characterize quasitriangular bialgebras. Recall that a k -bialgebra H , together with an element $R := R^1 \otimes R^2 \in H \otimes H$ is called a quasitriangular bialgebra, if the following axioms are satisfied.

$$\text{QT1} \quad \Delta^{\text{op}}(x)R = R\Delta(x)$$

$$\text{QT2} \quad R^1 r^1 \otimes R^2 \otimes r^2 = R^1 \otimes R_{(1)}^2 \otimes R_{(2)}^2$$

$$\text{QT3} \quad R^1 \otimes r^1 \otimes r^2 R^2 = R_{(1)}^1 \otimes R_{(2)}^1 \otimes R^2$$

$$\text{QT4} \quad \varepsilon(R^1) \otimes R^2 = 1$$

$$\text{QT5} \quad \varepsilon(R^2) \otimes R^1 = 1$$

For $R^1 \otimes R^2$ invertible, the axioms QT4 and QT5 follow from QT2 and QT3 , respectively. It is well known that for a finite bialgebra H the axioms QT2 - QT5 are equivalent to $R : H^* \ni f \mapsto f(R^2)R^1 \in H^{\text{op}}$ being a bialgebra morphism. In our 2-category theoretical setup we can make a somewhat more general statement for arbitrary bialgebras.

Let $(\mathcal{C}, \otimes, \mathcal{I}, \tau)$ be a braided monoidal category and $(H, \nabla, \eta, \Delta, \varepsilon)$ a bialgebra in $\mathcal{C}$. Then $(H, \Delta, \varepsilon, \nabla, \eta)$ is a bialgebra in $\mathcal{C}^{\text{op}}$ and hence, since the neutral object is a bialgebra in $\mathcal{C}$ and $\mathcal{C}^{\text{op}}$ anyway, $H \boxtimes I$ and $I \boxtimes H$ are bialgebras in $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$. Now we can define a *prequasitriangular* bialgebra $(H, \nabla, \eta, \Delta, \varepsilon, R)$ consisting of a bialgebra $H \in \mathcal{C}$ and a bialgebra morphism $R := R^1 \boxtimes R^2 : I \boxtimes H \rightarrow H \boxtimes I \in \mathcal{C} \boxtimes \mathcal{C}^{\text{op}}\text{-Cat}$. By definition of the tensor product of k -linear additive categories $R \in \text{Hom}_{\mathcal{C}}(I, H) \otimes_k \text{Hom}_{\mathcal{C}^{\text{op}}}(H, I) = \text{Hom}_{\mathcal{C}}(I, H) \otimes_k \text{Hom}_{\mathcal{C}}(I, H)$. R is an algebra morphism, thus it commutes

$$\begin{array}{ccc} I \boxtimes H \otimes_{\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}} I \boxtimes H & \xrightarrow{R \otimes r} & H \boxtimes I \otimes_{\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}} H \boxtimes I \\ \nabla_{I \boxtimes H} \downarrow & & \downarrow \nabla_{H \boxtimes I} \\ I \boxtimes H & \xrightarrow{R} & H \boxtimes I \end{array}$$

We have $\nabla_{I \boxtimes H} = \nabla_I \boxtimes \Delta_H \in \text{Hom}_{\mathcal{C}}(I \otimes I, I) \otimes_k \text{Hom}_{\mathcal{C}}(H, H \otimes H)$ and therefore

$$R \circ \nabla_{H \boxtimes I} = R^1 \circ \nabla_I \boxtimes \Delta_H \circ R^2 \in \text{Hom}(I \otimes I, H) \otimes_k \text{Hom}(I, H \otimes H)$$

We have $\nabla_{H \boxtimes I} = \nabla_H \boxtimes \Delta_I \in \text{Hom}_{\mathcal{C}}(H \otimes H, H) \otimes_k \text{Hom}_{\mathcal{C}}(I, I \otimes I)$, whence

$$\nabla_{H \boxtimes I} \circ (R \otimes_{\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}} r) = \nabla_H \circ (R^1 \otimes r^1) \boxtimes (R^2 \otimes r^2) \circ \Delta_I \in \text{Hom}(I \otimes I, H) \otimes_k \text{Hom}(I, H \otimes H)$$

Hence we get the following diagram in $\mathcal{C} \boxtimes \mathcal{C}$.

$$\begin{array}{ccc}
 I \otimes I \boxtimes I \otimes I & \xrightarrow{R^1 \otimes r^1 \boxtimes R^2 \otimes r^2} & H \otimes H \boxtimes H \otimes H \\
 \begin{array}{c} 1 \boxtimes \Delta_I \nearrow \\ I \otimes I \boxtimes I \\ \searrow \nabla_I \boxtimes 1 \end{array} & & \begin{array}{c} \searrow \nabla \boxtimes 1 \\ H \otimes H \boxtimes H \\ \nearrow 1 \boxtimes \Delta \end{array} \\
 I \otimes I & \xrightarrow{R^1 \boxtimes R^2} & H \otimes H
 \end{array}$$

The functor $\otimes : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$ induces a k -linear map

$$\text{Hom}(A, B) \otimes_k \text{Hom}(A', B') \ni f \otimes_k g \mapsto f \otimes g \in \text{Hom}(A \otimes A', B \otimes B')$$

for all $A, A', B, B' \in \mathcal{C}$. Therefore, the last diagram implies (QT2), if we apply the functor $\otimes$. Analogously one shows that the compatibility of R with the unit implies QT4 and that $\mathcal{R}$ being a coalgebra morphism implies QT3 and QT5.

As noted above, we can define a tensor product of categories that are enriched over the monoidal categories Vec , of all vector spaces. For the categories $\mathcal{C} = \text{vec}$, resp. $\mathcal{C} = \text{Vec}$ we have $\text{Hom}_{\mathcal{C}}(I, H) \otimes_k \text{Hom}_{\mathcal{C}}(I, H) \cong \text{Hom}_{\mathcal{C}}(I, H \otimes H)$ and hence, R being a bialgebra morphism is in fact equivalent with the axioms QT2-QT5. In many other cases (e.g. the category of representations of a not necessarily semisimple bialgebra H) there is an injective embedding, at least.

Chapter 3

Hopf Categories

Monoidal categories can be interpreted as algebra objects in the 2-category $\mathfrak{C}_k$. Having supplied $\mathfrak{C}_k$ with a tensor product in the last chapter, we now also want to define coalgebra and Hopf algebra objects in $\mathfrak{C}_k$. This leads us to the notions *comonoidal category* or *2-coalgebra* resp. *bimonoidal category* or *2-bialgebra*. Furthermore, we want to develop additional algebraic structure for an algebra A , in a way that the category ${}_A\mathcal{M}$ of representations of A becomes a comonoidal, resp. a bimonoidal category. We presuppose that all occurring categories live in $\mathfrak{C}_k$. So, talking about a category $\mathcal{C}$ we always mean a k -linear additive category. Hence we can make use of the results in the last chapter and define functors etc. only on decomposable tensors.

3.1 Comonoidal Categories

We could give a conceptual definition of comonoidal categories by saying that comonoidal categories in $\mathfrak{C}_k$ are algebra objects in the dual 2-category $\mathfrak{C}_k^{\text{op}}$. Yet, we want to make things a bit more explicit and explain comonoidal structures in elementary terms.

3.1. Definition: A comonoidal category $(\mathcal{C}, \blacklozenge, \Gamma, \alpha_c, \lambda_c, \rho_c)$ consists of a category $\mathcal{C} \in \mathfrak{C}_k$, two k -linear functors

$$\blacklozenge : \mathcal{C} \rightarrow \mathcal{C} \boxtimes \mathcal{C} \quad \text{and} \quad \Gamma : \mathcal{C} \rightarrow \text{vec}$$

and the natural isomorphisms

$$\begin{aligned} \alpha_c : (1 \boxtimes \blacklozenge) \circ \blacklozenge &\Rightarrow (\blacklozenge \boxtimes 1) \circ \blacklozenge \\ \lambda_c : (\Gamma \boxtimes 1) \circ \blacklozenge &\Rightarrow \hat{\otimes}^{-1} \\ \rho_c : (1 \boxtimes \Gamma) \circ \blacklozenge &\Rightarrow \hat{\otimes}^{-1}, \end{aligned}$$

called coassociativity law and left- resp. right unitary law, such that the following pentagon and triangle relation are satisfied.

$$\begin{array}{ccc}
(1 \boxtimes 1 \boxtimes \blacklozenge)(1 \boxtimes \blacklozenge) & \xrightarrow{(1 \boxtimes 1 \boxtimes \blacklozenge)\alpha_c} & (\blacklozenge \boxtimes \blacklozenge)\blacklozenge \\
\downarrow (1 \boxtimes \alpha_c)\blacklozenge & & \downarrow (\blacklozenge \boxtimes 1 \boxtimes 1)\alpha_c \\
(1 \boxtimes \blacklozenge \boxtimes 1)(1 \boxtimes \blacklozenge) & & (\blacklozenge \boxtimes 1 \boxtimes 1)\alpha_c \\
\downarrow (1 \boxtimes \blacklozenge \boxtimes 1)\alpha_c & & \downarrow (\alpha_c \boxtimes 1)\blacklozenge \\
(1 \boxtimes \blacklozenge \boxtimes 1)(\blacklozenge \boxtimes 1) & \xrightarrow{(\alpha_c \boxtimes 1)\blacklozenge} & (\blacklozenge \boxtimes 1 \boxtimes 1)(\blacklozenge \boxtimes 1)\blacklozenge
\end{array}$$

$$\begin{array}{ccc}
(1 \boxtimes (\Gamma \boxtimes 1))(1 \boxtimes \blacklozenge) & \xrightarrow{(1 \boxtimes \Gamma \boxtimes 1)\alpha_c} & ((1 \boxtimes \Gamma) \boxtimes 1)(\blacklozenge \boxtimes 1)\blacklozenge \\
\searrow & & \swarrow \\
& \blacklozenge &
\end{array}$$

A comonoidal category is called *strict*, if all morphisms $\alpha_c, \lambda_c, \rho_c$ are the identities. Comonoidal categories are also called 2-coalgebras.

The well known coherence theorem for monoidal categories also holds here (c.f. [CY3]). In the sequel we will often assume comonoidal categories as strict. This coherence theorem and Proposition 2.2 allow to use a generalized Sweedler notation. We define

$$\blacklozenge(X) =: \bigoplus_X X_{(1)} \boxtimes X_{(2)}$$

but in most cases we will not write the sum symbol explicitly. If we nevertheless want to refer to α_c , say, we write things like:

$$\alpha_c : X_{(1)} \boxtimes (X_{(2)} \boxtimes X_{(3)}) \rightarrow (X_{(1)} \boxtimes X_{(2)}) \boxtimes X_{(3)}$$

in situations where this is possible, otherwise we use the more exact form

$$\alpha_{cX} : X_{(1)} \boxtimes X_{(2,1)} \boxtimes X_{(2,2)} \rightarrow X_{(1,1)} \boxtimes X_{(1,2)} \boxtimes X_{(2)}$$

3.2. Definition: A weak comonoidal functor $(\mathcal{F}, \xi, \zeta)$ between two comonoidal categories $(\mathcal{C}, \blacklozenge, \Gamma, \alpha_c, \lambda_c, \rho_c)$ and $(\mathcal{D}, \blacklozenge, \Gamma, \alpha_c, \lambda_c, \rho_c)$ consists of a k -linear functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$, two natural morphisms $\xi : (\mathcal{F} \boxtimes \mathcal{F}) \circ \blacklozenge_{\mathcal{C}} \Rightarrow \blacklozenge_{\mathcal{D}} \circ \mathcal{F}$ and $\zeta : \Gamma_{\mathcal{C}} \Rightarrow \Gamma_{\mathcal{D}} \circ \mathcal{F}$, such that the following diagrams commute:

$$\begin{array}{ccc}
\mathcal{F}(X_{(1)}) \boxtimes (\mathcal{F}(X_{(2)}) \boxtimes \mathcal{F}(X_{(3)})) & \xrightarrow{(1 \boxtimes \xi)} & \mathcal{F}(X_{(1)}) \boxtimes (\mathcal{F}(X_{(2)})_{(1)} \boxtimes \mathcal{F}(X_{(2)})_{(2)}) \xrightarrow{(1 \boxtimes \blacklozenge) \circ \xi} \mathcal{F}(X)_{(1)} \boxtimes (\mathcal{F}(X)_{(2)} \boxtimes \mathcal{F}(X)_{(3)}) \\
\downarrow (\mathcal{F} \boxtimes \mathcal{F} \boxtimes \mathcal{F})(\alpha_c) & & \downarrow \alpha_c \\
(\mathcal{F}(X_{(1)}) \boxtimes \mathcal{F}(X_{(2)})) \boxtimes \mathcal{F}(X_{(3)}) & \xrightarrow{(\xi \boxtimes 1)} & (\mathcal{F}(X_{(1)})_{(1)} \boxtimes \mathcal{F}(X_{(1)})_{(2)}) \boxtimes \mathcal{F}(X_{(2)}) \xrightarrow{(\blacklozenge \boxtimes 1) \circ \xi} (\mathcal{F}(X)_{(1)} \boxtimes \mathcal{F}(X)_{(2)}) \boxtimes \mathcal{F}(X)_{(3)}
\end{array}$$

$$\begin{array}{ccccc}
\Gamma(X_{(1)}) \boxtimes \mathcal{F}(X_{(2)}) & \xrightarrow{\zeta \boxtimes 1} & \Gamma(\mathcal{F}(X_{(1)})) \boxtimes \mathcal{F}(X_{(2)}) & \xrightarrow{\xi} & \Gamma(\mathcal{F}(X)_{(1)}) \boxtimes \mathcal{F}(X)_{(2)} \\
& \searrow (1 \boxtimes \mathcal{F})\lambda_c & & \swarrow \lambda_c & \\
& & \mathcal{F}(X) & & \\
\mathcal{F}(X_{(1)}) \boxtimes \Gamma(X_{(2)}) & \xrightarrow{1 \boxtimes \zeta} & \mathcal{F}(X_{(1)}) \boxtimes \hat{\Gamma}(\mathcal{F}(X_{(2)})) & \xrightarrow{\xi} & \mathcal{F}(X)_{(1)} \boxtimes \Gamma(\mathcal{F}(X)_{(2)}) \\
& \searrow (\mathcal{F} \boxtimes 1)\rho_c & & \swarrow \rho_c & \\
& & \mathcal{F}(X) & &
\end{array}$$

If all morphisms ξ, ζ are isomorphisms, then $(\mathcal{F}, \xi, \zeta)$ is called a comonoidal functor.

3.3. Definition: A natural transformation $\chi : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ is called comonoidal, if the following diagrams commute:

$$\begin{array}{ccc}
\mathcal{F}(X_{(1)}) \boxtimes \mathcal{F}(X_{(2)}) & \xrightarrow{\xi_{\mathcal{F}}} & \mathcal{F}(X)_{(1)} \boxtimes \mathcal{F}(X)_{(2)} \\
\chi \boxtimes \chi \downarrow & & \downarrow \blacklozenge(\chi) \\
\mathcal{G}(X_{(1)}) \boxtimes \mathcal{G}(X_{(2)}) & \xrightarrow{\xi_{\mathcal{G}}} & \mathcal{G}(X)_{(1)} \boxtimes \mathcal{G}(X)_{(2)}
\end{array}
\quad
\begin{array}{ccc}
& \Gamma(X) & \\
\zeta_{\mathcal{F}} \swarrow & & \searrow \zeta_{\mathcal{G}} \\
\Gamma(\mathcal{F}(X)) & \xrightarrow{\Gamma(\chi)} & \Gamma(\mathcal{G}(X))
\end{array}$$

- 3.4. Examples for comonoidal categories:**
1. If $\mathcal{V} \in 2\text{-vec}$ is a 2-vector space with basis $\mathcal{B} := \{B_1, \dots, B_n\}$, then $(\mathcal{V}, \blacklozenge, \Gamma)$ together with the functors $\blacklozenge : \mathcal{V} \ni B_i \mapsto B_i \boxtimes B_i \in \mathcal{V} \boxtimes \mathcal{V}$ and $\Gamma : \mathcal{V} \ni B_i \mapsto k \in 2\text{-vec}$ is a comonoidal category. In particular, vec with $\blacklozenge : \text{vec} \ni k \mapsto k \boxtimes k \in \text{vec} \boxtimes \text{vec}$ is a comonoidal category.
 2. It is well known that k -algebras can be understood as k -linear categories with one single object. In this sense, k -bialgebras are one object comonoidal categories in $k\text{-lin}$.
 3. Let (A, ∇, η) be a semisimple finite k -algebra. In the last chapter, we saw that for semisimple algebras holds ${}_A\mathcal{M} \boxtimes_A \mathcal{M} \simeq {}_{A \otimes A}\mathcal{M}$. If A is commutative, then $\nabla : A \otimes A \rightarrow A$ is an algebra morphism and hence induces a functor

$${}_A\mathcal{M} \rightarrow {}_{A \otimes A}\mathcal{M} \simeq {}_A\mathcal{M} \boxtimes_A \mathcal{M}$$

The algebra morphism $\eta : k \rightarrow A$ induces a functor

$${}_A\mathcal{M} \rightarrow {}_k\mathcal{M} \simeq \text{vec}.$$

Since ∇ is associative and η is unitary, we see that α_c, λ_c and ρ_c can be chosen trivial, where trivial means identity up to the equivalence ${}_{A \otimes A}\mathcal{M} \simeq {}_A\mathcal{M} \boxtimes_A \mathcal{M}$. Thus, ${}_A\mathcal{M}$ is a comonoidal category. If $f : A \rightarrow B$ is an algebra morphism between semisimple commutative algebras, then the induced functor $\mathcal{F} : {}_B\mathcal{M} \rightarrow {}_A\mathcal{M}$ together with the trivial structures is a comonoidal functor. Note that an inner morphism $b : f \Rightarrow g \in B$ between algebra morphisms $f, g : A \rightarrow B$ (c.f. 1.7) induces a comonoidal natural transformation only in case $b : I \rightarrow B$ is an algebra morphism, which can only be achieved in case $b = 1_B$.

4. In the same way we can equip $\mathcal{M}^C$ with a comonoidal category structure for a cocommutative cosemisimple coalgebra C .
5. If $(\mathcal{V}, \otimes, I)$ is a monoidal 2-vector space, then $\mathcal{V}^*$ is a comonoidal 2-vector space, with $\blacklozenge := \otimes^* : \mathcal{V}^* \rightarrow (\mathcal{V} \boxtimes \mathcal{V})^* \simeq \mathcal{V}^* \boxtimes \mathcal{V}^*$. Since there is a natural equivalence $\mathcal{V} \simeq \mathcal{V}^*$, $M \mapsto \text{Hom}_{\mathcal{V}}(M, -)$, the comonoidal structure can be pulled back to $\mathcal{V}$. For $M \in \mathcal{V}$, then $\blacklozenge(M) = M_{(1)} \boxtimes M_{(2)} \in \mathcal{V} \boxtimes \mathcal{V}$ is defined as the representing object of the functor $\text{Hom}_{\mathcal{V}}(M, - \otimes -) : \mathcal{V} \boxtimes \mathcal{V} \rightarrow \text{vec}$ and $\Gamma(M) = \text{Hom}_{\mathcal{V}}(M, I)$. How does this comultiplication look in the case of a category of modules of a semisimple Hopf algebra $(H, \Delta, \nabla, \eta, \varepsilon, S)$? For algebras A, B with $A \subset B$ a subalgebra and an A -module X and a B -module Y , there is a well known adjunction $\text{Hom}_A(X, Y) \cong \text{Hom}_B(B \otimes_A X, Y)$. We know that $\Delta : H \hookrightarrow H \otimes H$ is a subalgebra. This yields the isomorphism

$$\text{Hom}_H(M, X \otimes Y) \cong \text{Hom}_{H \otimes H}((H_{\bullet} \otimes H_{\bullet}) \otimes_H M, X \otimes Y)$$

Using the H -right module isomorphism

$$\begin{aligned} H \otimes H_{\bullet} &\rightarrow H_{\bullet} \otimes H_{\bullet} \\ m \otimes n &\mapsto mn_{(1)} \otimes n_{(2)} \\ mS(n_{(1)}) \otimes n_{(2)} &\leftarrow m \otimes n \end{aligned}$$

we can write $\blacklozenge(M)$ in yet another form.

$$\begin{aligned} (H_{\bullet} \otimes H_{\bullet}) \otimes_H M &\cong H \otimes H_{\bullet} \otimes_H M &\cong H \otimes M \\ h \otimes g \otimes_H m &\mapsto hS(g_{(1)}) \otimes g_{(2)} \otimes_H m &\mapsto hS(g_{(1)}) \otimes g_{(2)} m \\ h \otimes 1 \otimes_H m &\leftarrow h \otimes 1 \otimes_H m &\leftarrow h \otimes m \end{aligned}$$

The $(H \otimes H)$ -module structure on $\blacklozenge(M) \cong H \otimes M$ thus reads:

$$(H \otimes H) \otimes (H \otimes M) \ni (h \otimes h') \otimes (g \otimes m) \mapsto hgS(h'_{(1)}) \otimes h'_{(2)} m \in H \otimes M$$

3.5. Lemma: *The monoidal (resp. comonoidal) categories in $\mathfrak{C}_k$ together with the monoidal functors (resp. comonoidal functors) and monoidal natural transformations (resp. comonoidal natural transformations) comprise a strongly involutory monoidal 2-subcategory of $\mathfrak{C}_k$.*

PROOF:

This is proved in a more general form in [DS], since $\mathfrak{C}_k$ is a braided Gray monoid. Yet, for later use we state the monoidal and comonoidal structures on the tensor product of two monoidal and comonoidal categories, respectively. Again we make use of Proposition 2.2 that allows us to define all functors and natural morphisms only on decomposable tensors in $\mathcal{C} \boxtimes \mathcal{D}$.

Let $\mathcal{C}, \mathcal{D}$ be monoidal categories. Then, $\mathcal{C} \boxtimes \mathcal{D}$ is again monoidal, where the tensor product is defined for $C, C' \in \mathcal{C}$ and $D, D' \in \mathcal{D}$: $(C \boxtimes D) \otimes_{\mathcal{C} \boxtimes \mathcal{D}} (C' \boxtimes D') := (C \otimes C') \boxtimes (D \otimes D')$. The neutral object is $I_{\mathcal{C}} \boxtimes I_{\mathcal{D}}$, the associativity law, the left and right unitary law read $\alpha_{\mathcal{C}} \boxtimes \alpha_{\mathcal{D}}$,

$\lambda_{\mathcal{C}} \boxtimes \lambda_{\mathcal{D}}$ and $\rho_{\mathcal{C}} \boxtimes \rho_{\mathcal{D}}$, respectively. If $(\mathcal{C}, \tau_{\mathcal{C}})$ and $(\mathcal{D}, \tau_{\mathcal{D}})$ are braided, then so is $\mathcal{C} \boxtimes \mathcal{D}$ with braiding $\tau_{\mathcal{C}} \boxtimes \tau_{\mathcal{D}}$.

Now let $\mathcal{C}, \mathcal{D}$ be comonoidal categories. The $\mathcal{C} \boxtimes \mathcal{D}$ is again comonoidal, with $\blacklozenge(C \boxtimes D) := (C_{(1)} \boxtimes D_{(1)}) \boxtimes (C_{(2)} \boxtimes D_{(2)})$ and $\Gamma(C \boxtimes D) := \Gamma(C) \otimes_{\text{vec}} \Gamma(D)$ for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$. The coassociativity law and the left and right counitary law read $\alpha_{c\mathcal{C}} \boxtimes \alpha_{c\mathcal{D}}$, $\lambda_{c\mathcal{C}} \boxtimes \lambda_{c\mathcal{D}}$ and $\rho_{c\mathcal{C}} \boxtimes \rho_{c\mathcal{D}}$, respectively. $\square$

3.2 Cobraided Comonoidal Categories

Commutative algebra objects in $\mathfrak{C}_k$ are braided categories (c.f. [DS]). Now we define the analogous notion for comonoidal categories. Let $(\mathcal{C}, \blacklozenge, \Gamma, \alpha_c, \lambda_c, \rho_c) \in \mathfrak{C}_k$ be a comonoidal k -linear additive category. Let $\mathcal{T} : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C} \boxtimes \mathcal{C}$ denote the involutor, defined in the last chapter and define

$$\begin{aligned} \blacklozenge^{\text{cos}} &:= \mathcal{T} \circ \blacklozenge : \mathcal{C} \ni X \mapsto X_{(2)} \boxtimes X_{(1)} \in \mathcal{C} \boxtimes \mathcal{C} \\ \alpha_c^{\text{cos}} &:= (1 \boxtimes \mathcal{T})(\mathcal{T} \boxtimes 1)(1 \boxtimes \mathcal{T})\alpha_c^{-1}, \quad \lambda_c^{\text{cos}} := \rho_c, \quad \rho_c^{\text{cos}} := \lambda_c \end{aligned}$$

Then, $(\mathcal{C}, \blacklozenge^{\text{cos}}, \Gamma, \alpha_c^{\text{cos}}, \lambda_c^{\text{cos}}, \rho_c^{\text{cos}})$ is again a comonoidal category, called the *cosymmetric* category.

3.6. Definition: Let $(\mathcal{C}, \blacklozenge, \Gamma, \alpha_c, \lambda_c, \rho_c)$ be a comonoidal category and let $\sigma : \blacklozenge \Rightarrow \mathcal{T} \circ \blacklozenge$ be a natural transformation given for any $X \in \mathcal{C}$ by morphisms $\sigma_X : X_{(1)} \boxtimes X_{(2)} \rightarrow X_{(2)} \boxtimes X_{(1)}$. $(\mathcal{C}, \sigma)$ is called a *weakly cobraided comonoidal category*, if the following diagrams commute for any $X \in \mathcal{C}$,

$$\begin{array}{c} \text{(QR1)} \quad \begin{array}{ccccc} X_{(1)} \boxtimes X_{(2,1)} \boxtimes X_{(2,2)} & \xrightarrow{(1 \boxtimes \blacklozenge)\sigma} & X_{(2)} \boxtimes X_{(1,1)} \boxtimes X_{(1,2)} & \xrightarrow{\alpha^{-1}} & X_{(2,2)} \boxtimes X_{(1)} \boxtimes X_{(2,1)} \\ \downarrow \alpha & & & & \uparrow (\mathcal{T} \boxtimes 1)(1 \boxtimes \sigma) \\ X_{(1,1)} \boxtimes X_{(1,2)} \boxtimes X_{(2)} & \xrightarrow{(\sigma \boxtimes 1)\blacklozenge} & X_{(1,2)} \boxtimes X_{(1,1)} \boxtimes X_{(2)} & \xrightarrow{\alpha^{-1}} & X_{(2,1)} \boxtimes X_{(1)} \boxtimes X_{(2,2)} \end{array} \\ \\ \text{(QL1)} \quad \begin{array}{ccccc} X_{(1,1)} \boxtimes X_{(1,2)} \boxtimes X_{(2)} & \xrightarrow{(\blacklozenge \boxtimes 1)\sigma} & X_{(2,1)} \boxtimes X_{(2,2)} \boxtimes X_{(1)} & \xrightarrow{\alpha} & X_{(1,2)} \boxtimes X_{(2)} \boxtimes X_{(1,1)} \\ \downarrow \alpha^{-1} & & & & \uparrow (1 \boxtimes \mathcal{T})(\sigma \boxtimes 1) \\ X_{(1)} \boxtimes X_{(2,1)} \boxtimes X_{(2,2)} & \xrightarrow{1 \boxtimes \sigma} & X_{(1)} \boxtimes X_{(2,2)} \boxtimes X_{(2,1)} & \xrightarrow{\alpha} & X_{(1,1)} \boxtimes X_{(2)} \boxtimes X_{(1,2)} \end{array} \\ \\ \text{(QR2)} \quad \begin{array}{ccc} X_{(1)} \boxtimes \Gamma(X_{(2)}) & \xrightarrow{(1 \boxtimes \Gamma)\sigma} & X_{(2)} \boxtimes \Gamma(X_{(1)}) \\ \searrow \rho_c & & \swarrow \lambda_c \\ & X & \end{array} \quad \text{(QL2)} \quad \begin{array}{ccc} \Gamma(X_{(1)}) \boxtimes X_{(2)} & \xrightarrow{(\Gamma \boxtimes 1)\sigma} & \Gamma(X_{(2)}) \boxtimes X_{(1)} \\ \searrow \lambda_c & & \swarrow \rho_c \\ & X & \end{array} \end{array}$$

If σ is a isomorphism, then $(\mathcal{C}, \sigma)$ is called a *cobraided comonoidal category*. If $\sigma^2 = \text{id} : \blacklozenge \Rightarrow \blacklozenge$, then $(\mathcal{C}, \sigma)$ is called *cosymmetric*.

We will use the following convention for σ . We denote by $\sigma_{j|j+1}$ the application of σ on the formal component $X_{(j)}$ and $X_{(j+1)}$. Equivalently, we write for the morphism $(\blacklozenge \boxtimes 1) \circ \sigma : (\blacklozenge \boxtimes 1)\blacklozenge \Rightarrow (\blacklozenge \boxtimes 1)T\blacklozenge = (1 \boxtimes T)(T \boxtimes 1)(1 \boxtimes \blacklozenge)\blacklozenge(X)$:

$$\sigma_{1|2,3} : X_{(1)} \boxtimes X_{(2)} \boxtimes X_{(3)} = (\blacklozenge \boxtimes 1)\blacklozenge(X) \Rightarrow (\blacklozenge \boxtimes 1)T\blacklozenge(X) = X_{(2)} \boxtimes X_{(3)} \boxtimes X_{(1)}.$$

Making use of the coherence laws we will write the diagrams (QR1) and (QL1) in the following way:

$$\begin{array}{ccc} X_{(1)} \boxtimes X_{(2)} \boxtimes X_{(3)} & \xrightarrow{\sigma_{1,2|3}} & X_{(3)} \boxtimes X_{(1)} \boxtimes X_{(2)} \\ \sigma_{1|2} \searrow & & \nearrow \sigma_{2|3} \\ & X_{(2)} \boxtimes X_{(1)} \boxtimes X_{(3)} & \end{array} \quad \begin{array}{ccc} X_{(1)} \boxtimes X_{(2)} \boxtimes X_{(2)} & \xrightarrow{\sigma_{1|2,3}} & X_{(2)} \boxtimes X_{(3)} \boxtimes X_{(1)} \\ \sigma_{2|3} \searrow & & \nearrow \sigma_{1|2} \\ & X_{(1)} \boxtimes X_{(3)} \boxtimes X_{(2)} & \end{array}$$

(QR1) (QL1)

If σ is an isomorphism, it can be shown as in the case of monoidal categories that (QR2) (resp. (QL2)) follows from (QR1) (resp. (QL1)).

Any cobraiding yields a solution of the Yang Baxter equation, namely:

$$\sigma_{1|2}\sigma_{2|3}\sigma_{1|2} = \sigma_{2|3}\sigma_{1|2}\sigma_{2|3} : X_{(1)} \boxtimes X_{(2)} \boxtimes X_{(3)} \rightarrow X_{(3)} \boxtimes X_{(2)} \boxtimes X_{(1)}$$

A geometrical interpretation is more subtle, however. The well known action of the braid group on any braided monoidal category ‘flip the tensors at the positions i and $i+1$, no matter what tensors there are located’ becomes substituted by an action ‘flip the i th and the $(i+1)$ th cotensor, no matter where they are located’ on a cobraided comonoidal category.

3.7. Definition: A comonoidal functor $(\mathcal{F}, \xi, \zeta)$ between two cobraided categories $(\mathcal{C}, \sigma_{\mathcal{C}})$ and $(\mathcal{D}, \sigma_{\mathcal{D}})$ is called cobraided, if for any pair $X, Y \in \mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(X_{(1)}) \boxtimes \mathcal{F}(X_{(2)}) & \xrightarrow{\xi} & \mathcal{F}(X)_{(1)} \boxtimes \mathcal{F}(X)_{(2)} \\ (\mathcal{F} \boxtimes \mathcal{F}) \circ \sigma_{\mathcal{C}} \downarrow & & \downarrow \sigma_{\mathcal{D}} \\ \mathcal{F}(X_{(2)}) \boxtimes \mathcal{F}(X_{(1)}) & \xrightarrow{\xi} & \mathcal{F}(X)_{(2)} \boxtimes \mathcal{F}(X)_{(1)} \end{array}$$

The following facts, whose corresponding statements for monoidal categories are well known (see [JS1]), also hold in the case of comonoidal categories.

There is a bijection between the cobraidings σ for $\blacklozenge$ and the pairs (ξ, ζ) that endow $(\blacklozenge, \xi, \zeta)$ with the structure of a comonoidal functor. Note that, in analogy to the case of monoidal categories, $(\Gamma, \lambda_c, \text{id})$ is a comonoidal functor, anyway. A cobraiding is a cosymmetry if and only if it is a comonoidal transformation. A comonoidal functor $(\mathcal{F}, \xi, \zeta)$ is cobraided, if and only if ξ is a comonoidal transformation.

3.3 Bimonoidal and Hopf Categories

In this section we define categories that carry both, a monoidal, as well as a comonoidal structure, such that both structures are compatible in an appropriate way. The definition of such *bimonoidal categories* was started in [CF]. Furthermore we supply these *2-bialgebras* with antipode functors. It is convenient to think of the unit object of a monoidal category simultaneously as an object $I \in \mathcal{C}$ and as a functor $\Lambda : \text{vec} \rightarrow \mathcal{C}$.

3.8. Definition: Let $\mathcal{C} \in \mathfrak{C}$ be a category supplied with a monoidal structure $(\mathcal{C}, \otimes, \Lambda, \alpha, \lambda, \rho)$ and a comonoidal structure $(\mathcal{C}, \blacklozenge, \Gamma, \alpha_c, \lambda_c, \rho_c)$. Furthermore let the following natural isomorphisms be given.

- $\Xi : (\otimes \boxtimes \otimes)(1 \boxtimes T \boxtimes 1)(\blacklozenge \boxtimes \blacklozenge) \Rightarrow \blacklozenge \otimes$,
i.e. for all $X, Y \in \mathcal{C} : (X_{(1)} \otimes Y_{(1)}) \boxtimes (X_{(2)} \otimes Y_{(2)}) \xrightarrow{\Xi} (X \otimes Y)_{(1)} \boxtimes (X \otimes Y)_{(2)}$
- $\Xi_0 : \Lambda \boxtimes \Lambda \Rightarrow \blacklozenge \Lambda$, i.e. an isomorphism $I \boxtimes I \xrightarrow{\Xi_0} I_{(1)} \boxtimes I_{(2)}$
- $\Theta : \otimes_{\text{vec}}(\Gamma \boxtimes \Gamma) \Rightarrow \Gamma \otimes$, i.e. for all $X, Y \in \mathcal{C}$ an isomorphism $\Gamma(X) \otimes \Gamma(Y) \xrightarrow{\Theta} \Gamma(X \otimes Y)$
- $\Theta_0 : \text{id}_{\text{vec}} \Rightarrow \Gamma \Lambda$, i.e. an isomorphism $k \xrightarrow{\Theta_0} \Gamma(I)$

Then, $(\mathcal{C}, \Xi, \Xi_0, \Theta, \Theta_0)$ is called a *bimonoidal category* or a *2-bialgebra*, if the following 12 diagrams commute.

$$\begin{array}{c}
 (1) \\
 \begin{array}{ccc}
 X_{(1)} \otimes (Y_{(1)} \otimes Z_{(1)}) \boxtimes X_{(2)} \otimes (Y_{(2)} \otimes Z_{(2)}) & \xrightarrow{\alpha \boxtimes \alpha} & (X_{(1)} \otimes Y_{(1)}) \otimes Z_{(1)} \boxtimes (X_{(2)} \otimes Y_{(2)}) \otimes Z_{(2)} \\
 \downarrow \blacklozenge(X) \otimes \Xi & & \downarrow \Xi \otimes \blacklozenge(Z) \\
 X_{(1)} \otimes (Y \otimes Z)_{(1)} \boxtimes X_{(2)} \otimes (Y \otimes Z)_{(2)} & & (X \otimes Y)_{(1)} \otimes Z_{(1)} \boxtimes (X \otimes Y)_{(2)} \otimes Z_{(2)} \\
 \downarrow \Xi & & \downarrow \Xi \\
 (X \otimes (Y \otimes Z))_{(1)} \boxtimes (X \otimes (Y \otimes Z))_{(2)} & \xrightarrow{\blacklozenge(\alpha)} & ((X \otimes Y) \otimes Z)_{(1)} \boxtimes ((X \otimes Y) \otimes Z)_{(2)}
 \end{array} \\
 \\
 \begin{array}{ccc}
 I \otimes X_{(1)} \boxtimes I \otimes X_{(2)} & \xrightarrow{\Xi_0 \otimes \blacklozenge(X)} & I_{(1)} \otimes X_{(1)} \boxtimes I_{(2)} \otimes X_{(2)} \xrightarrow{\Xi} (I \otimes X)_{(1)} \boxtimes (I \otimes X)_{(2)} \\
 \searrow \lambda \boxtimes \lambda & & \swarrow \blacklozenge(\lambda) \\
 & X_{(1)} \boxtimes X_{(2)} &
 \end{array} \\
 (2) \\
 \\
 \begin{array}{ccc}
 X_{(1)} \otimes I \boxtimes X_{(2)} \otimes I & \xrightarrow{\blacklozenge(X) \otimes \Xi_0} & X_{(1)} \otimes I_{(1)} \boxtimes X_{(2)} \otimes I_{(2)} \xrightarrow{\Xi} (X \otimes I)_{(1)} \boxtimes (X \otimes I)_{(2)} \\
 \searrow \rho \boxtimes \rho & & \swarrow \blacklozenge(\rho) \\
 & X_{(1)} \boxtimes X_{(2)} &
 \end{array} \\
 (3)
 \end{array}$$

$$(4) \quad \begin{array}{ccccc} \Gamma(X) \otimes (\Gamma(Y) \otimes \Gamma(Z)) & \xrightarrow{1 \otimes \Theta} & \Gamma(X) \otimes \Gamma(Y \otimes Z) & \xrightarrow{\Theta} & \Gamma(X \otimes (Y \otimes Z)) \\ \downarrow \alpha_{\text{vec}} & & & & \downarrow \Gamma(\alpha) \\ (\Gamma(X) \otimes \Gamma(Y)) \otimes \Gamma(Z) & \xrightarrow{\Theta \otimes 1} & \Gamma(X \otimes Y) \otimes \Gamma(Z) & \xrightarrow{\Theta} & \Gamma((X \otimes Y) \otimes Z) \end{array}$$

$$(5) \quad \begin{array}{ccccc} k \otimes \Gamma(X) & \xrightarrow{\Theta_0 \otimes 1} & \Gamma(I) \otimes \Gamma(X) & \xrightarrow{\Theta} & \Gamma(I \otimes X) \\ & \searrow \lambda_{\text{vec}} & & \swarrow \Gamma(\lambda) & \\ & & \Gamma(X) & & \end{array}$$

$$(6) \quad \begin{array}{ccccc} \Gamma(X) \otimes k & \xrightarrow{1 \otimes \Theta_0} & \Gamma(X) \otimes \Gamma(I) & \xrightarrow{\Theta} & \Gamma(X \otimes I) \\ & \searrow \rho_{\text{vec}} & & \swarrow \Gamma(\rho) & \\ & & \Gamma(X) & & \end{array}$$

(7)

$$\begin{array}{ccc} X_{(1)} \otimes Y_{(1)} \boxtimes (X_{(2)} \otimes Y_{(2)} \boxtimes X_{(3)} \otimes Y_{(3)}) & \xrightarrow{\alpha_c \otimes c \boxtimes c \boxtimes c \otimes \alpha_c} & (X_{(1)} \otimes Y_{(1)} \boxtimes X_{(2)} \otimes Y_{(2)}) \boxtimes X_{(3)} \otimes Y_{(3)} \\ \downarrow X_{(1)} \otimes Y_{(1)} \boxtimes \Xi & & \Xi \boxtimes X_{(3)} \otimes Y_{(3)} \downarrow \\ X_{(1)} \otimes Y_{(1)} \boxtimes (X_{(2)} \otimes Y_{(2)})_{(1)} \boxtimes (X_{(2)} \otimes Y_{(2)})_{(2)} & & (X_{(1)} \otimes Y_{(1)})_{(1)} \boxtimes (X_{(1)} \otimes Y_{(1)})_{(2)} \boxtimes X_{(2)} \otimes Y_{(2)} \\ \downarrow (1 \boxtimes \blacklozenge) \Xi & & (\blacklozenge \boxtimes 1) \Xi \downarrow \\ (X \otimes Y)_{(1)} \boxtimes ((X \otimes Y)_{(2)} \boxtimes (X \otimes Y)_{(3)}) & \xrightarrow{\alpha_c} & ((X \otimes Y)_{(1)} \boxtimes ((X \otimes Y)_{(2)} \boxtimes (X \otimes Y)_{(3)}) \end{array}$$

$$(8) \quad \begin{array}{ccccc} \Gamma(X_{(1)}) \otimes \Gamma(Y_{(1)}) \boxtimes X_{(2)} \otimes Y_{(2)} & \xrightarrow{\Theta \boxtimes 1} & \Gamma(X_{(1)} \otimes Y_{(1)}) \boxtimes X_{(2)} \otimes Y_{(2)} & \xrightarrow{(\Gamma \boxtimes 1) \Xi} & \Gamma(X \otimes Y)_{(1)} \boxtimes (X \otimes Y)_{(2)} \\ & \searrow \lambda_c \otimes \lambda_c & & \swarrow \lambda_c & \\ & & X \boxtimes Y & & \end{array}$$

$$(9) \quad \begin{array}{ccccc} X_{(1)} \boxtimes Y_{(1)} \boxtimes \Gamma(X_{(2)}) \otimes \Gamma(Y_{(2)}) & \xrightarrow{1 \otimes \Theta} & X_{(1)} \otimes Y_{(1)} \boxtimes \Gamma(X_{(2)} \otimes Y_{(2)}) & \xrightarrow{(1 \boxtimes \Gamma) \Xi} & (X \otimes Y)_{(1)} \boxtimes \Gamma(X \otimes Y)_{(2)} \\ & \searrow \rho_c \otimes \rho_c & & \swarrow \rho_c & \\ & & X \boxtimes X & & \end{array}$$

$$(10) \quad \begin{array}{ccccc} I \boxtimes (I \boxtimes I) & \xrightarrow{1 \boxtimes \Xi_0} & I \boxtimes (I_{(1)} \boxtimes I_{(2)}) & \xrightarrow{(1 \boxtimes \blacklozenge) \Xi_0} & I_{(1)} \boxtimes (I_{(2)} \boxtimes I_{(3)}) \\ \downarrow \Lambda \boxtimes \Lambda \boxtimes \Lambda(\alpha_{c \text{vec}}) & & & & \downarrow \alpha_c \\ (I \boxtimes I) \boxtimes I & \xrightarrow{\Xi_0 \boxtimes 1} & (I_{(1)} \boxtimes I_{(2)}) \boxtimes I & \xrightarrow{(\blacklozenge \boxtimes 1) \Xi_0} & (I_{(1)} \boxtimes I_{(2)}) \boxtimes I_{(3)} \end{array}$$

$$(11) \quad \begin{array}{ccccc} k \boxtimes I & \xrightarrow{\Theta_0 \boxtimes 1} & \Gamma(I) \boxtimes I & \xrightarrow{(\Gamma \boxtimes 1) \Xi_0} & \Gamma(I_{(1)}) \boxtimes I_{(2)} \\ & \searrow \cong & & \swarrow \lambda_c & \\ & & I & & \end{array}$$

$$(12) \quad \begin{array}{ccccc} I \boxtimes k & \xrightarrow{1 \boxtimes \Theta_0} & I \boxtimes \Gamma(I) & \xrightarrow{(1 \boxtimes \Gamma)\Xi_0} & I_{(1)} \boxtimes \Gamma(I_{(2)}) \\ & \searrow \cong & & \swarrow \rho_c & \\ & & I & & \end{array}$$

For bialgebras the multiplicativity of comultiplication and counit is equivalent to the comultiplicativity of multiplication and unit. In analogy the following theorem holds.

3.9. Theorem: *Let $\mathcal{C}$ be a monoidal and comonoidal category equipped with the natural transformations $\Xi, \Xi_0, \Theta, \Theta_0$ as in the definition above. The following are equivalent:*

1. $\mathcal{C}$ is a bimonoidal category
2. $(\blacklozenge, \Xi, \Xi_0)$ and $(\Gamma, \Theta, \Theta_0)$ are monoidal functors and $(\otimes, \Xi, \Theta)$ and $(\Lambda, \Xi_0, \Theta_0)$ are comonoidal functors.
3. $(\blacklozenge, \Xi, \Xi_0)$ and $(\Gamma, \Theta, \Theta_0)$ are monoidal functors and α_c, λ_c and ρ_c are monoidal natural transformations
4. $(\otimes, \Xi, \Theta)$ and $(\Lambda, \Xi_0, \Theta_0)$ are comonoidal functors and α, λ and ρ are comonoidal natural transformations

Note that condition (3) says that $\mathcal{C}$ is a coalgebra object in the braided monoidal 2-category (see lemma 3.5) of monoidal categories. In the same way, condition (4) says that $\mathcal{C}$ is an algebra object in the braided monoidal 2-category of comonoidal categories.

PROOF:

The diagrams (1)-(3) are exactly the coherence diagrams for $(\blacklozenge, \Xi, \Xi_0)$ being a monoidal functor. The diagrams (4)-(6) are the coherence diagrams for $(\Gamma, \Theta, \Theta_0)$ being a monoidal functor. The diagrams (7)-(9) make $(\otimes, \Xi, \Theta)$ a comonoidal functor and (10)-(12) are the diagrams for $(\Lambda, \Xi_0, \Theta_0)$ being a comonoidal functor. This proves $1. \Leftrightarrow 2.$

If $(\blacklozenge, \Xi, \Xi_0)$ is a monoidal functor, then so are $((1 \boxtimes \blacklozenge)\blacklozenge, \Xi \circ (\otimes \boxtimes \Xi), \Xi_0 \circ (\Lambda \boxtimes \Xi_0))$ and $((\blacklozenge \boxtimes 1)\blacklozenge, \Xi \circ (\Xi \boxtimes \otimes), \Xi_0 \circ (\Xi_0 \boxtimes \Lambda))$ and the diagrams (7) and (10) are the coherence diagrams for α_c being a monoidal natural transformation. In the same way, (8) and (11) are the coherence diagrams for λ_c being a monoidal natural transformation and the diagrams (9) and (12) make ρ_c a monoidal natural transformation. Using the $1. \Leftrightarrow 2.$, this proves $1. \Leftrightarrow 3.$

The same applies for $1. \Leftrightarrow 4.$ The coherence diagrams for α being a comonoidal natural transformation are diagrams (1) and (4). Those for λ are the diagrams (2) and (5) and those for ρ are the diagrams (3) and (6). Hence the theorem is proved. $\square$

It is well known [JS1] that a braiding τ in a braided monoidal category $(\mathcal{C}, \tau)$ is a monoidal transformation, if and only if τ is a symmetry. This holds analogously for cibraided comonoidal categories. In a 2-bialgebra a braiding can be a comonoidal morphism, however.

3.10. Definition: Let $(\mathcal{C}, \Xi, \Xi_0, \Theta, \Theta_0)$ be a bimonoidal category.

1. If $(\mathcal{C}, \tau)$ is a braided monoidal category, then $\mathcal{C}$ is called *commutative*, if the following diagrams commute.

$$\begin{array}{ccc} \diamond(X) \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \diamond(Y) & \xrightarrow{\Xi_{X,Y}} & \diamond(X \otimes Y) \\ \tau_{\mathcal{C} \boxtimes \mathcal{C}} \downarrow & & \downarrow \diamond(\tau) \\ \diamond(Y) \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \diamond(X) & \xrightarrow{\Xi_{Y,X}} & \diamond(Y \otimes X) \end{array} \quad \begin{array}{ccc} \Gamma(X) \otimes_{\text{vec}} \Gamma(Y) & \xrightarrow{\Theta_{X,Y}} & \Gamma(X \otimes Y) \\ \tau_{\text{vec}} \downarrow & & \downarrow \Gamma(\tau) \\ \Gamma(Y) \otimes_{\text{vec}} \Gamma(X) & \xrightarrow{\Theta_{Y,X}} & \Gamma(Y \otimes X) \end{array}$$

In this case τ is called a *comonoidal braiding*

2. If $(\mathcal{C}, \sigma)$ is a cobraided comonoidal category, then $\mathcal{C}$ is called *cocommutative*, if the following diagrams commute.

$$\begin{array}{ccc} \diamond(X) \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \diamond(Y) & \xrightarrow{\Xi_{X,Y}} & \diamond(X \otimes Y) \\ \sigma_X \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \sigma_Y \downarrow & & \downarrow \sigma_{X \otimes Y} \\ \mathcal{T} \diamond(X) \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \mathcal{T} \diamond(Y) & \xrightarrow{\mathcal{T} \Xi_{X,Y}} & \mathcal{T} \diamond(X \otimes Y) \end{array} \quad \begin{array}{ccc} & I \boxtimes I & \\ \Xi_0 \swarrow & & \searrow \mathcal{T}(\Xi_0) \\ I_{(1)} \boxtimes I_{(2)} & \xrightarrow{\sigma} & I_{(2)} \boxtimes I_{(1)} \end{array}$$

In this case σ is called a *monoidal cobraiding*

Note that by lemma 3.5 any instance of $\mathcal{T}$ is a monoidal as well as a comonoidal functor, which we consider here as strict.

3.11. Theorem: Let $(\mathcal{C}, \Xi, \Xi_0, \Theta, \Theta_0)$ be a bimonoidal category.

- If $(\mathcal{C}, \tau)$ is a braided category, the following are equivalent:
 1. $(\mathcal{C}, \tau)$ is commutative.
 2. τ is a comonoidal natural transformation.
 3. $(\diamond, \Xi, \Xi_0)$ and $(\Gamma, \Theta, \Theta_0)$ are braided functors.
 4. $\Xi, \Xi_0, \Theta, \Theta_0$ are monoidal natural transformations.
- If $(\mathcal{C}, \sigma)$ is a cobraided category, then the following are equivalent:
 1. $(\mathcal{C}, \sigma)$ is cocommutative.
 2. σ is a monoidal natural transformation.
 3. $(\otimes, \Xi, \Theta)$ and $(\Lambda, \Xi_0, \Theta_0)$ are cobraided functors.
 4. $\Xi, \Xi_0, \Theta, \Theta_0$ are comonoidal natural transformations.

PROOF:

Obvious by Definition. The equivalence of (3) and (4) each hold by the remark following definition 3.7. $\square$

3.12. Definition: Let $(\mathcal{C}, \otimes, \Lambda, \blacklozenge, \Gamma, \Xi^c, \Xi_0^c, \Theta^c, \Theta_0^c)$ and $(\mathcal{D}, \otimes, \Lambda, \blacklozenge, \Gamma, \Xi^{\mathcal{D}}, \Xi_0^{\mathcal{D}}, \Theta^{\mathcal{D}}, \Theta_0^{\mathcal{D}})$ be bimonoidal categories. A bimonoidal functor is a 5-tuple $(\mathcal{F}, \xi, \zeta, \xi^c, \zeta^c)$, such that $(\mathcal{F}, \xi, \zeta)$ is a monoidal functor, $(\mathcal{F}, \xi^c, \zeta^c)$ is a comonoidal functor and the following four diagrams commute.

$$(1) \quad \begin{array}{ccc} \mathcal{F}(X_{(1)} \otimes Y_{(2)}) \boxtimes \mathcal{F}(X_{(2)} \otimes Y_{(2)}) & \xrightarrow{(\mathcal{F} \boxtimes \mathcal{F})\Xi^c} & \mathcal{F}((X \otimes Y)_{(1)}) \boxtimes \mathcal{F}((X \otimes Y)_{(2)}) \\ \uparrow \xi \boxtimes \xi & & \downarrow \xi_{X \otimes Y}^c \\ \mathcal{F}(X_{(1)}) \otimes \mathcal{F}(Y_{(1)}) \boxtimes \mathcal{F}(X_{(2)}) \otimes \mathcal{F}(Y_{(2)}) & & \mathcal{F}(X \otimes Y)_{(2)} \boxtimes \mathcal{F}(X \otimes Y)_{(2)} \\ \downarrow \xi_X^c \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \xi_Y^c & & \uparrow \blacklozenge(\xi) \\ \mathcal{F}(X)_{(1)} \otimes \mathcal{F}(Y)_{(1)} \boxtimes \mathcal{F}(X)_{(2)} \otimes \mathcal{F}(Y)_{(2)} & \xrightarrow{\Xi^{\mathcal{D}}} & (\mathcal{F}(X) \otimes \mathcal{F}(Y))_{(1)} \boxtimes (\mathcal{F}(X) \otimes \mathcal{F}(Y))_{(2)} \end{array}$$

$$(2) \quad \begin{array}{ccc} I^{\mathcal{D}} \boxtimes I^{\mathcal{D}} & \xrightarrow{\zeta \boxtimes \zeta} & \mathcal{F}(I^c) \boxtimes \mathcal{F}(I^c) \\ \Xi_0^{\mathcal{D}} \downarrow & & \downarrow (\mathcal{F} \boxtimes \mathcal{F})\Xi_0^c \\ I_{(1)}^{\mathcal{D}} \boxtimes I_{(2)}^{\mathcal{D}} & \xrightarrow{\blacklozenge(\zeta)} \mathcal{F}(I^c)_{(1)} \boxtimes \mathcal{F}(I^c)_{(2)} & \xleftarrow{\xi^c} \mathcal{F}(I_{(1)}^c) \boxtimes \mathcal{F}(I_{(2)}^c) \end{array}$$

$$(3) \quad \begin{array}{ccc} \Gamma_{\mathcal{C}}(X) \otimes \Gamma_{\mathcal{C}}(Y) & \xrightarrow{\zeta^c \otimes \zeta^c} & \Gamma_{\mathcal{D}}(\mathcal{F}(X)) \otimes \Gamma_{\mathcal{D}}(\mathcal{F}(Y)) \\ \Theta^c \downarrow & & \downarrow \Theta^{\mathcal{D}} \\ \Gamma_{\mathcal{C}}(X \otimes Y) & \xrightarrow{\zeta^c} \Gamma_{\mathcal{D}}(\mathcal{F}(X \otimes Y)) & \xleftarrow{\Gamma(\xi)} \Gamma_{\mathcal{D}}(\mathcal{F}(X) \otimes \mathcal{F}(Y)) \end{array}$$

$$(4) \quad \begin{array}{ccc} k & \xrightarrow{\Theta_0^{\mathcal{D}}} & \Gamma_{\mathcal{D}}(I^{\mathcal{D}}) \\ \Theta_0^c \downarrow & & \downarrow \Gamma_{\mathcal{D}}(\zeta) \\ \Gamma_{\mathcal{C}}(I^c) & \xrightarrow{\zeta_{I^c}^c} & \Gamma_{\mathcal{D}}(\mathcal{F}(I^c)) \end{array}$$

3.13. Theorem: Let $\mathcal{C}, \mathcal{D}$ be bimonoidal categories and $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ a monoidal and comonoidal functor. The following are equivalent.

1. $\mathcal{F}$ is a bimonoidal functor
2. ξ, ζ are comonoidal natural transformations.
3. ξ^c, ζ^c are monoidal transformations.

Note that condition 2. says that $(\mathcal{F}, \xi, \zeta)$ is a morphism of algebra objects in the 2-category of coalgebra objects in $\mathfrak{C}_k$. In the same way, condition 3. says that $(\mathcal{F}, \xi^c, \zeta^c)$ is a morphism of coalgebra objects in the 2-category of algebra objects in $\mathfrak{C}_k$.

PROOF:

By lemma 3.5 and theorem 3.9 $\xi^e : (\mathcal{F} \boxtimes \mathcal{F}) \circ \blacklozenge \Rightarrow \blacklozenge \circ \mathcal{F}$ and $\zeta^e : \Gamma \Rightarrow \Gamma \circ \mathcal{F}$ are indeed natural transformations between monoidal functors. The diagrams (1) and (2) now show the monoidality of ξ^e , the diagrams (3) and (4) show the monoidality of ζ^e . This proves the equivalence of 1. and 2. Analogously, we know that $\xi : \otimes \circ (\mathcal{F} \boxtimes \mathcal{F}) \Rightarrow \mathcal{F} \circ \otimes$ and $\zeta : \Lambda \Rightarrow \mathcal{F} \circ \Lambda$ are natural transformations between comonoidal functors. The diagrams (1) and (3) are the coherence diagrams for ξ being comonoidal and the diagrams (2) and (4) are the coherence diagrams for ζ being comonoidal. $\square$

3.14. Definition: A natural transformation $\eta : (\mathcal{F}, \xi_{\mathcal{F}}, \zeta_{\mathcal{F}}, \xi_{\mathcal{F}}^e, \zeta_{\mathcal{F}}^e) \Rightarrow (\mathcal{G}, \xi_{\mathcal{G}}, \zeta_{\mathcal{G}}, \xi_{\mathcal{G}}^e, \zeta_{\mathcal{G}}^e)$ is called bimonoidal, if $\eta : (\mathcal{F}, \xi_{\mathcal{F}}, \zeta_{\mathcal{F}}) \Rightarrow (\mathcal{G}, \xi_{\mathcal{G}}, \zeta_{\mathcal{G}})$ is monoidal and $\eta : (\mathcal{F}, \xi_{\mathcal{F}}^e, \zeta_{\mathcal{F}}^e) \Rightarrow (\mathcal{G}, \xi_{\mathcal{G}}^e, \zeta_{\mathcal{G}}^e)$ is comonoidal.

Now we supply 2-bialgebras with antipodes. For simplicity we do this only for strict 2-bialgebras.

3.15. Definition: A 2-bialgebra $\mathcal{H}$, together with a functor $\mathcal{S} : \mathcal{H} \rightarrow \mathcal{H}$ and two natural isomorphisms $\sigma_1 : \otimes \circ (\mathcal{S} \boxtimes 1) \circ \blacklozenge \Rightarrow I \circ \Gamma$ and $\sigma_2 : I \circ \Gamma \Rightarrow \otimes \circ (1 \boxtimes \mathcal{S}) \circ \blacklozenge$ is called a Hopf category or a 2-Hopf algebra, if the following identities hold.

$$(C \xrightarrow{\sigma_2 \otimes 1} C_{(1)} \otimes S(C_{(2)}) \otimes C_{(3)} \xrightarrow{1 \otimes \sigma_1} C) = \text{id}_C \quad (3.1)$$

$$(S(C) \xrightarrow{1 \otimes \sigma_2} S(C_{(1)}) \otimes C_{(2)} \otimes S(C_{(3)}) \xrightarrow{\sigma_1 \otimes 1} S(C)) = \text{id}_{S(C)} \quad (3.2)$$

The triple $(\mathcal{S}, \sigma_1, \sigma_2)$ is called the antipode for $\mathcal{H}$.

Note that the two coherence axioms (3.1) and (3.2) are not really a restriction. The natural isomorphisms σ_1 and σ_2 constitute a category equivalence

$$\begin{aligned} \mathcal{H} \boxtimes \mathcal{H} &\rightarrow \mathcal{H} \otimes \mathcal{H} \\ M \boxtimes N &\mapsto M \otimes N_{(1)} \boxtimes N_{(2)} \\ M \otimes S(N_{(1)}) \boxtimes N_{(2)} &\leftarrow M \otimes N \end{aligned}$$

which can be chosen as an equivalence adjunction, anyway.

3.16. Lemma: Let $(\mathcal{H}, \otimes, I, \blacklozenge, \Gamma)$ be a bimonoidal category. If $(\mathcal{S}, \sigma_{1,\mathcal{S}}, \sigma_{2,\mathcal{S}}) : \mathcal{H} \rightarrow \mathcal{H}$ and $(\mathcal{T}, \sigma_{1,\mathcal{T}}, \sigma_{2,\mathcal{T}}) : \mathcal{H} \rightarrow \mathcal{H}$ are antipodes for $\mathcal{H}$ then there is a unique natural isomorphism $i : \mathcal{S} \Rightarrow \mathcal{T}$, satisfying

$$\begin{array}{ccc} \mathcal{S}(C_{(1)}) \otimes C_{(2)} & \xrightarrow{i \otimes 1} & \mathcal{T}(C_{(1)}) \otimes C_{(2)} \\ \sigma_{1,\mathcal{S}} \searrow & & \nearrow \sigma_{1,\mathcal{T}} \\ & \Gamma(C) \hat{\otimes} I & \end{array} \quad \begin{array}{ccc} C_{(1)} \otimes \mathcal{S}(C_{(2)}) & \xrightarrow{1 \otimes i} & C_{(1)} \otimes \mathcal{T}(C_{(2)}) \\ \sigma_{2,\mathcal{S}} \searrow & & \nearrow \sigma_{2,\mathcal{T}} \\ & \Gamma(C) \hat{\otimes} I & \end{array}$$

PROOF:

We may choose i as the composition

$$\mathcal{S}(C) \xrightarrow{1 \otimes \sigma_{2,\tau}} \mathcal{S}(C_{(1)}) \otimes C_{(2)} \otimes T(C_{(3)}) \xrightarrow{\sigma_{1,\mathcal{S}} \otimes 1} T(C)$$

Then, i fulfills both equations. On the other hand, if $g : \mathcal{S} \Rightarrow T$ is any other solution we have

$$\begin{array}{ccc} \mathcal{S}(C) & \xrightarrow{1 \otimes \sigma_{2,\mathcal{S}}} & \mathcal{S}(C_{(1)}) \otimes C_{(2)} \otimes \mathcal{S}(C_{(3)}) \\ g \downarrow & & \downarrow g \otimes 1 \\ T(C) & \xrightarrow{1 \otimes \sigma_{2,\mathcal{S}}} & T(C_{(1)}) \otimes C_{(2)} \otimes \mathcal{S}(C_{(3)}) \end{array} \quad \begin{array}{c} \searrow \sigma_{1,\mathcal{S}} \otimes 1 \\ \mathcal{S}(C) \\ \nearrow \sigma_{1,\mathcal{T}} \otimes 1 \end{array}$$

Since $(\sigma_{1,\mathcal{S}} \otimes 1) \circ (1 \otimes \sigma_{2,\mathcal{S}}) = \text{id}_{\mathcal{S}(C)}$ and $(\sigma_{1,\mathcal{T}} \otimes 1) \circ (1 \otimes \sigma_{2,\mathcal{S}}) = i^{-1}$, we get $i^{-1} \circ g = \text{id}$, whence $g = i$ and therefore i is unique. $\square$

Using its uniqueness we can define a bimonoidal structure on the antipode functor $\mathcal{S} : \mathcal{H} \rightarrow \mathcal{H}^{\text{sym}, \text{cos}}$ by

$$\begin{aligned} \xi : \mathcal{S}(C) \otimes^{\text{sym}} \mathcal{S}(D) &\xrightarrow{1 \otimes \sigma_{2,C \otimes D}} \mathcal{S}(D_{(1)}) \otimes \mathcal{S}(C_{(1)}) \otimes C_{(2)} \otimes D_{(2)} \otimes \mathcal{S}(C_{(1)} \otimes D_{(1)}) \xrightarrow{1 \otimes \sigma_{2,C} \otimes 1} \\ &\mathcal{S}(D_{(1)}) \otimes D_{(2)} \otimes \mathcal{S}(C \otimes D_{(3)}) \xrightarrow{\sigma_{2,D} \otimes 1} \mathcal{S}(C \otimes D) \end{aligned}$$

$$\zeta : I \xrightarrow{\sigma_2} I_{(1)} \otimes \mathcal{S}(I_{(2)}) \cong \mathcal{S}(I)$$

$$\begin{aligned} \xi^c : \mathcal{S}(C_{(1)}) \boxtimes \mathcal{S}(C_{(2)}) &\xrightarrow{1 \otimes \blacklozenge^{\text{cos}}(\sigma_2)} (\mathcal{S}(C_{(1)}) \boxtimes \mathcal{S}(C_{(2)})) \otimes_{\mathcal{H} \boxtimes \mathcal{H}} \blacklozenge^{\text{cos}}(C_{(3)} \otimes \mathcal{S}(C_{(4)})) \cong \\ &\mathcal{S}(C_{(1)}) \otimes C_{(4)} \otimes \mathcal{S}(C_{(5)})_{(2)} \boxtimes \mathcal{S}(C_{(2)}) \otimes C_{(3)} \otimes \mathcal{S}(C_{(5)})_{(1)} \xrightarrow{\sigma_{1 \otimes 1} \boxtimes \sigma_{1 \otimes 1}} \mathcal{S}(C)_{(2)} \boxtimes \mathcal{S}(C)_{(1)} \end{aligned}$$

$$\zeta^c : \Gamma(C) \xrightarrow{\Gamma(\sigma_2 \otimes 1)} \Gamma(C_{(1)} \otimes \mathcal{S}(C_{(2)}) \otimes C_{(3)}) \cong \Gamma(C_{(1)}) \otimes \Gamma(\mathcal{S}(C_{(2)})) \otimes \Gamma(C_{(3)}) \cong \Gamma(\mathcal{S}(C))$$

These coherence morphisms are, it is easily verified, unique with the property that they are compatible with σ_1 and σ_2 . Hence they fulfill all necessary coherence axioms.

3.17. Examples for 2-Hopf algebras:

1. We are in the situation of example 3.4.2., but with A a bialgebra, additionally. The structure morphisms $\Delta : A \rightarrow A \otimes A$ and $\varepsilon : A \rightarrow k$ induce functors $\otimes : {}_A\mathcal{M} \boxtimes {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}$ and $\mathcal{I} : \text{vec} \rightarrow {}_A\mathcal{M}$, respectively, which supply ${}_A\mathcal{M}$ with the structure of a monoidal category. Since ∇ and η are bialgebra morphisms, plainly the generated functors $\blacklozenge$ and Γ are monoidal functors. The coherence morphisms α_c, λ_c and ρ_c are induced by the coalgebra morphism $\eta : k \rightarrow A$, whence they are monoidal transformations. By theorem 3.9, ${}_A\mathcal{M}$ is a bimonoidal category. Since A is commutative, ${}_A\mathcal{M}$ is a cocommutative bimonoidal category with the trivial cosymmetric cobraiding. Note that all monoidal cobraidings are induced by invertible grouplike elements $c \in A$, which satisfy axioms $QR1$ and $QR2$ of definition 3.6. These axioms read $c^2 = c$, whence there is only the trivial cobraiding.

If additionally A is equipped with an antipode $S : A \rightarrow A$, then S is an algebra morphism and hence induces a functor ${}_S\mathcal{M} : {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}$. Obviously the following compositions of functors are isomorphic. (As a matter of fact they are even equal.)

$${}_A\mathcal{M} \xrightarrow{\nabla^{\mathcal{M}}} {}_{A \otimes A}\mathcal{M} \xrightarrow{s \otimes \text{id}^{\mathcal{M}}} {}_{A \otimes A}\mathcal{M} \xrightarrow{\Delta^{\mathcal{M}}} {}_A\mathcal{M} \cong {}_A\mathcal{M} \xrightarrow{\eta^{\mathcal{M}}} \text{vec} \xrightarrow{\epsilon^{\mathcal{M}}} {}_A\mathcal{M}$$

Analogously, the category M^C of comodules of a cocommutative cosemisimple bialgebra, resp. Hopf algebra C is also a cocommutative 2-bialgebra, resp. 2-Hopf algebra.

Let us regard two examples for this construction.

2. Let G be a finite abelian group and $H := k(G)$ the group algebra. Then, all simple representations $(N_g)_{g \in G}$ are one dimensional and the comonoidal structure on the basis is grouplike, i.e. $\blacklozenge : {}_H\mathcal{M} \ni N_g \mapsto N_g \boxtimes N_g \in {}_{H \otimes H}\mathcal{M}$. To see this note that the k -isomorphism $\cdot : N_g \otimes N_g \cong N_g$ is an $H \otimes H$ -module morphism. Indeed, let $c_N : H \rightarrow k$ denote the character of the simple representation N_g , then for $g, g' \in H$ and $x, y \in N$ we have $c(g)x \cdot c(g')y = c(gg')x \cdot y$.
3. Let G be a finite group and $\mathcal{M}^{kG}$ the category of G -graded vector spaces. Thus $\mathcal{M}^{kG}$ is the category of comodules of the cocommutative, cosemisimple Hopf algebra $k(G)$ and is hence a Hopf category which, it is easy to check, is equivalent to the so called categorification of G , as introduced in [CY2]. Again we see that the cotensor product is defined grouplike on the homogenous 1-dimensional generators of $\mathcal{M}^{kG}$.

In the next section we will examine in more generality under what assumptions the category of representations of an algebra A is supplied with a bimonoidal structure.

3.4 Cogebras and Representations of Bigbras

An algebra A in a monoidal category $\mathcal{C}$ has to be equipped with a compatible coalgebra structure, to lift the tensor product of $\mathcal{C}$ to a tensor product for ${}_A\mathcal{M}$. In this section we examine, what algebraic structures are suited for inducing a comonoidal, resp. a bimonoidal structure on ${}_A\mathcal{M}$. The theory of algebras, coalgebras and Hopf algebras in arbitrary monoidal categories has been examined for quite a long time [Par1, Maj]. We define an algebra (A, μ, η) in a monoidal category $(\mathcal{A}, \otimes, I)$ as a weak monoidal functor $(A, \mu, \eta) : \text{vec} \rightarrow \mathcal{A}$. (Weak means that μ and η are not necessarily isomorphisms.) This is the appropriate point of view, since it allows a direct transfer to comonoidal categories. We call the corresponding objects in comonoidal categories *cogebras*. Cogebras are to algebras as grouplikes are to idempotents. Note that the dual notion of algebra is that of a comonoidal functor $\omega : \mathcal{C} \rightarrow \text{vec}$. We now want to study some properties of such objects.

3.18. Definition: Let $(\mathcal{C}, \blacklozenge, \Gamma, \alpha_c, \lambda_c, \rho_c)$ be a comonoidal category. A cogeбра is a weak comonoidal functor $(C, \nu, \iota) : \text{vec} \rightarrow \mathcal{C}$. A comonoidal natural transformation $f : (C, \nu, \iota) \Rightarrow (D, \nu', \iota')$ between two cogebras is called a cogeбра morphism.

Hence, a cogeбра consists of an object $C \in \mathcal{C}$, a morphism $\nu : A \boxtimes A \rightarrow \mathcal{A}_{(1)} \boxtimes A_{(2)}$ in $\mathcal{C} \boxtimes \mathcal{C}$ and a morphism $\iota : k \rightarrow \Gamma(A)$ in vec , such that the following diagrams commute.

$$\begin{array}{ccc}
 A \boxtimes A \boxtimes A & \xrightarrow{1 \boxtimes \nu} & A \boxtimes A_{(1)} \boxtimes A_{(2)} \xrightarrow{(1 \boxtimes \diamond)\nu} A_{(1)} \boxtimes A_{(2,1)} \boxtimes A_{(2,2)} \\
 \downarrow \nu \boxtimes 1 & & \downarrow \alpha_c \\
 A_{(1)} \boxtimes A_{(2)} \boxtimes A & \xrightarrow{(\diamond \boxtimes 1)\nu} & A_{(1,1)} \boxtimes A_{(1,2)} \boxtimes A_{(2)}
 \end{array}$$

$$\begin{array}{ccc}
 & A & \\
 \lambda_c \nearrow & \downarrow \iota \boxtimes 1 & \nwarrow \rho_c \\
 \Gamma(A_{(1)}) \boxtimes A_{(2)} & \xrightarrow{\nu} & \Gamma(A) \boxtimes A
 \end{array}
 \qquad
 \begin{array}{ccc}
 & A & \\
 1 \boxtimes \iota \downarrow & & \nwarrow \rho_c \\
 A \boxtimes \Gamma(A) & \xrightarrow{\nu} & A_{(1)} \boxtimes \Gamma(A_{(2)})
 \end{array}$$

We refer to these diagrams by saying that ν is *dual associative* and ι is *dual unitary*. The morphism ν is called the *dual multiplication* and ι the *dual unit* of the cogeбра C .

If (C, ν, ι) and (C', ν', ι') are cogebras in $\mathcal{C}$ and $f \in \text{Mor}(A, A')$, then f is a cogeбра morphism, if and only if the following diagrams commute.

$$\begin{array}{ccc}
 A \boxtimes A & \xrightarrow{f \boxtimes f} & A' \boxtimes A' \\
 \downarrow \nu & & \downarrow \nu' \\
 A_{(1)} \boxtimes A_{(2)} & \xrightarrow{\diamond(f)} & A'_{(1)} \boxtimes A'_{(2)}
 \end{array}
 \qquad
 \begin{array}{ccc}
 & k & \\
 \iota \swarrow & & \searrow \iota' \\
 \Gamma(A) & \xrightarrow{\Gamma(f)} & \Gamma(A')
 \end{array}$$

The $\mathcal{C}$ -cogebras, together with the cogeбра morphisms comprise a category $\mathcal{C}\text{-Cog}$.

A cogeбра (A, ν, ι) in a cibraided comonoidal category is called *dual commutative*, if the corresponding comonoidal functor is cibraided. In a diagrammatic notation this can be expressed the following way.

$$\begin{array}{ccc}
 & A \boxtimes A & \\
 \nu \swarrow & & \searrow T(\nu) \\
 A_{(1)} \boxtimes A_{(2)} & \xrightarrow{\tau} & A_{(2)} \boxtimes A_{(1)}
 \end{array}$$

3.19. Example: 1. If $(\mathcal{C}, \Xi, \Xi_0, \Theta, \Theta_0)$ is a bimonoidal category, then (I, Ξ_0, Θ_0) is a cogeбра in $\mathcal{C}$.

2. If $\mathcal{C}$ is a monoidal category, whose underlying category is a finite 2-vector space, then any weak monoidal functor $(\omega, \xi, \zeta) : \mathcal{C} \rightarrow \text{vec}$ is a cogeбра in $\mathcal{C}^*$.

3. In vec , cogebras agree with algebras, since a morphism $\nu : A \boxtimes A \rightarrow \blacklozenge(A)$ is by definition an element in $\text{Hom}_{\text{vec}}(A, A) \otimes \text{Hom}_{\text{vec}}(A, k) \cong \text{Hom}_{\text{vec}}(A \otimes A, A)$. Recall that the comonoidal structure on vec is the inverse functor of the category equivalence $\boxtimes : \text{vec} \boxtimes \text{vec} \rightarrow \text{vec}$. Under this isomorphism, dual associativity and dual unity correspond to the usual associativity and unity of algebras.

The second example shows that unlike in the case of algebras, for a cogeбра it is not unusual to have invertible structure morphisms. In a cibraided category, the functor $\blacklozenge$ is always comonoidal and thus $\blacklozenge(C)$ is a cogeбра, if C is. Hence, in this case the category of $\mathcal{C}$ -cogebras is comonoidal. As in the case of algebras in monoidal categories it can be proved that ν is a cogeбра morphism, if and only if (A, ν, ι) is dual commutative. As indicated in the introduction, we can also define dual cogebras as the dual notion to cogebras.

Now we address the main problem of this section. Under what assumptions is the category of representations of an algebra a comonoidal category? We therefor need a new notion. We are interested in objects in a bimonoidal category $\mathcal{C}$ that carry both an algebra and a compatible cogeбра structure. With remark 3.5 it follows immediately that for $(A, \mu_A, \eta_A) \in \mathcal{C}\text{-Alg}$ and $(B, \mu_B, \eta_B) \in \mathcal{D}\text{-Alg}$ also $A \boxtimes B \in (\mathcal{C} \boxtimes \mathcal{D})\text{-Alg}$ and analogously for cogebras.

3.20. Definition: Let $(\mathcal{C}, \Xi, \Xi_0, \Theta, \Theta_0)$ be a bimonoidal category. A bigebra $(B, \nabla, \eta, \nu, \iota)$ consists of an algebra (B, ∇, η) and a cogeбра (B, ν, ι) , such that the following equivalent conditions hold.

1. ∇ and η are cogeбра morphisms.
2. ν and ι are algebra morphisms.

A morphism $f : A \rightarrow B$ between bigebras $(A, \nabla, \eta, \nu, \iota)$ and $(B, \nabla, \eta, \nu, \iota)$ is called a bigebra morphism, if it is simultaneously an algebra and a cogeбра morphism.

The claimed equivalence is a direct consequence of theorem 3.13. Nevertheless, we give the required commutative diagrams.

$$\begin{array}{c}
 (1) \quad \begin{array}{ccccc}
 B \otimes B \boxtimes B \otimes B & \xrightarrow{\nabla \boxtimes \nabla} & B \boxtimes B & \xrightarrow{\nu} & B_{(1)} \boxtimes B_{(2)} \\
 \downarrow = & & & & \uparrow \blacklozenge(\nabla) \\
 B \boxtimes B \otimes_{\mathcal{C} \boxtimes \mathcal{C}} B \boxtimes B & \xrightarrow{\nu \otimes \nu} & \blacklozenge(B) \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \blacklozenge(B) & \xrightarrow{\Xi} & \blacklozenge(B \otimes B)
 \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 (2) \quad \begin{array}{ccc}
 I \boxtimes I & \xrightarrow{\eta \boxtimes \eta} & A \boxtimes A \\
 \Xi_0 \downarrow & & \downarrow \nu \\
 I_{(1)} \boxtimes I_{(2)} & \xrightarrow{\blacklozenge(\eta)} & A_{(1)} \boxtimes A_{(2)}
 \end{array} & (3) \quad \begin{array}{ccc}
 k & \xrightarrow{\iota \boxtimes \iota} & \Gamma(A) \boxtimes \Gamma(A) \\
 \iota \downarrow & & \downarrow \Theta \\
 \Gamma(A) & \xrightarrow{\Gamma(\nabla)} & \Gamma(A \otimes A)
 \end{array}
 \end{array}$$

$$(4) \quad \begin{array}{ccc} & k & \\ \Theta_0 \swarrow & & \searrow \iota \\ \Gamma(I) & \xrightarrow{\Gamma(\eta)} & \Gamma(A) \end{array}$$

As in the proof of theorem 3.13, the diagrams (1) and (2) assert that ν is an algebra morphism, (3) and (4) assert the same for ι . That ∇ and η are cogebramorphisms is expressed by the diagrams (1) and (3) for ∇ and (2) and (4) for η .

Since, as we have seen in example 3.19, in *vec* cogebras agree with algebras, a k -bigebramorphisms with respect ∇_1 and η_1 . It is well known that this is only possible in case $\nabla_1 = \nabla_2$ and $\eta_1 = \eta_2$ and A is commutative. Hence the only k -bigebras are the commutative k -algebras.

In the following we denote by ${}_A\mathcal{M}$ the category of A -modules for an algebra A in the monoidal category $\mathcal{C}$. It is known that for any $\mathcal{C} \in \mathfrak{C}_k$ and any $A \in \mathcal{C}\text{-Alg}$ the category of A -modules ${}_A\mathcal{M}$ is again k -linear additive. Note that if $(\mathcal{F}, \xi, \zeta) : \mathcal{C} \rightarrow \mathcal{D}$ is a monoidal functor between monoidal categories $\mathcal{C}, \mathcal{D}$ and $A \in \mathcal{C}\text{-Alg}$, then $\mathcal{F}$ induces a functor $\mathcal{F} : {}_A\mathcal{M} \rightarrow {}_{\mathcal{F}(A)}\mathcal{M}$ between the categories of modules that we also denote by $\mathcal{F}$.

3.21. Theorem: *Let $(\mathcal{C}, \Xi, \Xi_0, \Theta, \Theta_0)$ be a bimonoidal category and $(A, \nabla, \eta, \nu, \iota)$ a $\mathcal{C}$ -bigebramorphisms with respect ∇_1 and η_1 . It is well known that this is only possible in case $\nabla_1 = \nabla_2$ and $\eta_1 = \eta_2$ and A is commutative. Hence the only k -bigebras are the commutative k -algebras.*

PROOF:

We define the cotensor product for ${}_A\mathcal{M}$ as the composition

$${}_A\mathcal{M} \xrightarrow{\diamond} {}_{\diamond(A)}\mathcal{M} \xrightarrow{\nu^{\mathcal{M}}} {}_{A \boxtimes A}\mathcal{M} \simeq {}_A\mathcal{M} \boxtimes {}_A\mathcal{M}$$

and the counit as the composition

$${}_A\mathcal{M} \xrightarrow{\Gamma} {}_{\Gamma(A)}\mathcal{M} \xrightarrow{\iota^{\mathcal{M}}} {}_k\mathcal{M} \simeq \text{vec}.$$

Both compositions are welldefined, since $\diamond$ and Γ are monoidal functors. Additionally, ν and ι are algebra morphisms and hence induce monoidal functors. For $M \in {}_A\mathcal{M}$ the $A \boxtimes A$ -module structure on $\diamond(A)$ hence is given by

$$A \boxtimes A \otimes_{\mathcal{C} \boxtimes \mathcal{C}} M_{(1)} \boxtimes M_{(2)} \xrightarrow{\nu^{\boxtimes 1}} A_{(1)} \boxtimes A_{(2)} \otimes_{\mathcal{C} \boxtimes \mathcal{C}} M_{(1)} \boxtimes M_{(2)} \xrightarrow{\Xi} (A \otimes M)_{(1)} \boxtimes (A \otimes M)_{(2)} \xrightarrow{\diamond^{(\mu)}} M_{(1)} \boxtimes M_{(2)}$$

and the k -module structure on $\Gamma(M)$ by

$$\Gamma(M) \xrightarrow{\iota^{\otimes 1}} \Gamma(A) \otimes \Gamma(M) \xrightarrow{\Theta} \Gamma(A \otimes M) \xrightarrow{\mu} \Gamma(M).$$

That the coherence morphisms $\alpha_c, \lambda_c, \rho_c$ from $\mathcal{C}$ are in fact module morphisms, can be seen easily from the fact that ν is dual associative and ι is dual unitary. $\square$

Let under hypothesis of the theorem $f : (A, \nabla, \eta, \nu, \iota) \rightarrow (B, \nabla, \eta, \nu, \iota)$ be a bigebra morphism. Since we presupposed ${}_A \boxtimes {}_A \mathcal{M}$ having the universal property of the tensor product, we also know ${}_f \boxtimes {}_f \mathcal{M} \cong {}_f \mathcal{M} \boxtimes {}_f \mathcal{M}$. Therefore, the induced functor ${}_f \mathcal{M} : {}_B \mathcal{M} \rightarrow {}_A \mathcal{M}$ carries a strict comonoidal structure. Plainly, if $f, g : A \rightarrow B$ are bigebra morphisms and $R : f \Rightarrow g$ is an inner morphism, then the induced natural transformation is a comonoidal transformation, if $R : I \rightarrow B$ is a cogebra morphism.

3.22. Theorem: *If under the hypotheses of theorem 3.21 the bimonoidal category $(\mathcal{C}, \sigma)$ is cocommutative, then σ is a cobraiding for ${}_A \mathcal{M}$, if and only if A is dual commutative.*

PROOF:

Let A be dual commutative. We show that σ is an $A \boxtimes A$ -module morphism.

$$\begin{array}{ccccccc}
 A \boxtimes A \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \blacklozenge(M) & \xrightarrow{\nu \boxtimes 1} & \blacklozenge(A) \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \blacklozenge(M) & \xrightarrow{\Xi} & \blacklozenge(A \otimes M) & \xrightarrow{\blacklozenge(\mu)} & \blacklozenge(M) \\
 \downarrow 1 \boxtimes \sigma & & \downarrow \sigma \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \sigma & & \downarrow \sigma & & \downarrow \sigma \\
 A \boxtimes A \otimes_{\mathcal{C} \boxtimes \mathcal{C}} T \circ \blacklozenge(M) & \xrightarrow{T(\nu) \boxtimes 1} & T \blacklozenge(A) \otimes_{\mathcal{C} \boxtimes \mathcal{C}} T \blacklozenge(M) & \xrightarrow{T\Xi} & T \blacklozenge(A \otimes M) & \xrightarrow{T \circ \blacklozenge(\mu)} & T \blacklozenge(M)
 \end{array}$$

The left square commutes, since A is dual commutative as a cogebra. The central part commutes, since the category $\mathcal{C}$ is cocommutative. The right part commutes, since σ is natural. On the other hand, if this diagram commutes, dual commutativity of A follows by setting $M = A$. $\square$

Note that the difficulties with defining commutative or braided bialgebras in braided monoidal categories do not occur for dual commutative bigebras in bimonoidal categories.

3.23. Definition: *Let $(A, \nabla, \eta, \Delta, \varepsilon)$ be a bialgebra and $(A, \nabla, \eta, \nu, \iota)$ be a bigebra in a commutative bimonoidal category. If Δ, ε and ν, ι are compatible, (meaning that Δ and ε are cogebra morphisms or, equivalently, ν and ι are coalgebra morphisms) then $(A, \nabla, \eta, \Delta, \varepsilon, \nu, \iota)$ is called a trialgebra.*

3.24. Theorem: *Let $(\mathcal{C}, \otimes, I, \blacklozenge, \Gamma, \tau, \Xi, \Xi_0, \Theta, \Theta_0)$ be a commutative bimonoidal category and $(A, \nabla, \eta, \Delta, \varepsilon, \nu, \iota)$ be a trialgebra in $\mathcal{C}$. If for the category ${}_A \mathcal{M}$ of A -modules holds ${}_A \boxtimes {}_A \mathcal{M} \simeq {}_A \mathcal{M} \boxtimes {}_A \mathcal{M}$, then $\Xi, \Xi_0, \Theta, \Theta_0$ are A -module morphisms and hence ${}_A \mathcal{M}$ together with the bimonoidal structure of $\mathcal{C}$ is bimonoidal.*

PROOF:

We showed already that the underlying monoidal and comonoidal structure of $\mathcal{C}$ can be pulled back to ${}_A \mathcal{M}$. We still have to show that the coherence morphisms $\Xi, \Xi_0, \Theta, \Theta_0$ from $\mathcal{C}$ are in fact $A \boxtimes A$ -module morphisms. We do this only for Ξ . Let (X, μ) and (Y, μ) be A -modules. The following diagram proves that $\Xi : \blacklozenge(X) \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \blacklozenge(Y) \rightarrow \blacklozenge(X \otimes Y)$ is an $A \boxtimes A$ -module morphism. In this diagram, tensor symbols, not occurring inside parentheses denote the tensor product in $\mathcal{C} \boxtimes \mathcal{C}$. The left and the right vertical compositions are the $A \boxtimes A$ -module structures on $\blacklozenge(X) \otimes \blacklozenge(Y)$ and $\blacklozenge(X \otimes Y)$, respectively, as defined in theorem 3.21.

$$\begin{array}{ccc}
A \boxtimes A \otimes \blacklozenge(X) \otimes \blacklozenge(Y) & \xrightarrow{A \boxtimes A \otimes \Xi} & A \boxtimes A \otimes \blacklozenge(X \otimes Y) \\
\Delta_{A \boxtimes A} \otimes 1 \otimes 1 \downarrow & & \downarrow \nu \otimes 1 \\
A \boxtimes A \otimes A \boxtimes A \otimes \blacklozenge(X) \otimes \blacklozenge(Y) & & \blacklozenge(A) \otimes \blacklozenge(X \otimes Y) \\
\downarrow 1 \otimes \tau_{A \boxtimes A, \blacklozenge(X)} \otimes 1 & \searrow \nu \otimes \nu \otimes 1 \otimes 1 & \downarrow \Xi \\
A \boxtimes A \otimes \blacklozenge(X) \otimes A \boxtimes A \otimes \blacklozenge(Y) & \blacklozenge(A) \otimes \blacklozenge(A) \otimes \blacklozenge(X) \otimes \blacklozenge(Y) & \blacklozenge(\Delta) \otimes 1 \\
\downarrow \nu \otimes 1 \otimes \nu \otimes 1 & \downarrow 1 \otimes \tau \otimes 1 & \downarrow \Xi \otimes \Xi \\
\blacklozenge(A) \otimes \blacklozenge(X) \otimes \blacklozenge(A) \otimes \blacklozenge(Y) & \blacklozenge(A \otimes A) \otimes \blacklozenge(X \otimes Y) & \blacklozenge(A \otimes X \otimes Y) \\
\downarrow \Xi \otimes \Xi & \downarrow \Xi & \downarrow \blacklozenge(\Delta \otimes 1 \otimes 1) \\
\blacklozenge(A \otimes X) \otimes \blacklozenge(A \otimes Y) & \xrightarrow{\Xi} & \blacklozenge(A \otimes X \otimes A \otimes Y) \\
\downarrow \blacklozenge(\mu) \otimes \blacklozenge(\mu) & & \downarrow \blacklozenge(1 \otimes \tau \otimes 1) \\
\blacklozenge(X) \otimes \blacklozenge(Y) & \xrightarrow{\Xi} & \blacklozenge(X \otimes Y)
\end{array}$$

(I) (II)

The area (I) is the compatibility of Δ and ν , area (II) the commutativity of the category $\mathcal{C}$. The remaining inner parts commute by naturality of Ξ and τ , respectively. $\square$

It is possible to define quasi notions for bi- and trialgebras. An algebra (A, ∇, η) in a bimonoidal category $\mathcal{C}$, together with algebra morphisms $\nu : A \boxtimes A \rightarrow \blacklozenge(A)$ and $\iota : k \rightarrow \Gamma(A)$ is called a *quasi bigebra*, if ν is dual associative and ι is dual unitary up to an inner isomorphism. For dual associativity, say, such an inner isomorphism is given by an element $\aleph := \aleph^1 \boxtimes \aleph^2 \boxtimes \aleph^3 \in A_{(1)} \boxtimes A_{(2)} \boxtimes A_{(3)}$, satisfying $\aleph * (1 \boxtimes \blacklozenge)(\nu) \circ (1 \boxtimes \nu) = (\blacklozenge \boxtimes 1)(\nu) \circ (\nu \boxtimes 1) * \aleph$, where $*$ denotes the convolution product as in example 1.7. We define a new coassociator on the category ${}_A\mathcal{M}$ by $\tilde{\alpha}_e := \aleph * \alpha_e$. This constraint fulfills the dual associativity law, if $\aleph$ satisfies

$$(\blacklozenge \boxtimes 1 \boxtimes 1)(\aleph)(1 \boxtimes 1 \boxtimes \blacklozenge)(\aleph) = (\aleph^1 \boxtimes \aleph^2 \boxtimes \aleph^3 \boxtimes 1)(1 \boxtimes \blacklozenge \boxtimes 1)(\aleph)(1 \boxtimes \aleph^1 \boxtimes \aleph^2 \boxtimes \aleph^3).$$

If $(\mathcal{C}, \sigma)$ is cosymmetric we can define *dual quasi commutative* bigebbras by requiring the existence of an element $R := R^1 \boxtimes R^2 \in A \boxtimes A$, satisfying $R * (\sigma \circ \nu) = \mathcal{T}(\nu) * R$. We get a cobraiding $R * \sigma$ on ${}_A\mathcal{M}$, in case the following equations hold.

$$\begin{aligned}
(1 \boxtimes \blacklozenge)(R) &= R_{1,3} R_{1,2} \\
(\blacklozenge \boxtimes 1)(R) &= R_{1,3} R_{2,3}
\end{aligned}$$

We can also get non trivial coherence morphisms $(\Xi, \Xi_0, \Theta, \Theta_0)$ by weakening the compatibility of dual multiplication and comultiplication in A . We need four inner morphisms satisfying the appropriate axioms to include the twelve coherence axioms of a bimonoidal category.

In the dual situation, when comodules over coalgebras are considered, these quasi notions correspond to the non trivial categorifications as examined in [CY2].

Note that our abstract results on cogebras and bigebras can be used to give proofs for our examples of Hopf categories in 3.17. In general it seems to be difficult to find non-trivial examples for cogebras. In the category vec , as we have seen they agree with algebras. This is no surprise, since in vec , being the trivial Hopf category, tensor product and cotensor product are inverse functors. However we will see that cogebras also play an important role in the representation theory of monoidal categories, which hints that there should be interesting examples.

Of course there is also an appropriate notion of representation of a cogeбра. Let $C \in \mathcal{C}$ be a cogeбра. A representation of C consists of an object $K \in \mathcal{C}$, together with a morphism $C \boxtimes K \rightarrow K_{(1)} \boxtimes K_{(2)}$, such that some obvious diagrams commute. Representations of cogebras C comprise a category $C\text{-Rep}$. To equip the category $C\text{-Rep}$ with the underlying comonoidal structure of $\mathcal{C}$, the presence of a compatible dual cogeбра structure will not do. Rather a second cogeбра structure, compatible with the first is needed. This can only be obtained in case C is dual commutative. Therefore, to obtain comonoidal categories, categories of representations of cogebras are not appropriate. However, if C carries a compatible coalgebra structure (presupposed $\mathcal{C}$ is a bimonoidal category) then $C\text{-Rep}$ becomes a monoidal category. Now, in case C is a cocommutative coalgebra in a commutative category $\mathcal{C}$, then $C\text{-Rep}$ becomes supplied with the underlying braiding of $\mathcal{C}$. We mention this here, since the category of representations of a bialgebra in a braided category is supplied with the underlying braiding only in case this braiding is a symmetry.

Chapter 4

Representations of Monoidal Categories

In example 1.7, we assigned to an algebra in a monoidal category the category of representations of this algebra and noticed that this process was described by a 2-functor. In this chapter we want to examine the 2-category of representations of a 2-algebra. However, we do not work within an arbitrary Gray-Monoid, but within our base 2-category $\mathfrak{C}_k$, where 2-algebras agree with monoidal categories. Our main theorem 4.10 does not hold in a general Gray monoid. As we have seen in chapter 1, monoidal categories (resp. strict monoidal categories) can be interpreted as bicategories (resp. 2-categories) with just one element. Via this identification monoidal functors (resp. strict monoidal functors) correspond to pseudo functors (resp. 2-functors) and we defined lax quasi monoidal transformations between monoidal functors as lax transformations between the corresponding pseudo functors and $\otimes$ -modifications as modifications between lax transformations. Thereby, a subtrcategory Mon of Bicat was defined. Now, making use of this identification we assign to a monoidal category $\mathcal{C}$ the 2-category of pseudofunctors $\text{Pseud}(\mathcal{C}, \mathfrak{C}_k)$, which we consider as the 2-category ${}_{\mathcal{C}}\mathfrak{M}$ of representations of $\mathcal{C}$. A monoidal functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ then induces a 2-functor $\text{Pseud}(\mathcal{F}, \mathfrak{C}_k) : {}_{\mathcal{D}}\mathfrak{M} \rightarrow {}_{\mathcal{C}}\mathfrak{M}$. This assignment can be extended to lax quasi monoidal transformations and $\otimes$ -modifications, which then induce pseudo natural transformations and modifications, respectively. This defines a representable trifunctor $\text{Pseud}(-, \mathfrak{C}_k) : \text{Mon} \rightarrow \text{Bicat}$. Since, however we have not defined tricategories and trifunctors, we use these tricategorical considerations only on a heuristical level and define representations of monoidal categories in elementary terms. Our approach here parallels the situation in one lower dimension, where the category of representations of an algebra A can be defined as the functor category $\text{Fun}(A, \text{vec})$, identifying A with a one object category.

Representations of monoidal categories were first defined in [Par1], where they were called $\mathcal{C}$ -categories. We fix a monoidal category $\mathcal{C}$ that we identify with its one object bicategory. Following the remarks after definition 1.6, we see that $\text{Pseud}(\mathcal{C}, \mathfrak{C}_k)$ is always a 2-category even if $\mathcal{C}$ is only a bicategory, i.e. a non strict monoidal category. We now define the data of this

2-category in elementary terms. Using the inner hom-functor for $\boxtimes$ from theorem 2.8 we can define a pseudofunctor $\mathcal{C} \rightarrow \mathfrak{C}_k$ the following way.

4.1. Definition: Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. A $\mathcal{C}$ -category (or $\mathcal{C}$ -module category) is a quadruple $(\mathcal{M}, \odot, a, l)$, consisting of a category $\mathcal{M} \in \mathfrak{C}_k$, a functor $\odot : \mathcal{C} \boxtimes \mathcal{M} \rightarrow \mathcal{M}$ and two natural isomorphisms

$$\begin{aligned} a_{C,D,M} : C \odot (D \odot M) &\rightarrow (C \otimes D) \odot M \\ l_M : M &\rightarrow I \odot M \end{aligned}$$

called the associativity law and the unitary law, respectively, such that the following diagrams commute.

$$\begin{array}{ccc} C \odot (D \odot (E \odot M)) & \xrightarrow{1 \odot a} C \odot ((D \otimes E) \odot M) & \xrightarrow{a} (C \otimes (D \otimes E)) \odot M \\ \downarrow a & & \downarrow \alpha \odot 1 \\ (C \otimes D) \odot (E \odot M) & \xrightarrow{a} & ((C \otimes D) \otimes E) \odot M \end{array}$$

$$\begin{array}{ccc} C \odot (I \odot M) & \xrightarrow{a} & (C \otimes I) \odot M \\ \swarrow 1 \otimes l & & \searrow \rho \odot 1 \\ & C \odot M & \end{array}$$

A $\mathcal{C}$ -category is called strict, if a and l are the identical morphisms, respectively.

Note that strict $\mathcal{C}$ -categories correspond to strict pseudofunctors. Compatible functors between $\mathcal{C}$ -categories correspond to lax natural transformations between the corresponding pseudofunctors.

4.2. Definition: Let $(\mathcal{M}, \odot_{\mathcal{M}}, a, l)$ and $(\mathcal{N}, \odot_{\mathcal{N}}, a, l)$ be $\mathcal{C}$ -categories. A weak $\mathcal{C}$ -functor (or weak $\mathcal{C}$ -module functor) $(\mathcal{F}, \tilde{\mathcal{F}})$ consists of a functor $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{N}$ and a natural morphism $\tilde{\mathcal{F}} : \mathcal{F}(C \odot M) \rightarrow C \odot \mathcal{F}(M)$, such that the following diagrams commute.

$$\begin{array}{ccc} \mathcal{F}(C \odot (D \odot M)) & \xrightarrow{\tilde{\mathcal{F}}} C \odot \mathcal{F}(D \odot M) & \xrightarrow{1 \odot \tilde{\mathcal{F}}} C \odot (D \odot \mathcal{F}(M)) \\ \downarrow \mathcal{F}(a) & & \downarrow a \\ \mathcal{F}((C \otimes D) \odot M) & \xrightarrow{\tilde{\mathcal{F}}} & (C \otimes D) \odot \mathcal{F}(M) \end{array}$$

$$\begin{array}{ccc} \mathcal{F}(I \odot M) & \xrightarrow{\tilde{\mathcal{F}}} & I \odot \mathcal{F}(M) \\ \swarrow l & & \searrow \mathcal{F}(l) \\ & \mathcal{F}(M) & \end{array}$$

A weak $\mathcal{C}$ -functor is called a $\mathcal{C}$ -functor, if all morphisms $\tilde{\mathcal{F}}$ are isomorphisms. A $\mathcal{C}$ -functor is called strict, if all morphisms $\tilde{\mathcal{F}}$ are the identical morphisms. We often call $\tilde{\mathcal{F}}$ the $\mathcal{C}$ -structure belonging to $\mathcal{F}$.

Obviously, strict $\mathcal{C}$ -functors correspond to 2-natural transformations. If $(\mathcal{F}, \tilde{\mathcal{F}}) : \mathcal{M} \rightarrow \mathcal{M}'$ and $(\mathcal{G}, \tilde{\mathcal{G}}) : \mathcal{M}' \rightarrow \mathcal{M}''$ are $\mathcal{C}$ -functors then a composition is defined by

$$(\mathcal{G}, \tilde{\mathcal{G}}) \circ (\mathcal{F}, \tilde{\mathcal{F}}) := (\mathcal{G} \circ \mathcal{F}, \tilde{\mathcal{G}}_{\mathcal{F}(-)} \circ \mathcal{G}(\tilde{\mathcal{F}}))$$

corresponding to the weak composition of lax transformations. Note that colax natural transformations correspond to weak $\mathcal{C}$ -functors with coherence morphisms defined in the opposite direction.

4.3. Definition: Let $(\mathcal{F}, \tilde{\mathcal{F}}), (\mathcal{G}, \tilde{\mathcal{G}}) : \mathcal{M} \rightarrow \mathcal{N}$ be $\mathcal{C}$ -functors. A natural transformation $\chi : \mathcal{F} \rightarrow \mathcal{G}$ is called a $\mathcal{C}$ -transformation, if for all $C \in \mathcal{C}$ and all $M \in \mathcal{M}$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(C \odot M) & \xrightarrow{\tilde{\mathcal{F}}} & C \odot \mathcal{F}(M) \\ \chi_{C \odot M} \downarrow & & \downarrow C \odot \chi \\ \mathcal{G}(C \odot M) & \xrightarrow{\tilde{\mathcal{G}}} & C \odot \mathcal{G}(M) \end{array}$$

Two $\mathcal{C}$ -categories $\mathcal{M}, \mathcal{N}$ are called equivalent, if the corresponding pseudofunctors are equivalent. This means there are $\mathcal{C}$ -functors $(\mathcal{F}, \tilde{\mathcal{F}}) : \mathcal{M} \rightarrow \mathcal{N}$ and $(\mathcal{G}, \tilde{\mathcal{G}}) : \mathcal{N} \rightarrow \mathcal{M}$ and $\mathcal{C}$ -natural isomorphisms $\chi : \mathcal{F} \circ \mathcal{G} \Rightarrow \text{id}$ and $\rho : \mathcal{G} \circ \mathcal{F} \Rightarrow \text{id}$. For $\mathcal{C}$ -categories a coherence theorem can be proved, similar to the one for monoidal categories. (As a matter of fact, both coherence theorems follow from the coherence theorem for 2-categories.)

We denote the 2-category $\text{Pseud}(\mathcal{C}, \mathfrak{C}_k)$ also by $\mathcal{C}\text{-Cat}$ or ${}_{\mathcal{C}}\mathfrak{M}$ and the 2-category of strict $\mathcal{C}$ -categories (but arbitrary $\mathcal{C}$ -functors) by $\text{Pseud}(\mathcal{C}, \mathfrak{C}_k)^s$ or ${}_{\mathcal{C}}\mathfrak{M}^s$. Now, let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \otimes, I, \alpha, \lambda, \rho)$ be monoidal categories and let $(\mathcal{F}, \xi, \zeta) : \mathcal{C} \rightarrow \mathcal{D}$ be a monoidal functor. We identify $\mathcal{F}$ with its corresponding pseudofunctor. We know that $\mathcal{F}$ induces a 2-functor

$$\mathcal{F}^* := {}_{\mathcal{F}}\mathfrak{M} := \text{Pseud}(\mathcal{F}, \mathfrak{C}_k) : \text{Pseud}(\mathcal{D}, \mathfrak{C}_k) \rightarrow \text{Pseud}(\mathcal{C}, \mathfrak{C}_k)$$

How does this 2-functor look in detail? It is easily verified that its data are given as in the following lemma.

4.4. Lemma: A monoidal functor $(\mathcal{F}, \xi, \zeta) : \mathcal{C} \rightarrow \mathcal{D}$ induces a 2-functor $\mathcal{F}^* : \mathcal{D}\text{-Cat} \rightarrow \mathcal{C}\text{-Cat}$, given by the following data.

- (i) For any $(\mathcal{M}, \odot, a, l) \in \mathcal{D}\text{-Cat}$ an object $(\mathcal{M}, \odot \circ (\mathcal{F} \boxtimes \mathcal{M}), \tilde{a}, \tilde{l}) \in \mathcal{C}\text{-Cat}$, where $\tilde{a}$ and $\tilde{l}$ are defined for $C, D \in \mathcal{C}$ and $M \in \mathcal{M}$ in the following way.

$$\begin{aligned} \tilde{a}_{C,D,M} &: \mathcal{F}(C) \odot (\mathcal{F}(D) \odot M) \xrightarrow{a} (\mathcal{F}(C) \otimes \mathcal{F}(D)) \odot M \xrightarrow{\xi \odot M} \mathcal{F}(C \otimes D) \odot M \\ \tilde{l} &: M \xrightarrow{l} I_{\mathcal{D}} \odot M \xrightarrow{\zeta} \mathcal{F}(I_{\mathcal{C}}) \odot M \end{aligned}$$

(ii) For any $\mathcal{M}, \mathcal{N} \in \mathcal{D}\text{-Cat}$ the functor $\mathcal{F}_{\mathcal{M}, \mathcal{N}}^*$ is given by

$$\begin{aligned} \mathcal{F}_{\mathcal{M}, \mathcal{N}}^* : \mathcal{D}\text{-Cat}(\mathcal{M}, \mathcal{N}) &\rightarrow \mathcal{C}\text{-Cat}(\mathcal{M}, \mathcal{N}) \\ (\mathcal{H}, \tilde{\mathcal{H}}) &\mapsto (\mathcal{H}, \tilde{\mathcal{H}}_{\mathcal{F}}) \\ \chi : (\mathcal{H}, \tilde{\mathcal{H}}) \Rightarrow (\mathcal{H}', \tilde{\mathcal{H}}') &\mapsto \chi : (\mathcal{H}, \tilde{\mathcal{H}}_{\mathcal{F}}) \Rightarrow (\mathcal{H}', \tilde{\mathcal{H}}'_{\mathcal{F}}) \end{aligned}$$

where $\tilde{\mathcal{H}}_{\mathcal{F}}$ is given for $C \in \mathcal{C}$ and $M \in \mathcal{M}$ by $\tilde{\mathcal{H}}_{\mathcal{F}} : \mathcal{H}(\mathcal{F}(C) \odot M) \rightarrow \mathcal{F}(C) \odot \mathcal{H}(M)$.

(iii) The coherence morphisms belonging to $\text{Pseud}(\mathcal{F}, \mathfrak{C}_k)$ are the identities.

PROOF:

This is a routine verification. $\square$

If $\mathcal{F}$ and $\mathcal{G}$ are monoidally equivalent, then $\mathcal{F}^*$ and $\mathcal{G}^*$ are equivalent as 2-functors. For a $\mathcal{D}$ -category $\mathcal{M}$, we will often denote the $\mathcal{C}$ -category $\mathcal{F}^*(\mathcal{M})$ by $\mathcal{M}_{\mathcal{F}}$. Clause (iii) might be a bit surprising. Since a monoidal functor corresponds to a pseudofunctor via our identification, one might expect that $\mathcal{F}^*$ is a pseudofunctor as well and 2-functors are induced only by strict monoidal functors. The remark following definition 1.6 tells us, however that we always get 2-functors here. Note that if $\mathcal{F}$ is strict monoidal, $\mathcal{F}^*$ can be restricted to the strict $\mathcal{C}$ -categories $\mathcal{F}^* : {}_{\mathcal{D}}\mathfrak{M}^s \rightarrow {}_{\mathcal{C}}\mathfrak{M}^s$.

Our next goal is to show that if we handle with pseudo notions only, our to be trihomomorphism Pseud is full and faithful on the level of 2- and 3-morphisms. To put it more exactly, we show that there is category equivalence between the quasi monoidal transformations $\mathcal{F} \Rightarrow \mathcal{G}$ and the pseudo natural transformations $\mathcal{F}^* \Rightarrow \mathcal{G}^*$. We proceed in several steps. First we define a colax natural transformation for any lax quasi monoidal transformation. Since there are many examples for monoidal transformations that are not isomorphisms we do not want to abandon the lax-philosophy, yet.

4.5. Lemma: Let $(X, \chi) : (\mathcal{F}, \xi_{\mathcal{F}}, \zeta_{\mathcal{F}}) \Rightarrow (\mathcal{G}, \xi_{\mathcal{G}}, \zeta_{\mathcal{G}}) : \mathcal{C} \rightarrow \mathcal{D}$ be a lax quasi monoidal transformation and $(\mathcal{M}, \odot, a, l) \in \mathcal{D}\text{-Cat}$. Then

$$\begin{aligned} (X \odot -, \tilde{\chi})_{\mathcal{M}} : \mathcal{M}_{\mathcal{F}} &\rightarrow \mathcal{M}_{\mathcal{G}} \\ M &\mapsto X \odot M \\ f : M \rightarrow N &\mapsto X \odot f : X \odot M \rightarrow X \odot N \end{aligned}$$

is a weak $\mathcal{C}$ -functor with $\mathcal{C}$ -structure $\tilde{\chi}$ defined as the following composition.

$$X \odot (\mathcal{F}(C) \odot M) \xrightarrow{a} (X \otimes \mathcal{F}(C)) \odot M \xrightarrow{\chi \odot M} (\mathcal{G}(C) \otimes X) \odot M \xrightarrow{a^{-1}} \mathcal{G}(C) \odot (X \odot M)$$

PROOF:

The functor property is plain by the definition of $\odot$ and the coherence of $\tilde{\chi}$ is a direct consequence of the lax quasi monoidality of χ . A more conceptual proof can be given, by arguing

that $(X \odot -, \tilde{\chi})$ is the strong composition of the pseudofunctor $\mathcal{M} : \mathcal{D} \rightarrow \mathfrak{C}_k$ with the lax transformation $(X, \chi) : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$. $\square$

Now we endow this family of 1-morphisms in $\mathcal{C}\text{-Cat}$ with the structure of a colax natural transformation (cf. the remark following definition 1.5). Recall that we can not get rid of the co , unless we withdraw to the pseudo case.

4.6. Theorem: *Let $(X, \chi) : (\mathcal{F}, \xi_{\mathcal{F}}, \zeta_{\mathcal{F}}) \Rightarrow (\mathcal{G}, \xi_{\mathcal{G}}, \zeta_{\mathcal{G}}) : \mathcal{C} \rightarrow \mathcal{D}$ be a lax quasi monoidal transformation. Then, a colax natural transformation $((X \odot -, \tilde{\chi}), \tilde{X}_-) : \mathcal{F}^* \Rightarrow \mathcal{G}^* : \mathcal{D}\text{-Cat} \rightarrow \mathcal{C}\text{-Cat}$ is given by*

1. *for each $\mathcal{M} \in \mathcal{D}\text{-Cat}$, the weak $\mathcal{C}$ -functor $((X \odot -, \tilde{\chi}))_{\mathcal{M}} : \mathcal{M}_{\mathcal{F}} \rightarrow \mathcal{M}_{\mathcal{G}}$.*
2. *for any $\mathcal{M}, \mathcal{N} \in \mathcal{D}\text{-Cat}$, a transformation natural in the weak $\mathcal{D}$ -functor $(\mathcal{H}, \tilde{\mathcal{H}}) : \mathcal{M} \rightarrow \mathcal{N}$, given by the $\mathcal{C}$ -transformation $\tilde{X}_{\mathcal{H}} := \tilde{\mathcal{H}}_{X, -} : \mathcal{G}^*(\mathcal{H}) \circ (X \odot -) \Rightarrow (X \odot -) \circ \mathcal{F}^*(\mathcal{H})$.*

PROOF:

We have to show that for any pair $\mathcal{M}, \mathcal{N} \in \mathcal{D}\text{-Cat}$ and any weak $\mathcal{D}$ -functor $(\mathcal{H}, \tilde{\mathcal{H}}) : \mathcal{M} \rightarrow \mathcal{N}$ the morphism

$$\tilde{X}_{\mathcal{H}}^{\mathcal{M}, \mathcal{N}}(M) : \mathcal{G}^*(\mathcal{H}) \circ (X \odot M) = \mathcal{H}(X \odot M) \xrightarrow{\tilde{\mathcal{H}}_{X, M}} X \odot \mathcal{H}(M) = (X \odot -) \circ \mathcal{F}^*(\mathcal{H})(M)$$

is a 2-morphism in $\mathcal{C}\text{-Cat}$, i.e. a $\mathcal{C}$ -natural transformation. Furthermore we have to check naturality in $(\mathcal{H}, \tilde{\mathcal{H}})$ and the two coherence diagrams from definition 1.4.

The following diagram states that $\tilde{X}_{\mathcal{H}} : \mathcal{G}^*(\mathcal{H}) \circ (X \odot -) \Rightarrow (X \odot -) \circ \mathcal{F}^*(\mathcal{H})$ is a $\mathcal{C}$ -transformation for any $(\mathcal{H}, \tilde{\mathcal{H}}) : \mathcal{M} \rightarrow \mathcal{N} \in \mathcal{D}\text{-Cat}$.

$$\begin{array}{ccccc} \mathcal{H}(X \odot \mathcal{F}(C) \odot M) & \xrightarrow{\mathcal{H}(\tilde{\chi} \odot M)} & \mathcal{H}(\mathcal{G}(C) \odot X \odot M) & \xrightarrow{\tilde{\mathcal{H}}_{\mathcal{G}(C), X \odot M}} & \mathcal{G}(C) \odot \mathcal{H}(X \odot M) \\ \tilde{\mathcal{H}}_{X, \mathcal{F}(C) \odot M} \downarrow & \searrow & \searrow & & \downarrow \mathcal{G}(C) \odot \tilde{\mathcal{H}}_{X, M} \\ X \odot \mathcal{H}(\mathcal{F}(C) \odot M) & \xrightarrow{X \odot \tilde{\mathcal{H}}_{\mathcal{F}(C), M}} & X \odot \mathcal{F}(C) \odot \mathcal{H}(M) & \xrightarrow{\tilde{\chi} \odot \mathcal{H}(M)} & \mathcal{G}(C) \odot X \odot \mathcal{H}(M) \end{array}$$

The diagonal morphisms occurring are defined to be $\tilde{\mathcal{H}}_{X \odot \mathcal{F}(C), M}$ and $\tilde{\mathcal{H}}_{\mathcal{G}(C) \odot X, M}$, respectively. The central sector commutes because of the naturality of $\tilde{\mathcal{H}}$.

Next we have to verify that $\tilde{X}_{\mathcal{H}}$ is natural in $\mathcal{H}$. Let $\alpha : (\mathcal{H}, \tilde{\mathcal{H}}) \Rightarrow (\mathcal{H}', \tilde{\mathcal{H}}') : \mathcal{M} \rightarrow \mathcal{N}$ be a $\mathcal{D}$ -natural transformation. The coherence diagram for α being $\mathcal{D}$ -natural is equivalent with $\tilde{X}_{\mathcal{H}}$ being natural.

$$\begin{array}{ccc} \mathcal{H}(X \odot M) & \xrightarrow{\alpha = \mathcal{G}^*(\alpha) \circ (X \odot -)} & \mathcal{H}'(X \odot M) \\ \tilde{\mathcal{H}} = \tilde{X}_{\mathcal{H}} \downarrow & & \downarrow \tilde{\mathcal{H}}' = \tilde{X}_{\mathcal{H}'} \\ X \odot \mathcal{H}(M) & \xrightarrow{X \odot \alpha = (X \odot -) \circ \mathcal{F}^*(\alpha)} & X \odot \mathcal{H}'(M) \end{array}$$

It remains to show that $X \odot -$ satisfies the two coherence diagrams of definition 1.4. The first follows from the definition of composition of $\mathcal{D}$ -functors $\mathcal{H}' \circ \mathcal{H} = \mathcal{H}'(\tilde{\mathcal{H}}) \circ \tilde{\mathcal{H}}'_{\mathcal{H}(-)}$, the second from the fact that the $\mathcal{D}$ -functor $\text{Id} : \mathcal{M} \rightarrow \mathcal{M}$ has $\tilde{\text{Id}} = \text{id} : X \odot \text{Id}(M) \rightarrow \text{Id}(X \odot M)$ as its $\mathcal{C}$ -structure. $\square$

Recall from remark 1.18 that 2-natural transformations $\mathcal{F} \Rightarrow \mathcal{G}$ correspond to pairs (X, id) . If we consider strict $\mathcal{D}$ -categories, then the colax natural transformation corresponding to a 2-natural transformation therefore has a family of strict $\mathcal{C}$ -functors as underlying 1-morphisms. Let us assume for a moment we consider only strict $\mathcal{D}$ -functors. Then, $\tilde{\mathcal{H}}$ is the identity for any $\mathcal{D}$ -functor $\mathcal{H}$ and hence for any lax quasi monoidal transformation (X, χ) the family of weak $\mathcal{C}$ -functors $(X \odot -, \tilde{\chi})_{\mathcal{M}} : \mathcal{M}_{\mathcal{F}} \Rightarrow \mathcal{M}_{\mathcal{G}}$ defines a 2-natural transformation. In case we consider strict $\mathcal{D}$ -categories but arbitrary $\mathcal{D}$ -functors, the monoidal transformations χ (i.e. $X = I_{\mathcal{D}}$) induce 2-natural transformations, since $\tilde{\mathcal{H}}_{I, M} = \text{id} : \mathcal{H}(I \odot M) \rightarrow I \odot \mathcal{H}(M)$, for any $\mathcal{D}$ -functor $(\mathcal{H}, \tilde{\mathcal{H}})$. Due to the coherence theorems we can assume that even in the case of nonstrict $\mathcal{D}$ -categories, a monoidal transformation induces a 2-natural transformation

4.7. Corollary: *A monoidal transformation $\chi : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ between monoidal functors induces a 2-natural transformation, given by the $\mathcal{C}$ -functor $(\text{Id}, \tilde{\chi}) : \mathcal{M}_{\mathcal{F}} \Rightarrow \mathcal{M}_{\mathcal{G}}$ for any $\mathcal{M} \in \mathcal{D}\text{-Cat}$, where the $\mathcal{C}$ -structure $\tilde{\chi}$ is defined by*

$$\text{Id}(\mathcal{F}(C) \odot M) \xrightarrow{\chi \odot M} \mathcal{G}(C) \odot \text{Id}(M)$$

It is easy to verify that the copseudo natural transformation $(I \odot -, \tilde{\chi})$, given by the theorem is isomorphic to the 2-natural transformation $(\text{Id}, \tilde{\chi})$ via the $\mathcal{C}$ -isomorphism $l : \text{Id}(-) \rightarrow I \odot -$.

4.8. Theorem: *Let $f : (X, \chi) \Rightarrow (Y, \psi)$ be a $\otimes$ -modification. Then, for any $\mathcal{M} \in \mathcal{D}\text{-Cat}$ the morphism $f_{\mathcal{M}}(M) := (f \odot M) : X \odot M \rightarrow Y \odot M$ is a $\mathcal{C}$ -natural transformation and the family $(f_{\mathcal{M}})_{\mathcal{M} \in \mathcal{D}\text{-Cat}}$ comprises a modification $f : ((X \odot -, \tilde{\chi}), \tilde{X}_-) \Rightarrow ((Y \odot -, \tilde{\psi}), \tilde{Y}_-)$ between the two colax natural transformations.*

PROOF:

For any $\mathcal{M} \in \mathcal{D}\text{-Cat}$, the morphism $f_{\mathcal{M}} : X \odot - \Rightarrow Y \odot - : \mathcal{M}_{\mathcal{F}} \rightarrow \mathcal{M}_{\mathcal{G}}$ is a $\mathcal{C}$ -natural transformation, since for any $M \in \mathcal{M}$ the following diagram commutes.

$$\begin{array}{ccc} X \odot \mathcal{F}(C) \odot M & \xrightarrow{\tilde{\chi} \odot M} & \mathcal{G}(C) \odot X \odot M \\ f \odot \mathcal{F}(C) \odot M \downarrow & & \downarrow \mathcal{G}(C) \odot f \odot M \\ Y \odot \mathcal{F}(C) \odot M & \xrightarrow{\tilde{\psi} \odot M} & \mathcal{G}(C) \odot Y \odot M \end{array}$$

This is the coherence diagram of the $\otimes$ -modification f , tensored with M from the right.

Furthermore, $(f_{\mathcal{M}})_{\mathcal{M} \in \mathcal{D}\text{-Cat}}$ is a modification, since for any $(\mathcal{H}, \tilde{\mathcal{H}}) : \mathcal{M} \rightarrow \mathcal{N} \in \mathcal{D}\text{-Cat}$ the diagram

$$\begin{array}{ccc}
\mathcal{H}(X \odot M) & \xrightarrow{\mathcal{H}(f \odot M) = \mathcal{G}^*(H) \circ f_{\mathcal{M}}(M)} & \mathcal{H}(Y \odot M) \\
\tilde{\mathcal{H}}_{X,M} = \tilde{X}_{\mathcal{H}} \downarrow & & \downarrow \tilde{\mathcal{H}}_{Y,M} = \tilde{Y}_{\mathcal{H}} \\
X \odot \mathcal{H}(M) & \xrightarrow{f \odot \mathcal{H}(M) = f_{\mathcal{M}} \circ \mathcal{F}^*(\mathcal{H})(M)} & Y \odot \mathcal{H}(M)
\end{array}$$

commutes by naturality of $\tilde{\mathcal{H}}$. $\square$

Now we want to prove that any pseudo natural transformation $\mathcal{F}^* \Rightarrow \mathcal{G}^*$ is isomorphic to one, induced by a quasi monoidal transformation. Therefore we have to strictify our assumptions and allow only $\mathcal{C}$ -functors with invertible coherence morphisms. From now on we denote by $\text{Pseud}(\mathcal{C}, \mathfrak{C}_k)$ the 2-category consisting of pseudofunctors, *pseudo* natural transformations and modifications.

4.9. Corollary: *If, under the hypotheses of the theorem 4.6, (X, χ) is a quasi monoidal transformation, then it induces a pseudo natural transformation*

$$((X \odot -, \tilde{\chi}), \tilde{X}_-) : \mathcal{F}^* \Rightarrow \mathcal{G}^* : \mathcal{D}\text{-Cat} \rightarrow \mathcal{C}\text{-Cat},$$

if we define $\tilde{X}_{\mathcal{H}} := \tilde{\mathcal{H}}_{X,-}^{-1}$ for all $\mathcal{D}$ -functors $(\mathcal{H}, \tilde{\mathcal{H}}) : \mathcal{M} \rightarrow \mathcal{N}$.

PROOF:

Obvious. $\square$

Now we state our main theorem asserting that the assignment, defined by the last two theorems defines an equivalence of categories.

Note that $\mathcal{D} \in \mathcal{D}\text{-Cat}$ with the regular representation. Then, $(- \odot M, a_{-, -, M}^{-1}) : \mathcal{D} \rightarrow \mathcal{M}$ is a $\mathcal{D}$ -functor, for any $M \in \mathcal{M} \in \mathcal{D}\text{-Cat}$ and hence for $(\mathcal{H}, \tilde{\mathcal{H}}) : \mathcal{M} \rightarrow \mathcal{N}$ an arbitrary $\mathcal{D}$ -functor, $\mathcal{H}(- \otimes M)$ and $- \otimes \mathcal{H}(M) : \mathcal{D} \rightarrow \mathcal{N}$ are $\mathcal{D}$ -functors, as well (with the obvious $\mathcal{D}$ -structures). Moreover, the morphism $\tilde{\mathcal{H}}^{-1} : - \otimes \mathcal{H}(M) \Rightarrow \mathcal{H}(- \otimes M)$ is a $\mathcal{D}$ -natural transformation.

4.10. Theorem: *Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ be monoidal functors between monoidal categories. Then, the assignment*

$$\begin{aligned}
\mathcal{J} : \text{QMon}(\mathcal{F}, \mathcal{G}) &\rightarrow \text{PsNat}(\mathcal{F}^*, \mathcal{G}^*) \\
(X, \chi) &\mapsto ((X \odot -, \tilde{\chi}), \tilde{X}_-) \\
f &\mapsto (f_{\mathcal{M}})_{\mathcal{M} \in \mathcal{D}\text{-Cat}}
\end{aligned}$$

defines a category equivalence.

PROOF:

First we observe that $\mathcal{J}$ in fact is a functor. Indeed, if $g : (X, \chi) \Rightarrow (Y, \psi)$ and $f : (Y, \psi) \Rightarrow (Z, \zeta)$ are $\otimes$ -modifications, then the equation $(f \circ g)_{\mathcal{M}} = f_{\mathcal{M}} \circ g_{\mathcal{M}}$ holds for all $\mathcal{M} \in \mathcal{D}\text{-Cat}$ and obviously, $\mathcal{J}(\text{id}_{(X, \chi)}) = \text{id}_{\mathcal{J}(X, \chi)}$.

$\mathcal{J}$ is faithful, since for $f, g : (X, \chi) \rightarrow (X', \chi')$ in $\mathcal{D}$ with $f \neq g$ also $f \otimes I_{\mathcal{D}} \neq g \otimes I_{\mathcal{D}} : X \otimes I_{\mathcal{D}} \rightarrow X' \otimes I_{\mathcal{D}}$. $\mathcal{J}$ is also full, since for $\psi : X \odot - \Rightarrow X' \odot -$ an arbitrary modification, we can define $f := \psi_{\mathcal{D}}(I_{\mathcal{D}}) : X \cong X \otimes I_{\mathcal{D}} \rightarrow X' \otimes I_{\mathcal{D}} \cong X'$. Since $(- \odot M, a_{-, M}^{-1}) : \mathcal{D} \rightarrow \mathcal{M}$ is a $\mathcal{D}$ -functor for any $M \in \mathcal{M} \in \mathcal{D}\text{-Cat}$ we can apply the modification ψ and get the following diagram.

$$\begin{array}{ccc} X \odot (I \odot M) & \xrightarrow{\psi_{\mathcal{M}}(I \odot M)} & X' \odot (I \odot M) \\ \uparrow a^{-1} & & \uparrow a^{-1} \\ (X \otimes I) \odot M & \xrightarrow{\psi_{\mathcal{D}}(I) \odot M} & (X' \otimes I) \odot M \end{array}$$

By coherence of a follows $\psi_{\mathcal{M}}(M) = f \odot M : X \odot M \rightarrow X' \odot M$.

Now, let $((\lambda, \tilde{\lambda}), \lambda^1) : \mathcal{F}^* \Rightarrow \mathcal{G}^*$ be any pseudo natural transformation. We show that λ is isomorphic to a pseudo natural transformation $((X \odot -, \tilde{\chi}), \tilde{X})$, induced by a quasi monoidal transformation (X, χ) . We have $\mathcal{D} \in \mathcal{D}\text{-Cat}$ with the regular representation. We define $X := \lambda_{\mathcal{D}}(I_{\mathcal{D}})$. For any $A \in \mathcal{C}$ we regard the $\mathcal{D}$ -functor $\mathcal{K}_A : \mathcal{D} \ni D \mapsto D \otimes \mathcal{F}(A) \in \mathcal{D}$. Then, λ^1 supplies a 2-morphism natural in $\mathcal{K}_A$,

$$\begin{array}{ccc} \mathcal{D}_{\mathcal{F}} & \xrightarrow{\lambda_{\mathcal{D}}} & \mathcal{D}_{\mathcal{G}} \\ \mathcal{F}^*(\mathcal{K}_A) \downarrow & \uparrow \lambda_{\mathcal{K}_A}^1 & \downarrow \mathcal{G}^*(\mathcal{K}_A) \\ \mathcal{D}_{\mathcal{F}} & \xrightarrow{\lambda_{\mathcal{D}}} & \mathcal{D}_{\mathcal{G}} \end{array}$$

Starting in the left upper corner with the object $I_{\mathcal{D}}$, we get an isomorphism $\lambda_{\mathcal{K}_A}^1 : \lambda_{\mathcal{D}}(\mathcal{K}_A(I)) \rightarrow \mathcal{K}_A(\lambda_{\mathcal{D}}(I))$. We define χ as the following composition:

$$\begin{aligned} X \otimes \mathcal{F}(A) &= \mathcal{K}_A(X) = \mathcal{K}_A(\lambda_{\mathcal{D}}(I)) \xrightarrow{(\lambda_{\mathcal{K}_A}^1)^{-1}} \lambda_{\mathcal{D}}(\mathcal{K}_A(I)) = \lambda_{\mathcal{D}}(I \otimes \mathcal{F}(A)) \\ &\cong \lambda_{\mathcal{D}}(\mathcal{F}(A) \otimes I) \xrightarrow{\tilde{\lambda}} \mathcal{G}(A) \otimes \lambda_{\mathcal{D}}(I) = \mathcal{G}(A) \otimes X \end{aligned}$$

To show that (X, χ) satisfies the coherence axioms of quasi monoidal transformation, we notice that $\xi_{\mathcal{F}} : \mathcal{K}_B \circ \mathcal{K}_A \Rightarrow \mathcal{K}_{A \otimes B}$ is a $\mathcal{D}$ -isomorphism, i.e. a 2-isomorphism in $\mathcal{D}\text{-Cat}$ and hence compatible with λ^1 . Furthermore ξ is compatible with $\tilde{\lambda}$, since $(\lambda, \tilde{\lambda})$ is a $\mathcal{C}$ -functor. Therefore, (X, χ) is in fact a quasi monoidal transformation.

Now we show that $((\lambda, \tilde{\lambda}), \lambda^1)$ and $((X \odot -, \tilde{\chi}), \tilde{X})$ are isomorphic as pseudo natural transformations. We need an invertible modification

$$\alpha : ((\lambda, \tilde{\lambda}), \lambda^1) \Rightarrow ((X \odot -, \tilde{\chi}), \tilde{X}) : \mathcal{F}^* \Rightarrow \mathcal{G}^* : \mathcal{D}\text{-Cat} \rightarrow \mathcal{C}\text{-Cat}.$$

Any such modification is given by a family $(\alpha_{\mathcal{M}})_{\mathcal{M} \in \mathcal{D}\text{-Cat}}$ of $\mathcal{C}$ -natural isomorphisms

$$\alpha_{\mathcal{M}} : (\lambda, \tilde{\lambda}) \Rightarrow (X \odot -, \tilde{\chi}) : \mathcal{M}_{\mathcal{F}} \rightarrow \mathcal{M}_{\mathcal{G}}$$

such that for any $\mathcal{D}$ -functor $(\mathcal{H}, \tilde{\mathcal{H}}) : \mathcal{M} \rightarrow \mathcal{N}$ the following diagram commutes.

$$(*) \quad \begin{array}{ccc} & \lambda & \\ \mathcal{M}_{\mathcal{F}} & \Downarrow \mathfrak{a} & \mathcal{M}_{\mathcal{G}} \\ & \Uparrow \tilde{X}_{\mathcal{H}} \quad X \odot - & \\ \mathcal{H} \downarrow & & \downarrow \mathcal{H} \\ & \lambda & \\ \mathcal{N}_{\mathcal{F}} & \Downarrow \mathfrak{a} & \mathcal{N}_{\mathcal{G}} \\ & \Uparrow X \odot - & \end{array}$$

Using the $\mathcal{D}$ -functor $((- \odot M), a^{-1}) : \mathcal{D} \ni D \mapsto D \odot M \in \mathcal{M}$ we define:

$$\mathfrak{a}_{\mathcal{M}}(M) : \lambda_{\mathcal{M}}(M) \cong \lambda_{\mathcal{M}}(I \odot M) \xrightarrow{\lambda^1_{(- \odot M)}} (- \odot M)(\lambda_{\mathcal{D}}(I)) = X \odot M$$

By definition, $\lambda^1_{\mathcal{H}}$ is a $\mathcal{C}$ -isomorphism for any $\mathcal{D}$ -functor $\mathcal{H}$ and therefore also for the $\mathcal{D}$ -functors $(- \odot M, a^{-1})$, for $M \in \mathcal{M} \in \mathcal{D}\text{-Cat}$. It is an easy consequence that $\mathfrak{a}_{\mathcal{M}}$ is a $\mathcal{C}$ -isomorphism.

It remains to verify the property of a modification. Let $(\mathcal{H}, \tilde{\mathcal{H}}) : \mathcal{M} \rightarrow \mathcal{N}$ be a $\mathcal{D}$ -functor. Then, $\tilde{\mathcal{H}}^{-1} : - \otimes \mathcal{H}(M) \Rightarrow \mathcal{H}(- \otimes M)$ is a 2-morphism in $\mathcal{D}\text{-Cat}$ (i.e. a $\mathcal{D}$ -natural transformation). Because of the pseudo naturality of λ the following diagram commutes.

$$\begin{array}{ccccc} \mathcal{D} & \xrightarrow{\lambda_{\mathcal{D}}} & \mathcal{D} & & \\ \downarrow - \odot M & \searrow & \downarrow - \odot M & & \\ \mathcal{M} & \xrightarrow{\lambda_{\mathcal{M}}} & \mathcal{M} & & \\ \uparrow \lambda^1_{\mathcal{H}} & \nearrow & \uparrow \lambda^1_{\mathcal{H}} & & \\ \mathcal{N} & \xrightarrow{\lambda_{\mathcal{N}}} & \mathcal{N} & & \\ \uparrow \tilde{\mathcal{H}}^{-1} & \nearrow & \uparrow \tilde{\mathcal{H}}^{-1} & & \\ \mathcal{N} & \xrightarrow{\lambda_{\mathcal{N}}} & \mathcal{N} & & \end{array}$$

If we start in this diagram in the upper left corner with $I \in \mathcal{D}$, then the left face is the trivial coherence morphism and we get that the following diagram commutes for any $M \in \mathcal{M} \in \mathcal{D}\text{-Cat}$, which implies the required diagram (*).

$$\begin{array}{ccc} \lambda_{\mathcal{N}}(\mathcal{H}(M)) & \xrightarrow{\lambda^1_{- \odot \mathcal{H}(M)} = \mathfrak{a}_{\mathcal{N}}(\mathcal{H}(M))} & X \odot \mathcal{H}(M) \\ \lambda^1_{\mathcal{H}} \downarrow & & \downarrow \tilde{\mathcal{H}}^{-1} = \tilde{X}_{\mathcal{H}} \\ \mathcal{H}(\lambda_{\mathcal{M}}(M)) & \xrightarrow{\mathcal{H}(\lambda^1_{- \odot M}) = \mathcal{H}(\mathfrak{a}_{\mathcal{M}}(M))} & \mathcal{H}(X \odot M) \end{array}$$

□

Note that in particular the monoidal transformations $\chi : \mathcal{F} \Rightarrow \mathcal{G}$ are in bijection with the $\mathcal{C}$ -structures $\tilde{\chi}$, endowing the identity functor $\text{Id} : \mathcal{M}_{\mathcal{F}} \rightarrow \mathcal{M}_{\mathcal{G}}$ with the structure of a $\mathcal{C}$ -functor, and hence with the structure of a 2-natural transformation.

4.11. Theorem: *Let $\mathcal{C}, \mathcal{D}$ be monoidal categories and $\text{MonFu}(\mathcal{C}, \mathcal{D})$ be the 2-category of monoidal functors. The family of functors $\mathcal{J}_{\mathcal{F}, \mathcal{G}} : \text{QMon}(\mathcal{F}, \mathcal{G}) \rightarrow \text{PsNat}(\mathcal{F}^*, \mathcal{G}^*)$ endows the mapping*

$$\begin{aligned} \mathfrak{J} : \text{MonFu}(\mathcal{C}, \mathcal{D}) &\rightarrow \text{Pseud}(\mathcal{D}\text{-Cat}, \mathcal{C}\text{-Cat}) \\ \mathcal{F} &\mapsto \mathcal{F}^* \end{aligned}$$

with the structure of a pseudofunctor that is locally an equivalence.

PROOF:

To make a pseudofunctor out of $\mathfrak{J}$, we have to determine the natural isomorphisms ξ and ζ of definition 1.3. Let $(X, \chi) : \mathcal{F} \Rightarrow \mathcal{G}$ and $(Y, \psi) : \mathcal{G} \Rightarrow \mathcal{H}$ be two composable quasi monoidal transformations. For the natural isomorphisms $\xi : \mathcal{J}(X, \chi) \circ \mathcal{J}(Y, \psi) \rightarrow \mathcal{J}(Y \otimes X, (\psi \otimes 1)(1 \otimes \chi))$ we take the isomorphism $a_{Y, X, -} : Y \odot (X \odot M) \rightarrow (Y \otimes X) \odot M$ and for the isomorphism $\zeta : \text{Id}_{\mathcal{F}^*} \rightarrow \mathcal{J}(\text{Id}_{\mathcal{F}})$ we take $l : M \rightarrow I \odot M$. We have to verify that a and l are in fact 2-morphisms in $\text{Pseud}(\mathcal{D}\text{-Cat}, \mathcal{C}\text{-Cat})$, that a is natural in (X, χ) and (Y, ψ) and that the coherence diagrams of a pseudofunctor are satisfied. 2-morphisms in $\text{Pseud}(\mathcal{D}\text{-Cat}, \mathcal{C}\text{-Cat})$ are modifications, hence a and l have to be $\mathcal{C}$ -isomorphisms, which is a consequence of the coherence diagrams of definition 4.1. The coherence diagrams of a modification follow from the coherence diagrams of the definition of $\mathcal{C}$ -functors. The morphism a is natural in X and Y by definition, and the required coherence diagrams of definition 1.3 again are a consequence of the coherence diagrams of definition 4.1. $\square$

It is now an easy consequence that any monoidal category $\mathcal{C}$ can be reconstructed from the knowledge of the 2-category $\mathcal{C}\text{-Cat}$ and the forgetful functor $\omega : \mathcal{C}\text{-Cat} \rightarrow \mathfrak{C}_k$.

4.12. Corollary: *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and $\omega : \mathcal{C}\text{-Cat} \rightarrow \mathfrak{C}_k$ the forgetful 2-functor. Then there is a monoidal equivalence of categories.*

$$\mathcal{C} \stackrel{\otimes}{\simeq} \text{PsNat}(\omega, \omega)$$

PROOF:

The forgetful 2-functor $\omega : \mathcal{C}\text{-Cat} \rightarrow \mathfrak{C}_k = \text{vec-Cat}$ is induced by the (unique up to isomorphism) monoidal functor $\mathcal{I} : \text{vec} \rightarrow \mathcal{C}$. The quasi monoidal transformations are the pairs $(C, \text{id}_C) : \mathcal{I} \Rightarrow \mathcal{I}$ for any object $C \in \mathcal{C}$. Furthermore, any $f : C \rightarrow D \in \mathcal{C}$ is a $\otimes$ -modification, hence $\text{QMon}(\mathcal{I}, \mathcal{I}) \cong \mathcal{C}$ as monoidal categories. By theorem 4.10 $\mathcal{J} : \mathcal{C} \rightarrow \text{PsNat}(\omega, \omega)$ is a category equivalence. Monoidality of $\mathcal{J}$ is a consequence of theorem 4.11. $\square$

4.13. Corollary: *Let $\mathcal{C}$ be a monoidal category and let $\mathcal{Z}(\mathcal{C})$ denote the center of $\mathcal{C}$. Let $\mathfrak{Id} : \mathcal{C}\text{-Cat} \rightarrow \mathcal{C}\text{-Cat}$ be the identity 2-functor. We have:*

$$\mathcal{Z}(\mathcal{C}) \simeq \text{PsNat}(\mathfrak{Id}, \mathfrak{Id})$$

PROOF:

Obviously $\mathfrak{Id} : \mathcal{C}\text{-Cat} \rightarrow \mathcal{C}\text{-Cat}$ is induced by the monoidal functor $\text{Id} : \mathcal{C} \rightarrow \mathcal{C}$ and by remark 1.18 we know $\mathcal{Z}(\mathcal{C}) \simeq \text{QMon}(\text{Id}, \text{Id})$, whence the claim follows with theorem 4.10 and theorem 4.11. $\square$

Chapter 5

Gray Monoids as Representations of 2-Hopf Algebras

In this chapter we apply the theory, developed so far. We show how axioms for pseudo natural transformations with structure translate into axioms for the generating quasi monoidal transformations. This process is explained in detail in the second section for the 2-braid structures, while in the first section we merely state the results. We begin by showing that Gray monoids can be obtained as representations of bimonoidal categories. Bimonoidal functors then induce strict monoidal 2-functors and we weaken the concept of a bimonoidal functor to get general monoidal 2-functors. We prove that quasi monoidal transformations that carry a compatible cogeбра structure induce pseudo natural transformations endowed with a 2-monoidal structure. Furthermore we show that the 2-category of finite dimensional representations of a 2-Hopf algebra is rigid in an appropriate sense.

In the next section, we define ‘almost cocommutative’ bimonoidal categories, in such a way that their 2-categories of representations become braided monoidal 2-categories. In an almost cocommutative 2-bialgebra $\mathcal{C}$ a quasi monoidal transformation substitutes for the monoidal cobraiding of a cocommutative 2-bialgebra. We develop the axioms needed for such 2-quasitriangular structures $R \in \mathcal{C} \boxtimes \mathcal{C}$, as higher dimensional analogs of the well known axioms for quasitriangular bialgebras. In the last section we consider an example for a quasitriangular Hopf category. Of course we could apply the same considerations for the coassociativity law of a 2-bialgebra and define ‘quasi bimonoidal categories’ in a way that their 2-categories of representations become supplied with non trivial associativity constraints. Since we did not define monoidal bicategories in full generality, we will not go into this subject.

Two isomorphic monoidal functors, generate equivalent 2-functors. Thus we can confine ourselves to strict monoidal categories and strict monoidal functors in this chapter. By $(\mathcal{T}, \tilde{T}_1, \tilde{T}_2, \mathfrak{J})$ we denote the symmetric structure for $\mathfrak{C}_k$ as defined in chapter 2. Furthermore we provide $\mathfrak{C}_k$ with its Gray monoid structure, supposing the tensor product to be strictly associative and strictly unitary.

5.1 Representations of Hopf Categories

We begin this chapter showing that the 2-category of representations of a bimonoidal category carries the structure of a monoidal 2-category. We know from lemma 3.5 that given two monoidal categories $\mathcal{C}, \mathcal{D}$ also $\mathcal{C} \boxtimes \mathcal{D}$ is a monoidal category in $\mathfrak{C}_k$. The Gray functor $\boxtimes$ induces an embedding $\hat{\boxtimes} : \mathcal{C}\text{-Cat} \times \mathcal{D}\text{-Cat} \rightarrow \mathcal{C} \boxtimes \mathcal{D}\text{-Cat}$ that is also a Gray functor. If $(\mathcal{M}, \odot, a, l) \in \mathcal{C}\text{-Cat}$ and $(\mathcal{N}, \odot, a, l) \in \mathcal{D}\text{-Cat}$, we have

$$(\mathcal{M} \boxtimes \mathcal{N}, (\odot_{\mathcal{C}} \boxtimes \odot_{\mathcal{D}}) \circ (\mathcal{C} \boxtimes \mathcal{T}_{\mathcal{D}, \mathcal{M}} \boxtimes \mathcal{N}), a_{\mathcal{M} \boxtimes \mathcal{N}}, l_{\mathcal{M} \boxtimes \mathcal{N}}) \in \mathcal{C} \boxtimes \mathcal{D}\text{-Cat},$$

with the obvious definitions for $a_{\mathcal{M}, \mathcal{N}}$ and $l_{\mathcal{M}, \mathcal{N}}$. It is obvious, how $\hat{\boxtimes}$ is defined on 1- and 2-morphisms. Note that for $\mathcal{F} \in \mathcal{C}\text{-Cat}, \mathcal{G} \in \mathcal{D}\text{-Cat}$ the 2-morphism $\boxtimes_{\mathcal{F}, \mathcal{G}} \in \mathfrak{C}_k$ is a $\mathcal{C} \boxtimes \mathcal{D}$ -natural transformation and hence can be chosen as the Gray morphism for $\hat{\boxtimes}$.

Let $\blacklozenge : \mathcal{C} \rightarrow \mathcal{C} \boxtimes \mathcal{C}$ and $\Gamma : \mathcal{C} \rightarrow \text{vec}$ be monoidal functors. By lemma 4.4 the monoidal functor $\blacklozenge$ induces a 2-functor $\blacklozenge \mathcal{M} : \mathcal{C} \boxtimes \mathcal{C}\text{-Cat} \rightarrow \mathcal{C}\text{-Cat}$. We define a new pseudofunctor (that we also denote by $\boxtimes$)

$$\boxtimes : {}_{\mathcal{C}}\mathfrak{M} \times {}_{\mathcal{C}}\mathfrak{M} \xrightarrow{\hat{\boxtimes}} {}_{\mathcal{C} \boxtimes \mathcal{C}}\mathfrak{M} \xrightarrow{\blacklozenge} {}_{\mathcal{C}}\mathfrak{M}$$

The unit object is the category vec with the $\mathcal{C}$ -category structure, induced by $\Gamma : \mathcal{C} \rightarrow \text{vec}$. Hence it is the image of vec under the 2-functor

$$\Gamma \mathfrak{M} : \mathfrak{C}_k = {}_{\text{vec}}\mathfrak{M} \rightarrow {}_{\mathcal{C}}\mathfrak{M}.$$

Now let $(\mathcal{C}, \blacklozenge, \Gamma, \alpha_c, \rho_c, \lambda_c)$ be a 2-bialgebra. Theorem 3.9 claims that the coherence morphisms α_c, λ_c and ρ_c are monoidal natural transformations, and hence, by corollary 4.7, generate 2-natural transformations, whose underlying $\mathcal{C}$ -functors are as functors the identities. Then it is a routine verification to show that this endows $\mathcal{C}\text{-Cat}$ with the structure of a monoidal 2-category. Since we have not defined general monoidal 2-categories, we will not go into the details here. In the case of the Hopf category $\mathcal{M}^{kG}$ of example 3.17(3.), the different choices of the data α_c, ρ_c and λ_c correspond to the different categorifications of the group G in the language of [CY2]. They all induce coherence structures in the 2-category of representations of $\mathcal{M}^{kG}$, whose underlying functors are identities. By weakening α_c, λ_c and ρ_c to quasi monoidal transformations we could get monoidal 2-categories with non trivial associativity and unitary coherence morphisms, respectively. In case the comonoidal structure on $\mathcal{C}$ is strict, α_c, λ_c and ρ_c are the identities. Therefore also the induced 2-natural transformations are the identities and hence $\mathcal{C}\text{-Cat}$ becomes in fact a Gray monoid. Thus we have proved

5.1. Theorem: *Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \blacklozenge, \Gamma, \Xi, \Xi_0, \Theta, \Theta_0)$ be a bimonoidal category that is strict as a comonoidal category. Then $(\mathcal{C}\text{-Cat}, \boxtimes, I, \boxtimes_{-, -})$ is a Gray monoid. If $\mathcal{C}$ is a strict bimonoidal category, the strict $\mathcal{C}$ -categories $\mathcal{C}\text{-Cat}^s$ form a sub Gray monoid of $\mathcal{C}\text{-Cat}$.*

To ensure that we really deal with Gray monoids, all 2-bialgebras occurring from now are meant to be strict comonoidal.

Next, we examine under what conditions a monoidal functor between bimonoidal categories induces a monoidal 2-functor. It is to be expected that bimonoidal functors have this property.

5.2. Theorem: *Let $(\mathcal{F}, \xi_{\mathcal{F}}, \zeta_{\mathcal{F}}, \xi_{\mathcal{F}}^c, \zeta_{\mathcal{F}}^c) : \mathcal{C} \rightarrow \mathcal{D}$ be a bimonoidal functor. Then, the 2-functor $\mathcal{F}^* : {}_{\mathcal{D}}\mathfrak{M} \rightarrow {}_{\mathcal{C}}\mathfrak{M}$ carries a strict monoidal structure, if we endow it with the 2-natural transformations*

$$\Xi^{\mathcal{F}} = (\text{Id}, \tilde{\xi}^c) : \mathcal{M}_{\mathcal{F}} \boxtimes \mathcal{N}_{\mathcal{F}} \rightarrow (\mathcal{M} \boxtimes \mathcal{N})_{\mathcal{F}}$$

induced by the monoidal transformation ξ^c and the 1-morphism induced by ζ^c

$$\Upsilon := (\text{Id}, \tilde{\zeta}) : \text{vec}_{\Gamma_{\mathcal{C}}} \rightarrow \text{vec}_{\Gamma_{\mathcal{D}} \circ \mathcal{F}}$$

PROOF:

Let $\mathcal{M}, \mathcal{N} \in \mathcal{D}\text{-Cat}$. The $\mathcal{C}$ -category $\mathcal{F}^*(\mathcal{M}) \boxtimes \mathcal{F}^*(\mathcal{N}) = \mathcal{M}_{\mathcal{F}} \boxtimes \mathcal{N}_{\mathcal{F}}$ is by definition $\mathcal{M} \boxtimes \mathcal{N}$ as a category and carries the $\mathcal{C}$ -category structure induced by the monoidal functor $(\mathcal{F} \boxtimes \mathcal{F}) \circ \blacklozenge$. Likewise, $\mathcal{F}^*(\mathcal{M} \boxtimes \mathcal{N}) = (\mathcal{M} \boxtimes \mathcal{N})_{\mathcal{F}}$ equals $\mathcal{M} \boxtimes \mathcal{N}$ with the $\mathcal{C}$ -category structure induced by the monoidal functor $\blacklozenge \circ \mathcal{F}$. Therefore, the monoidal transformation $\xi^c : (\mathcal{F} \boxtimes \mathcal{F}) \circ \blacklozenge \Rightarrow \blacklozenge \circ \mathcal{F}$ induces a 2-natural transformation $\Xi^{\mathcal{F}} = (\text{Id}, \tilde{\xi}^c) : \mathcal{M}_{\mathcal{F}} \boxtimes \mathcal{N}_{\mathcal{F}} \rightarrow (\mathcal{M} \boxtimes \mathcal{N})_{\mathcal{F}}$. In a similar manner $\zeta^c : \Gamma_{\mathcal{C}} \Rightarrow \Gamma_{\mathcal{D}} \circ \mathcal{F}$ induces the 1-morphism $\Upsilon := (\text{Id}, \tilde{\zeta}) : \mathcal{I}_{\mathcal{C}}\mathfrak{M} = \text{vec}_{\Gamma_{\mathcal{C}}} \rightarrow \text{vec}_{\Gamma_{\mathcal{D}} \circ \mathcal{F}} = \mathcal{F}^*(\mathcal{I}_{\mathcal{D}}\mathfrak{M})$. The coassociativity law and the unitary laws for the structure morphisms ξ^c and ζ^c imply the diagrams A.1(1) and (2) commuting on the nose, whence the modifications ω , ζ^r and ζ^l can be chosen as identities. $\square$

This leads to an immediate weakening of the concept of a bimonoidal functor. To get nontrivial coherence modifications ω, ζ^l, ζ^r we have to replace the monoidal transformations $\xi_{\mathcal{F}}^c, \zeta_{\mathcal{F}}^c$ by quasi monoidal transformations. Then we can define ω, ζ^l, ζ^r as $\boxtimes$ -modifications. For simplicity we do this only for $\xi_{\mathcal{F}}^c$ and leave $\zeta_{\mathcal{F}}^c$ unchanged.

5.3. Definition: *A quasi bimonoidal functor consists of a monoidal functor $(\mathcal{F}, \xi, \zeta) : \mathcal{C} \rightarrow \mathcal{D}$ together with a quasi monoidal transformation*

$$(X, \tilde{\xi}^c) := (X^1 \boxtimes X^2, \tilde{\xi}^c) : (\mathcal{F} \boxtimes \mathcal{F}) \circ \blacklozenge_{\mathcal{C}} \Rightarrow \blacklozenge_{\mathcal{D}} \circ \mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \boxtimes \mathcal{D},$$

a monoidal transformation $\zeta^c : \Gamma_{\mathcal{C}} \Rightarrow \Gamma_{\mathcal{D}} \circ \mathcal{F}$ and, given two copies X, Y of the quasi monoidal transformation X , a $\boxtimes$ -modification, given by isomorphisms

$$\omega_{X,Y} : Y^1 \boxtimes Y_{(1)}^2 \otimes X^1 \boxtimes Y_{(2)}^2 \otimes X^2 \rightarrow Y_{(1)}^1 \otimes X^1 \boxtimes Y_{(2)}^1 \otimes X^2 \boxtimes Y^2$$

and $\boxtimes$ -modifications

$$\zeta^r : \Gamma(X^1) \hat{\otimes} X^2 \rightarrow I \quad \zeta^l : X^1 \hat{\otimes} \Gamma(X^2) \rightarrow I,$$

such that the following diagrams commute. In the first diagram we use three copies $X = Y = Z$ of the quasi monoidal transformation $X^1 \boxtimes X^2$. We denote the first application by X , the second by Y and the third by Z . The corners in the following diagrams correspond to the paths around the coherence diagrams in A.1. The arrows correspond to the faces. We do not write tensor product symbols $\otimes$ in this diagram.

$$\begin{array}{c}
\begin{array}{ccc}
& Z^1 \boxtimes Z_{(1)}^2 Y^1 \boxtimes Z_{(2)}^2 Y_{(1)}^2 X^1 \boxtimes Z_{(3)}^2 Y_{(2)}^2 X^2 & \\
(1 \boxtimes 1 \boxtimes \blacklozenge)(\omega_{Z,Y}) \otimes (1 \boxtimes 1 \boxtimes X) \swarrow & & \searrow (1 \boxtimes \blacklozenge^2)(Z^1 \boxtimes Z^2) \otimes (1 \boxtimes \omega_{Y,X}) \\
Z_{(1)}^1 Y^1 \boxtimes Z_{(2)}^1 Y^2 \boxtimes Z_{(1)}^2 X^1 \boxtimes Z_{(2)}^2 X^2 & & Z^1 \boxtimes Z_{(1)}^2 Y_{(1)}^1 X^1 \boxtimes Z_{(2)}^2 Y_{(2)}^1 X^2 \boxtimes Z_{(3)}^2 Y^2 \\
\downarrow \boxtimes_{\omega, \omega} = \text{id} & & \downarrow (1 \boxtimes \blacklozenge \boxtimes 1)(\omega_{Z,Y}) \otimes (1 \boxtimes X \boxtimes 1) \\
Z_{(1)}^1 X^1 \boxtimes Z_{(2)}^1 X^2 \boxtimes Z_{(1)}^2 Y^1 \boxtimes Z_{(2)}^2 Y^2 & & Z_{(1)}^1 Y^1 \boxtimes Z_{(2)}^1 Y_{(1)}^2 X^1 \boxtimes Z_{(3)}^1 Y_{(2)}^2 X^2 \boxtimes Z^2 \\
(\blacklozenge \boxtimes 1 \boxtimes 1)(\omega_{Z,Y}) \otimes (X \boxtimes 1 \boxtimes 1) \swarrow & & \searrow (\blacklozenge^2 \boxtimes 1)(Z^1 \boxtimes Z^2) \otimes \omega_{Y,X} \\
& Z_{(1)}^1 Y_{(1)}^1 X^1 \boxtimes Z_{(2)}^1 Y_{(2)}^1 X^2 \boxtimes Z_{(3)}^1 Y^2 \boxtimes Z^2 & \\
& \uparrow \zeta_X^l & \downarrow (\zeta_X^r)^{-1} \\
& X^1 \boxtimes X^2 & \\
& \swarrow \zeta_X^l & \searrow (\zeta_X^r)^{-1} \\
Y^1 \boxtimes \Gamma(Y_{(1)}^2) \Gamma(X^1) \boxtimes Y_{(2)}^2 X^2 & \xrightarrow{\omega} & Y_{(1)}^1 X^1 \boxtimes \Gamma(Y_{(2)}^1) \Gamma(X^2) \boxtimes Y^2
\end{array}
\end{array}$$

Applying theorem 4.10 and 4.11, it follows immediately from this definition that quasi bi-monoidal functors induce monoidal 2-functors.

For the definition of monoidal pseudo natural transformations we confine ourselves to strict monoidal pseudofunctors induced by bimonoidal functors. Let $\mathcal{F}, \mathcal{G}$ be bimonoidal functors. Under what conditions does there exist a monoidal structure on a pseudo natural transformation induced by a quasi monoidal transformation $(X, \chi) : \mathcal{F} \Rightarrow \mathcal{G}$? We have to define modifications

$$\begin{array}{ccc}
\mathcal{M}_{\mathcal{F}} \boxtimes \mathcal{N}_{\mathcal{F}} & \xrightarrow{\Xi^{\mathcal{F}}} & (\mathcal{M} \boxtimes \mathcal{N})_{\mathcal{F}} \\
\chi \boxtimes \chi \downarrow & \chi_2 \Rightarrow & \downarrow \chi \\
\mathcal{M}_{\mathcal{G}} \boxtimes \mathcal{N}_{\mathcal{G}} & \xrightarrow{\Xi^{\mathcal{G}}} & (\mathcal{M} \boxtimes \mathcal{N})_{\mathcal{G}}
\end{array}
\quad
\begin{array}{ccc}
& \text{vec}_{\Gamma_C} & \\
\Upsilon \swarrow & \chi^I \Rightarrow & \searrow \Upsilon \\
\text{vec}_{\Gamma_D(\mathcal{F})} & \xrightarrow{\Gamma(\chi)} & \text{vec}_{\Gamma_D(\mathcal{G})}
\end{array}$$

such that the axioms of definition A.2 are satisfied. We know that such modifications are induced by $\otimes$ -modifications

$$\begin{array}{ccc}
(\mathcal{F} \boxtimes \mathcal{F}) \circ \blacklozenge & \xrightarrow{\xi_{\mathcal{F}}^c} & \blacklozenge \circ \mathcal{F} \\
((X, \chi) \boxtimes (X, \chi)) \circ \blacklozenge \downarrow & \nu \Rightarrow & \downarrow \blacklozenge \circ (X, \chi) \\
(\mathcal{G} \boxtimes \mathcal{G}) \circ \blacklozenge & \xrightarrow{\xi_{\mathcal{G}}^c} & \blacklozenge \circ \mathcal{G}
\end{array}
\quad
\begin{array}{ccc}
& \Gamma_C & \\
\zeta_{\mathcal{F}}^c \swarrow & \iota \Rightarrow & \searrow \zeta_{\mathcal{G}}^c \\
\Gamma_D \circ \mathcal{F} & \xrightarrow{\Gamma \circ (X, \chi)} & \Gamma_D \circ \mathcal{G}
\end{array}$$

Again we first consider the special case $X = I$ of a monoidal transformation χ . In case χ is also comonoidal (and hence it is a bimonoidal transformation) we see that the 2-natural transformation $(\text{Id}, \tilde{\chi})$ is in fact strictly monoidal since by definition 3.3 these two diagrams commute on the nose.

In the general case, the $\boxtimes$ -modifications are given by morphisms $\nu : X \boxtimes X \rightarrow X_{(1)} \boxtimes X_{(2)}$ and $\iota : k \rightarrow \Gamma(X)$, fulfilling the coherence diagrams of a $\boxtimes$ -transformations. Note that the 2-morphisms ω and $\boxtimes_{-,-}$ in the coherence axioms of definition A.2 are identities. It can easily be seen that for the pseudo natural transformations, induced by ν and ι to satisfy the coherence axioms of definition A.2, we have to require the dual associativity law and the dual unitary laws of a cogeбра for ν and ι . Modifications between pseudo natural transformations are given by $\boxtimes$ -transformations $f : X \rightarrow Y$ and obviously the coherence axioms are the defining axioms for a cogeбра morphism. Thus we are led to the following definition.

5.4. Definition: Let $(\mathcal{F}, \xi_{\mathcal{F}}, \zeta_{\mathcal{F}}, \xi_{\mathcal{F}}^c, \zeta_{\mathcal{F}}^c) : \mathcal{C} \rightarrow \mathcal{D}$ and $(\mathcal{G}, \xi_{\mathcal{G}}, \zeta_{\mathcal{G}}, \xi_{\mathcal{G}}^c, \zeta_{\mathcal{G}}^c) : \mathcal{C} \rightarrow \mathcal{D}$ be bimonoidal functors.

1. A quasi monoidal transformation $(X, \chi) : \mathcal{F} \Rightarrow \mathcal{G}$ is called bimonoidal, in case there exists a cogeбра structure $(X, \nu, \iota) \in \mathcal{D}\text{-Cog}$ on X such that ν and ι are $\boxtimes$ -modifications.
2. A $\boxtimes$ -modification $f : (X, \chi) \Rightarrow (Y, \psi)$ is called bimonoidal, in case f is a cogeбра morphism.

The category of quasi bimonoidal transformations $\mathcal{F} \Rightarrow \mathcal{G}$ is denoted by $\text{QBim}(\mathcal{F}, \mathcal{G})$.

5.5. Theorem: For bimonoidal functors $(\mathcal{F}, \xi_{\mathcal{F}}, \zeta_{\mathcal{F}}, \xi_{\mathcal{F}}^c, \zeta_{\mathcal{F}}^c)$ and $(\mathcal{G}, \xi_{\mathcal{G}}, \zeta_{\mathcal{G}}, \xi_{\mathcal{G}}^c, \zeta_{\mathcal{G}}^c)$ we have

$$\text{PsNat}_{\boxtimes}(\mathcal{F}^*, \mathcal{G}^*) \simeq \text{QBim}(\mathcal{F}, \mathcal{G}).$$

This theorem holds as a consequence of theorem 4.10 and theorem 4.11. Compare this with the well known result in one lower algebraic dimension (c.f. 1.7). There, natural transformation are induced by inner algebra morphisms $r : f \Rightarrow g : B \rightarrow A$. If f, g are bialgebra morphisms, then r generates a monoidal transformation, in case it is grouplike. We have seen in the introduction that grouplikes are 0-dimensional cogebras. Thus, this result was to be expected.

As a last point of this section we show that the 2-category of finite dimensional representations of a Hopf category is rigid (in the sense of theorem 2.17). Assume we have given a Hopf category $(\mathcal{H}, \mathcal{S}, \sigma_1, \sigma_2)$ that is strict as a monoidal, as a comonoidal and as a bimonoidal category. Also we consider only strict $\mathcal{H}$ -categories. If the underlying category of $\mathcal{M} \in {}_{\mathcal{H}}\mathfrak{M}^s$ is a finite dimensional 2-vector space, then $\mathcal{M}^*$ is an $\mathcal{H}^{\text{sym}}$ -category.

$$\mathcal{H}^{\text{sym}} \boxtimes \mathcal{M}^* \ni C \boxtimes \mathcal{F} \mapsto \mathcal{F}(C \odot -) \in \mathcal{M}^*.$$

We already know that $\mathcal{S} : \mathcal{H} \rightarrow \mathcal{H}^{\text{sym}, \text{cos}}$ is a bimonoidal functor, whence $\mathcal{M}^*$ is an $\mathcal{H}$ -category as well and $*$: ${}_{\mathcal{H}}\mathfrak{M}^s \rightarrow {}_{\mathcal{H}}\mathfrak{M}^s$ is an anti monoidal 2-functor. Furthermore $(\text{db}, \sigma_2) : \text{vec} \rightarrow \mathcal{M} \boxtimes \mathcal{M}^*$ and $(\text{ev}, \sigma_1) : \mathcal{M}^* \boxtimes \mathcal{M} \rightarrow \text{vec}$ are $\mathcal{H}$ -functors and the transformations

$$(i) \quad (\mathcal{M} \xrightarrow{\text{db} \boxtimes 1} \mathcal{M} \boxtimes \mathcal{M}^* \boxtimes \mathcal{M} \xrightarrow{1 \boxtimes \text{ev}} \mathcal{M}) \cong \text{id}_{\mathcal{M}}$$

$$(ii) (\mathcal{M}^* \xrightarrow{1 \boxtimes \text{db}} \mathcal{M}^* \boxtimes \mathcal{M} \boxtimes \mathcal{M}^* \xrightarrow{\text{ev} \boxtimes 1} \mathcal{M}^*) \cong \text{id}_{\mathcal{M}^*}$$

are $\mathcal{H}$ -natural. To see this first note that σ_1 and σ_2 in fact serve as $\mathcal{H}$ -structures for the functors ev and db , respectively.

$$\begin{aligned} \text{ev}(C \odot (\mathcal{F} \boxtimes M)) &= \text{ev}(\mathcal{F}(\mathcal{S}(C_{(1)}) \odot -) \boxtimes C_{(2)} \odot M) \xrightarrow{\sigma_1} \Gamma(C) \hat{\otimes} \text{ev}(\mathcal{F} \boxtimes M) = C \odot \text{ev}(\mathcal{F} \boxtimes M) \\ \text{db}(C \odot \text{vec}) &= \Gamma(C) \hat{\otimes} M_i \boxtimes M^i \xrightarrow{\sigma_2} C_{(1)} \odot M_i \boxtimes M^i (S(C_2) \odot -) = C \odot \text{db}(\text{vec}) \end{aligned}$$

We do not need coherence morphisms other than σ_1, σ_2 , because of our strictness assumptions. That (ev, σ_1) and (db, σ_2) really fulfill the coherence axioms of a $\mathcal{H}$ -functor can easily be seen from the definition of the bimonoidal structure on $\mathcal{S}$. Coherence of (ev, σ_1) , say, follows from this diagram, which holds by definition of $\xi_{\mathcal{S}}$.

$$\begin{array}{ccc} \mathcal{S}(D_{(1)}) \otimes \mathcal{S}(C_{(1)}) \otimes C_{(2)} \otimes D_{(2)} & \xrightarrow{\sigma_{1,D} \circ \sigma_{1,C}} & \Gamma(C) \otimes \Gamma(D) \\ \Xi \circ (\xi_{\mathcal{S}} \otimes 1) \downarrow & & \downarrow \Theta \\ \mathcal{S}(C \otimes D)_{(1)} \otimes (C \otimes D)_{(2)} & \xrightarrow{\sigma_{1,C \otimes D}} & \Gamma(C \otimes D) \end{array}$$

That the natural transformations in (i) and (ii) are in fact $\mathcal{H}$ -transformations is a direct consequence of the coherence axioms of σ_1 and σ_2 .

5.2 Quasitriangular Bimonoidal Categories

In this section we develop the category of *quasitriangular structures* for any 2-bialgebra $\mathcal{C}$, which we prove to be equivalent to the category of 2-braid structures on ${}_{\mathcal{C}}\mathfrak{M}$. Let $(\mathcal{C}, \otimes, I, \blacklozenge, \Gamma)$ be a strictly bimonoidal category. The monoidal functors $\blacklozenge$ and $\mathcal{T} \circ \blacklozenge$ define by lemma 4.4 two 2-functors $\mathcal{C} \boxtimes \mathcal{C}\text{-Cat} \rightarrow \mathcal{C}\text{-Cat}$. By theorem 4.10 any pseudo natural transformation is induced by a quasi monoidal transformation $(R, \sigma) : \blacklozenge \Rightarrow \mathcal{T} \circ \blacklozenge$, where $R := R^1 \boxtimes R^2 \in \mathcal{C} \boxtimes \mathcal{C}$ is the underlying object and $\sigma_C : R^1 \otimes C_{(1)} \boxtimes R^2 \otimes C_{(2)} \rightarrow C_{(2)} \otimes R^1 \boxtimes C_{(1)} \otimes R_{(2)}$ is the natural isomorphism. The coherence diagrams for being a quasi monoidal transformation read:

(D 5.1)

$$\begin{array}{ccc} R^1 C_{(1)} D_{(1)} \boxtimes R^2 C_{(2)} D_{(2)} & \xrightarrow{\sigma \otimes_{\mathcal{C} \boxtimes \mathcal{C}} 1} C_{(2)} R^1 D_{(1)} \boxtimes C_{(1)} R^2 D_{(2)} & \xrightarrow{1 \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \sigma} C_{(2)} D_{(2)} R^1 \boxtimes C_{(1)} D_{(1)} R^2 \\ \downarrow 1 \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \Xi & & \Xi \otimes_{\mathcal{C} \boxtimes \mathcal{C}} 1 \downarrow \\ R^1 (CD)_{(1)} \boxtimes R^2 (CD)_{(2)} & \xrightarrow{\sigma} & (CD)_{(2)} R^1 \boxtimes (CD)_{(1)} R^2 \end{array}$$

(D 5.2)

$$\begin{array}{ccc} R^1 I \boxtimes R^2 I & \xrightarrow{\rho} R^1 \boxtimes R^2 & \xrightarrow{\lambda^{-1}} IR^1 \boxtimes IR^2 \\ \downarrow 1 \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \Xi_0 & & \downarrow \Xi_0 \otimes_{\mathcal{C} \boxtimes \mathcal{C}} 1 \\ R^1 I_{(1)} \boxtimes R^2 I_{(2)} & \xrightarrow{\sigma} & I_{(2)} R^1 \boxtimes I_{(1)} R^2 \end{array}$$

Here we omitted the tensor product symbols $\otimes$ as a spacesaving measure. By lemma 4.5 the functor

$$\begin{aligned} (R \odot -) : (\mathcal{M} \boxtimes \mathcal{N})_{\diamond} &\rightarrow (\mathcal{M} \boxtimes \mathcal{N})_{T\diamond} \\ M \boxtimes N &\mapsto R^1 \odot M \boxtimes R^2 \odot N \end{aligned}$$

together with $\tilde{\sigma}$, defined as the composition

$$\begin{aligned} R \odot (C \odot (M \boxtimes N)) &= (R^1 \otimes C_{(1)}) \odot M \boxtimes (R^2 \otimes C_{(2)}) \odot N \\ &\xrightarrow{\sigma \odot M \boxtimes N} (C_{(2)} \otimes R^1) \odot M \boxtimes (C_{(1)} \otimes R^2) \odot N = C \odot (R \odot (M \boxtimes N)) \end{aligned} \quad (5.1)$$

is a $\mathcal{C}$ -functor. By theorem 4.6 this family of $\mathcal{C}$ -functors becomes a pseudo natural transformation, if we define for all $\mathcal{C} \boxtimes \mathcal{C}$ -functors $(\mathcal{H}, \tilde{\mathcal{H}}) : \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{M}' \boxtimes \mathcal{N}'$:

$$R_{\mathcal{H}}(M \boxtimes N) : (R^1 \boxtimes R^2) \odot \mathcal{H}(M \boxtimes N) \xrightarrow{\tilde{\mathcal{H}}^{-1}} \mathcal{H}((R^1 \boxtimes R^2) \odot M \boxtimes N) \quad (5.2)$$

We can think of $\mathcal{T} \circ \diamond$ inducing the symmetric tensor product $\boxtimes^{\text{sym}}$ on $\mathcal{C}\text{-Cat}$, since $\mathcal{T}$ being strongly involutory induces a pseudo natural transformation $\mathcal{T} : (\mathcal{M} \boxtimes \mathcal{N})_{\diamond} \Rightarrow (\mathcal{N} \boxtimes \mathcal{M})_{T\diamond}$. By composing this with the pseudo natural morphism $((R \odot -, \tilde{\sigma}), \tilde{R})$ we get the pseudo natural transformation given for $\mathcal{M}, \mathcal{N} \in \mathcal{C}\text{-Cat}$, by the $\mathcal{C}$ -functor $\mathcal{R}_{\mathcal{M}, \mathcal{N}} : \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{N} \boxtimes \mathcal{M}$, $M \boxtimes N \mapsto R^2 \odot N \boxtimes R^1 \odot M$ with $\mathcal{C}$ -structure and pseudo natural structure as defined above, but twisted by the involutor $\mathcal{T}$.

First we examine the special case $R = I \boxtimes I$, where $\sigma : \diamond \rightarrow \mathcal{T} \circ \diamond$ is a monoidal isomorphism. Then, $(\mathcal{T}, \tilde{\sigma}) : \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{N} \boxtimes \mathcal{M}$, $M \boxtimes N \mapsto N \boxtimes M$ is a $\mathcal{C}$ -functor, with $\mathcal{C}$ -structure

$$\tilde{\sigma} : \mathcal{T}(C \odot (M \boxtimes N)) = C_{(2)} \odot N \boxtimes C_{(1)} \odot M \xrightarrow{\mathcal{T}(\sigma) \odot N \boxtimes M} C_{(1)} \odot N \boxtimes C_{(2)} \odot M = C \odot \mathcal{T}(M \boxtimes N)$$

5.6. Theorem: *Let $(\mathcal{C}, \otimes, \diamond, I, \Gamma)$ be a bimonoidal category. Then, $\sigma : \diamond \Rightarrow \mathcal{T}\diamond$ is a monoidal cobraiding for $\mathcal{C}$, if and only if $((\mathcal{T}, \tilde{\sigma}), \tilde{T}_1, \tilde{T}_2, \tilde{\mathcal{J}})$ is a 2-braiding for $\mathcal{C}\text{-Cat}$.*

PROOF:

$\tilde{T}_1, \tilde{T}_2, \tilde{\mathcal{J}}$ are defined as identities on decomposable functors and hence are $\mathcal{C}$ -morphisms, anyway. Let $\mathcal{M}, \mathcal{N}, \mathcal{P} \in \mathcal{C}\text{-Cat}$. It is enough to show that on decomposable functors $M \boxtimes N \boxtimes P \in \mathcal{M} \boxtimes \mathcal{N} \boxtimes \mathcal{P}$ the equations

$$\begin{aligned} (1 \boxtimes \mathcal{T}_{M,P}) \circ (\mathcal{T}_{M,N} \boxtimes 1) &= \mathcal{T}_{M,N \boxtimes P} : M \boxtimes N \boxtimes P \mapsto N \boxtimes P \boxtimes M \\ (\mathcal{T}_{M,P} \boxtimes 1) \circ (1 \boxtimes \mathcal{T}_{N,P}) &= \mathcal{T}_{M \boxtimes N, P} : M \boxtimes N \boxtimes P \mapsto P \boxtimes M \boxtimes N \end{aligned}$$

also hold as equations of $\mathcal{C}$ -functors. We show this for the first equation. The $\mathcal{C}$ -structure belonging to the left hand side results as the following composition:

$$\begin{aligned} (1 \boxtimes \mathcal{T})(\mathcal{T} \boxtimes 1)(C \odot (M \boxtimes N \boxtimes P)) &= C_{(2)} \odot N \boxtimes C_{(3)} \odot P \boxtimes C_{(1)} \odot M \\ &\downarrow \sigma_{1|2} \odot (N \boxtimes P \boxtimes M) \\ (1 \boxtimes \mathcal{T})(C \odot (\mathcal{T} \boxtimes 1)(M \boxtimes N \boxtimes P)) &= C_{(1)} \odot N \boxtimes C_{(3)} \odot P \boxtimes C_{(2)} \odot M \\ &\downarrow \sigma_{2|3} \odot (N \boxtimes P \boxtimes M) \\ C \odot (1 \boxtimes \mathcal{T})(\mathcal{T} \boxtimes 1)(M \boxtimes N \boxtimes P) &= C_{(1)} \odot N \boxtimes C_{(2)} \odot P \boxtimes C_{(3)} \odot M \end{aligned}$$

To the right hand side belongs the $\mathcal{C}$ -structure

$$\begin{aligned} \mathcal{T}_{\mathcal{M}, \mathcal{N} \boxtimes \mathcal{P}}(C \odot (M \boxtimes N \boxtimes P)) &= C_{(2)} \odot N \boxtimes C_{(3)} \odot P \boxtimes C_{(1)} \odot M \\ &\quad \downarrow \sigma_{1,2|3} \odot (N \boxtimes P \boxtimes M) \\ \mathcal{T}(C \odot M \boxtimes N \boxtimes P) &= C_{(1)} \odot N \boxtimes C_{(2)} \odot P \boxtimes C_{(3)} \odot M \end{aligned}$$

Both morphisms agree, if σ fulfills axiom $(QR1)$ of a cobraiding. On the other hand, if both morphisms agree, then also for $M = N = P = I \in \mathcal{C}$, whence σ fulfills axiom $(QR1)$ of a cobraiding. With an analogous reasoning we see that axiom $(QL1)$ is equivalent to the second equation. $\square$

We can interpret this result according to the quantum hierarchy. If a 1-bialgebra H is cocommutative, then the vector space twist has the property of being an H -module morphism. In the 2-dimensional setup, cocommutativity as well as being a module morphism, is no longer a property, but a new datum. The theorem now claims that the different cocommutativity data for $\mathcal{C}$, namely the set of monoidal cobraidings σ in $\mathcal{C}$, are in bijection with the different data for the symmetry in $\mathfrak{C}_k$ being a module morphism, namely set of $\mathcal{C}$ -structures $\tilde{\sigma}$ that complete the pseudo natural transformation $\mathcal{T}$ to a transformation in $\mathcal{C}\text{-Cat}$.

To get non trivial solutions of the Zamolodchikov equation, we have to define non trivial modifications $\tilde{T}_1, \tilde{T}_2$. If we start with a monoidal isomorphism $\sigma : \blacklozenge \Rightarrow \mathcal{T}\blacklozenge$, such modifications are generated by inner isomorphisms τ_1 and τ_2 such that the two Yang Baxter triangles $(QL1)$ and $(QL2)$ of definition 3.6 are satisfied up to τ_1 and τ_2 , respectively. However, τ_1 and τ_2 then are endomorphisms of the neutral object, which seems to be too restricted. Therefore, to get non trivial solutions we consider now the case of quasi monoidal transformations $(R, \sigma) : \blacklozenge \Rightarrow \mathcal{T} \circ \blacklozenge$. The price we pay is that the 2-braidings we obtain this way are in general not invertible.

We have seen that (R, σ) induces a pseudo natural transformation $((\mathcal{R}, \tilde{\sigma}), \mathcal{R}_-)$ that we now want to extend to a 2-braid structure. As in the cocommutative case, we have two pairs of pseudo natural transformations,

$$(1 \boxtimes \mathcal{R}_{\mathcal{M}, \mathcal{P}}) \circ (\mathcal{R}_{\mathcal{M}, \mathcal{N}} \boxtimes 1) \quad \text{and} \quad \mathcal{R}_{\mathcal{M}, \mathcal{N} \boxtimes \mathcal{P}} : \mathcal{M} \boxtimes \mathcal{N} \boxtimes \mathcal{P} \rightarrow \mathcal{N} \boxtimes \mathcal{P} \boxtimes \mathcal{M} \quad (5.3)$$

$$(\mathcal{R}_{\mathcal{M}, \mathcal{P}} \boxtimes 1) \circ (1 \boxtimes \mathcal{R}_{\mathcal{N}, \mathcal{P}}) \quad \text{and} \quad \mathcal{R}_{\mathcal{M} \boxtimes \mathcal{N}, \mathcal{P}} : \mathcal{M} \boxtimes \mathcal{N} \boxtimes \mathcal{P} \rightarrow \mathcal{P} \boxtimes \mathcal{M} \boxtimes \mathcal{N} \quad (5.4)$$

that are, however, no longer equal even as mere functors. The left pseudo natural transformation of (5.3)

$$(1 \boxtimes \mathcal{R})(\mathcal{R} \boxtimes 1)(M \boxtimes N \boxtimes P) = R^2 \odot N \boxtimes r^2 \odot P \boxtimes r^1 \otimes R^1 \odot M$$

is induced by the quasi monoidal transformation with the underlying object $r^1 \otimes R^1 \boxtimes R^2 \boxtimes r^2$, where $R = r$ are two copies of R . In the sequel we only consider the pseudo natural transformation $R : \blacklozenge^* \Rightarrow (\mathcal{T}\blacklozenge)^*$, induced by (R, σ) as defined on page 71, not taking into account any more the twist coming from $\mathcal{T}$. The corresponding isomorphism reads:

$$\begin{aligned} r^1 \otimes R^1 \otimes C_{(1)} \boxtimes R^2 \otimes C_{(2)} \boxtimes r^2 \otimes C_{(3)} &\xrightarrow{\sigma_{1|2}} r^1 \otimes C_{(2)} \otimes R^1 \boxtimes C_{(1)} \otimes R^2 \boxtimes r^2 \otimes C_{(3)} \\ &\xrightarrow{\sigma_{2|3}} C_{(3)} \otimes r^1 \otimes R^1 \boxtimes C_{(1)} \otimes R^2 \boxtimes C_{(2)} \otimes r^2 \end{aligned}$$

The right pseudo natural transformation of equation (5.3)

$$\mathcal{R}(M \boxtimes N \boxtimes P) = R_{(1)}^2 \odot N \boxtimes R_{(2)}^2 \odot P \boxtimes R^1 \odot M$$

is induced by $R^1 \boxtimes \blacklozenge(R^2)$ with the isomorphism

$$R^1 \otimes C_{(1)} \boxtimes R_{(1)}^2 \otimes C_{(2)} \boxtimes R_{(2)}^2 \otimes C_{(3)} \xrightarrow{\sigma_{1|2,3}} C_{(3)} \otimes R^1 \boxtimes C_{(1)} \otimes R_{(1)}^2 \boxtimes C_{(2)} \otimes R_{(2)}^2$$

where $\sigma_{1|2,3}$ denotes the morphism $(1 \boxtimes \Xi^{-1}) \circ (1 \boxtimes \blacklozenge)(\sigma) \circ (1 \boxtimes \Xi)$. Analogously, the two functors from equation (5.4) are induced by the quasi monoidal transformations $r^1 \boxtimes R^1 \boxtimes r^2 \otimes R^2$ with the isomorphism

$$\begin{aligned} r^1 \otimes C_{(1)} \boxtimes R^1 \otimes C_{(2)} \boxtimes r^2 \otimes R^2 \otimes C_{(3)} &\xrightarrow{\sigma_{2|3}} r^1 \otimes C_{(1)} \boxtimes C_{(3)} \otimes R^1 \boxtimes r^2 \otimes C_{(2)} \otimes R^2 \\ &\xrightarrow{\sigma_{1|2}} C_{(2)} \otimes r^1 \boxtimes C_{(3)} \otimes R^1 \boxtimes C_{(1)} \otimes r^2 \otimes R^2 \end{aligned}$$

and $\blacklozenge(R^1) \boxtimes R^2$ with isomorphism

$$R_{(1)}^1 \otimes C_{(1)} \boxtimes R_{(2)}^1 \otimes C_{(2)} \boxtimes R^2 \otimes C_{(3)} \xrightarrow{\sigma_{1,2|3}} C_{(2)} \otimes r^1 \boxtimes C_{(3)} \otimes R^1 \boxtimes C_{(1)} \otimes r^2 \otimes R^2$$

We have to connect each of these two pairs of quasi monoidal transformations by a $\boxtimes$ -modification $\tilde{R}_{1(-|-)}$ and $\tilde{R}_{2(-|-)}$, respectively, giving the modifications

$$\begin{array}{ccc} & \mathcal{N} \boxtimes \mathcal{M} \boxtimes \mathcal{P} & \\ \mathcal{R}_{\mathcal{M}, \mathcal{N} \boxtimes \mathcal{P}} \nearrow & \Downarrow \tilde{R}_{1(\mathcal{M}|\mathcal{N}, \mathcal{P})} & \nwarrow \mathcal{N} \boxtimes \mathcal{R}_{\mathcal{M}, \mathcal{P}} \\ \mathcal{M} \boxtimes \mathcal{N} \boxtimes \mathcal{P} & \xrightarrow{\mathcal{R}_{\mathcal{M}, \mathcal{N} \boxtimes \mathcal{P}}} & \mathcal{N} \boxtimes \mathcal{P} \boxtimes \mathcal{M} \end{array} \quad \begin{array}{ccc} & \mathcal{M} \boxtimes \mathcal{P} \boxtimes \mathcal{N} & \\ \mathcal{M} \boxtimes \mathcal{R}_{\mathcal{N}, \mathcal{P}} \nearrow & \Downarrow \tilde{R}_{2(\mathcal{M}, \mathcal{N}|\mathcal{P})} & \nwarrow \mathcal{R}_{\mathcal{M}, \mathcal{P} \boxtimes \mathcal{N}} \\ \mathcal{M} \boxtimes \mathcal{N} \boxtimes \mathcal{P} & \xrightarrow{\mathcal{R}_{\mathcal{M} \boxtimes \mathcal{N}, \mathcal{P}}} & \mathcal{P} \boxtimes \mathcal{M} \boxtimes \mathcal{N} \end{array}$$

By theorem 4.10 such $\boxtimes$ -modifications are given by isomorphisms

$$\begin{aligned} \tilde{R}_{1(-|-)} &:= \mu^1 \boxtimes \delta^2 : r^1 \otimes R^1 \boxtimes R^2 \boxtimes r^2 \Rightarrow R^1 \boxtimes R_{(1)}^2 \boxtimes R_{(2)}^2 \quad \text{and} \\ \tilde{R}_{2(-|-)} &:= \delta^1 \boxtimes \mu^2 : r^1 \boxtimes R^1 \boxtimes r^2 \otimes R^2 \Rightarrow R_{(1)}^1 \boxtimes R_{(2)}^1 \boxtimes R^2, \end{aligned}$$

respectively. The coherence for being a $\boxtimes$ -modification for $\tilde{R}_1$ reads:

$$\begin{array}{ccc} & r^1 \otimes C_{(2)} \otimes R^1 \boxtimes C_{(1)} \otimes R^2 \boxtimes r^2 \otimes C_{(3)} & \\ \nearrow \sigma_{1|2} & & \searrow \sigma_{2|3} \\ r^1 \otimes R^1 \otimes C_{(1)} \boxtimes R^2 \otimes C_{(2)} \boxtimes r^2 \otimes C_{(3)} & & C_{(3)} \otimes r^1 \otimes R^1 \boxtimes C_{(1)} \otimes R^2 \boxtimes C_{(2)} \otimes r^2 \\ \downarrow \mu^1 \boxtimes \delta^2 \otimes c \boxtimes c \boxtimes c \boxtimes \blacklozenge^2(C) & & \downarrow \blacklozenge^2(C) \otimes c \boxtimes c \boxtimes c \boxtimes \mu^1 \boxtimes \delta^2 \\ R^1 \otimes C_{(1)} \boxtimes R_{(1)}^2 \otimes C_{(2)} \boxtimes R_{(2)}^2 \otimes C_{(3)} & \xrightarrow{(1 \boxtimes \blacklozenge)(\sigma)} & C_{(3)} \otimes R^1 \boxtimes C_{(1)} \otimes R_{(1)}^2 \boxtimes C_{(2)} \otimes R_{(2)}^2 \end{array}$$

The coherence law for $\tilde{R}_2$ reads:

$$\begin{array}{c}
\text{(D 5.4)} \\
\begin{array}{ccc}
& r^1 \otimes C_{(1)} \boxtimes C_{(3)} \otimes R^1 \boxtimes r^2 \otimes C_{(2)} \otimes R^2 & \\
\sigma_{2|3} \nearrow & & \searrow \sigma_{1|2} \\
r^1 \otimes C_{(1)} \boxtimes R^1 \otimes C_{(2)} \boxtimes r^2 \otimes R^2 \otimes C_{(3)} & & C_{(2)} \otimes r^1 \boxtimes C_{(3)} \otimes R^1 \boxtimes C_{(1)} \otimes r^2 \otimes R^2 \\
\downarrow \delta^1 \boxtimes \mu^2 \otimes_{C \boxtimes C \boxtimes C} \blacklozenge^2(C) & & \downarrow \blacklozenge^2(C) \otimes_{C \boxtimes C \boxtimes C} \delta^1 \boxtimes \mu^2 \\
R_{(1)}^1 \otimes C_{(1)} \boxtimes R_{(2)}^1 \otimes C_{(2)} \boxtimes R^2 \otimes C_{(3)} & \xrightarrow{(\blacklozenge \boxtimes 1)(\sigma)} & C_{(2)} \otimes R_{(1)}^1 \boxtimes C_{(3)} \otimes R_{(2)}^1 \boxtimes C_{(1)} \otimes R^2
\end{array}
\end{array}$$

Furthermore we need two $\boxtimes$ -modifications giving for any $\mathcal{M} \in \mathcal{D}\text{-Cat}$ the $\mathcal{C}$ -natural isomorphisms $\mathfrak{r}_{1\mathcal{M}} : \text{Id}_{\mathcal{M}} \Rightarrow \mathcal{R}_{\mathcal{M}, \text{vec}} : \mathcal{M} = \mathcal{M} \boxtimes \text{vec} \rightarrow \text{vec} \boxtimes \mathcal{M} = \mathcal{M}$ and $\mathfrak{r}_{2\mathcal{M}} : \text{Id}_{\mathcal{M}} \Rightarrow \mathcal{R}_{\text{vec}, \mathcal{M}} : \mathcal{M} = \text{vec} \boxtimes \mathcal{M} \rightarrow \mathcal{M} \boxtimes \text{vec} = \mathcal{M}$, respectively. The $\mathcal{C}$ -functor $\text{Id}_{\mathcal{M}}$ corresponds to the object $I_{\mathcal{C}}$ by the equivalence of theorem 4.10 and the $\mathcal{C}$ -functors $\mathcal{R}_{\mathcal{M}, \text{vec}}$ and $\mathcal{R}_{\text{vec}, \mathcal{M}}$ to the objects $R^1 \boxtimes \Gamma(R^2)$ and $\Gamma(R^1) \boxtimes R^2$, respectively. Thus, the $\boxtimes$ -modifications are given by isomorphisms $\mathfrak{r}_1 : I_{\mathcal{C}} \rightarrow R^1 \boxtimes \Gamma(R^2)$ and $\mathfrak{r}_2 : I_{\mathcal{C}} \rightarrow \Gamma(R^1) \boxtimes R^2$, satisfying the coherence conditions for a $\boxtimes$ -modification which read:

$$\begin{array}{ccc}
& C & \\
\mathfrak{r}_1 \otimes C \swarrow & & \searrow C \otimes \mathfrak{r}_1 \\
R^1 \otimes C_{(1)} \boxtimes \Gamma(R^2 \otimes C_{(2)}) & \xrightarrow{(\Gamma \otimes 1)(\sigma)} & C_{(2)} \otimes R^1 \boxtimes \Gamma(C_{(1)} \otimes R^2)
\end{array}$$

$$\begin{array}{ccc}
& C & \\
\mathfrak{r}_2 \otimes C \swarrow & & \searrow C \otimes \mathfrak{r}_2 \\
\Gamma(R^1 \otimes C_{(1)}) \boxtimes R^2 \otimes C_{(2)} & \xrightarrow{(1 \otimes \Gamma)(\sigma)} & \Gamma(C_{(2)} \otimes R^1) \boxtimes C_{(1)} \otimes R^2
\end{array}$$

So far, we have defined all the data of definition 1.10. To make a 2-braiding out of our system $(((\mathcal{R}, \sigma), \mathcal{R}_-), \tilde{R}_{1(-|-)}, \tilde{R}_{2(-|-)}, \mathfrak{r}_1, \mathfrak{r}_2)$, we have to ensure that all coherence diagrams of a braided Gray monoid are satisfied. Therefore we regard the regular representation $\mathcal{C} \in \mathcal{C}\text{-Cat}$ and start with the decomposable tensor $I \boxtimes I \boxtimes I \boxtimes I \in \mathcal{C} \boxtimes \mathcal{C} \boxtimes \mathcal{C} \boxtimes \mathcal{C}$. We take four copies $R = r = T = t$ of the object $R \in \mathcal{C} \boxtimes \mathcal{C}$. We denote the first application by R , the second by r , etc. To indicate on which tensors a certain morphism acts, we write the objects of its source as lower indices to the morphism, assuming that all the remaining tensors are acted on by the identity. We now translate the axioms of a braided Gray monoid into axioms for our to be quasitriangular structure. We pursue the policy that we always translate first the different paths around a polyhedron. They give the corners in our new diagrams. The faces of the original axioms give the edges in the new diagrams.

First we regard the tetrahedron $(\bullet \boxtimes (\bullet \boxtimes \bullet \boxtimes \bullet))$ with $A = B = C = D = I$. To each of the four paths around this tetrahedron corresponds a composition of applications of the pseudo natural transformation $\mathcal{R}$, namely

$$\begin{aligned}
ABCD \rightarrow BACD \rightarrow BCAD \rightarrow BCDA & \hat{=} T^1 \otimes r^1 \otimes R^1 \boxtimes R^2 \boxtimes r^2 \boxtimes T^2 \\
ABCD \rightarrow BCAD \rightarrow BCDA & \hat{=} r^1 \otimes R^1 \boxtimes R_{(1)}^2 \boxtimes R_{(2)}^2 \boxtimes r^2
\end{aligned}$$

$$\begin{aligned}
ABCD \rightarrow BACD \rightarrow BCDA &\hat{=} r^1 \otimes R^1 \boxtimes R^2 \boxtimes r_{(1)}^2 \boxtimes r_{(2)}^2 \\
ABCD \rightarrow BCDA &\hat{=} R^1 \boxtimes R_{(1)}^2 \boxtimes R_{(2)}^2 \boxtimes R_{(3)}^2
\end{aligned}$$

The four faces of the tetrahedron correspond to four $\boxtimes$ -modifications between these objects. This yields the following commutative diagram.

$$\begin{array}{ccc}
T^1 \otimes r^1 \otimes R^1 \boxtimes R^2 \boxtimes r^2 \otimes T^2 & \xrightarrow{(\mu^1 \boxtimes \delta^2)_{R,r}} & r^1 \otimes R^1 \boxtimes R_{(1)}^2 \boxtimes R_{(2)}^2 \boxtimes r^2 \\
(\mu^1 \boxtimes \delta^2)_{r,T} \downarrow & & \downarrow (1 \boxtimes \blacklozenge \boxtimes 1)(\mu^1 \boxtimes \delta^2) \\
r^1 \otimes R^1 \boxtimes R^2 \boxtimes r_{(1)}^2 \boxtimes r_{(2)}^2 & \xrightarrow{(1 \boxtimes 1 \boxtimes \blacklozenge)(\mu^1 \boxtimes \delta^2)} & R^1 \boxtimes R_{(1)}^2 \boxtimes R_{(2)}^2 \boxtimes R_{(3)}^2
\end{array}$$

In the same way we get from $((\bullet \boxtimes \bullet \boxtimes \bullet) \boxtimes \bullet)$ the following diagram

$$\begin{array}{ccc}
T^1 \boxtimes r^1 \boxtimes R^1 \boxtimes T^2 \otimes r^2 \otimes R^2 & \xrightarrow{(\delta^1 \boxtimes \mu^2)_{R,r}} & r^1 \boxtimes R_{(1)}^1 \boxtimes R_{(1)}^1 \boxtimes r^2 \otimes R^2 \\
(\delta^1 \boxtimes \mu^2)_{r,T} \downarrow & & \downarrow (1 \boxtimes \blacklozenge \boxtimes 1)(\delta^1 \boxtimes \mu^2) \\
r_{(1)}^1 \boxtimes r_{(2)}^1 \boxtimes R^1 \boxtimes r^2 \otimes R^2 & \xrightarrow{(\blacklozenge \boxtimes 1 \boxtimes 1)(\delta^1 \boxtimes \mu^2)} & R_{(1)}^1 \otimes R_{(2)}^1 \otimes R_{(3)}^1 \boxtimes R^2
\end{array}$$

In the diagram $((\bullet \boxtimes \bullet) \boxtimes (\bullet \boxtimes \bullet))$ occurs the Gray morphism $\boxtimes_{A,B}$. We have seen that this morphism is the identity on decomposable tensors. Therefore, the identity morphism occurs in the corresponding diagram for the quasitriangular structure. Here it only permutes the order of the applications of R . As a spacesaving measure we quit writing tensor product symbols $\otimes$. To keep track what is going on we write next to any application of a $\boxtimes$ -modification the corresponding number of the 2-morphism in the original diagram.

$$\begin{array}{ccc}
t^1 r^1 \boxtimes T^1 R^1 \boxtimes r^2 R^2 \boxtimes t^2 T^2 & \xleftarrow{\text{Id} \hat{=} 6.} & t^1 T^1 \boxtimes r^1 R^1 \boxtimes T^2 R^2 \boxtimes t^2 r^2 \\
& \searrow (\delta^1 \boxtimes \mu^2)_{R,r} \hat{=} 1. & \searrow (\mu^1 \boxtimes \delta^2)_{R,r} \hat{=} 3. \\
T^1 R_{(1)}^1 \boxtimes r^1 R_{(2)}^1 \boxtimes R^2 \boxtimes T^2 r^2 & & T^1 r^1 \boxtimes R^1 \boxtimes r^2 R_{(1)}^2 \boxtimes T^2 R_{(2)}^2 \\
\downarrow (\delta^1 \boxtimes \mu^2)_{r,T} \hat{=} 2. & & \downarrow (\mu^1 \boxtimes \delta^2)_{r,T} \hat{=} 4. \\
r_{(1)}^1 R_{(1)}^1 \boxtimes r_{(2)}^1 R_{(2)}^1 \boxtimes R^2 \boxtimes r^2 & & r^1 \boxtimes R^1 \boxtimes r_{(1)}^2 R_{(1)}^2 \boxtimes r_{(2)}^2 R_{(2)}^2 \\
& \searrow \blacklozenge(\mu^1) \boxtimes \delta^2 \hat{=} 7. & \swarrow \delta^1 \boxtimes \blacklozenge(\mu^2) \hat{=} 5. \\
& R_{(1)}^1 \boxtimes R_{(2)}^1 \boxtimes R_{(1)}^2 \boxtimes R_{(2)}^2 &
\end{array}$$

In the last coherence diagram $S^+ = S^-$ occur the 2-morphisms $\mathcal{R}_{(-,\mathcal{R})}^{-1} = 1 \boxtimes \sigma$ and $\mathcal{R}_{(\mathcal{R},-)} = \sigma^{-1} \boxtimes 1$, as defined in (5.2). The left-hand way around the following diagram corresponds to the left face of diagram $(S^+ = S^-)$ of definition 1.10, the right-hand way to the right face.

$$\begin{array}{ccc}
& r^1 R^1 \boxtimes T^1 R^2 \boxtimes T^2 r^2 & \\
& \swarrow (\mu^1 \boxtimes \delta^2)_{R,r} \quad \searrow (T \boxtimes 1)(\delta^1 \boxtimes \mu^2)_{r,T} & \\
R^1 \boxtimes r^1 R^2_{(1)} \boxtimes r^2 R^2_{(2)} & & r^1_{(2)} R^1 \boxtimes r^1_{(1)} R^2 \boxtimes r^2 \\
\downarrow 1 \boxtimes \sigma & & \downarrow \sigma^{-1} \boxtimes 1 \\
r^1 \boxtimes r^2_{(2)} R^1 \boxtimes r^2_{(1)} R^2 & & r^1 R^2_{(1)} \boxtimes r^2 R^2_{(2)} \boxtimes R^1 \\
\swarrow (1 \boxtimes T)(\mu^1 \boxtimes \delta^2)^{-1}_{r,T} \quad \searrow (\delta^1 \boxtimes \mu^2)^{-1}_{R,r} & & \\
& T^1 r^1 \boxtimes T^2 R^1 \boxtimes r^2 R^2 &
\end{array}$$

Next we figure out the coherence diagrams for the unit. We write for each axiom its hieroglyph and the corresponding composition of morphisms. For the first four axioms it is understood that each composition has to agree with the identity.

$$\begin{array}{ll}
(\bullet \boxtimes (I \boxtimes \bullet)) & R^1 \boxtimes R^2 \xrightarrow{\tau_1} r^1 \otimes R^1 \boxtimes \Gamma(R^2) \boxtimes r^2 \xrightarrow{\mu^1 \boxtimes \delta^2} R^1 \boxtimes \Gamma(R^2_{(1)}) \boxtimes R^2_{(2)} \cong R^1 \boxtimes R^2 \\
(\bullet \boxtimes (\bullet \boxtimes I)) & R^1 \boxtimes R^2 \xrightarrow{\tau_1} r^1 \otimes R^1 \boxtimes R^2 \boxtimes \Gamma(r^2) \xrightarrow{\mu^1 \boxtimes \delta^2} R^1 \boxtimes R^2_{(1)} \boxtimes \Gamma(R^2_{(2)}) \cong R^1 \boxtimes R^2 \\
((\bullet \boxtimes I) \boxtimes \bullet) & R^1 \boxtimes R^2 \xrightarrow{\tau_2} r^1 \boxtimes \Gamma(R^1) \boxtimes r^2 \otimes R^2 \xrightarrow{\delta^1 \boxtimes \mu^2} R^1_{(1)} \boxtimes \Gamma(R^1_{(2)}) \boxtimes R^2 \cong R^1 \boxtimes R^2 \\
((I \boxtimes \bullet) \boxtimes \bullet) & R^1 \boxtimes R^2 \xrightarrow{\tau_2} \Gamma(r^1) \boxtimes R^1 \boxtimes r^2 \otimes R^2 \xrightarrow{\delta^1 \boxtimes \mu^2} \Gamma(R^1_{(1)}) \boxtimes R^1_{(2)} \boxtimes R^2 \cong R^1 \boxtimes R^2
\end{array}$$

The remaining axioms read:

$$\begin{array}{ll}
((\bullet \boxtimes \bullet) \boxtimes I) & I \boxtimes I \xrightarrow{\tau_1 \boxtimes \tau_1} r^1 \boxtimes R^1 \boxtimes \Gamma(r^2 \otimes R^2) \xrightarrow{\delta^1 \boxtimes \Gamma(\mu^2)} R^1_{(1)} \boxtimes R^1_{(2)} \boxtimes \Gamma(R^2) = \\
& I \boxtimes I \xrightarrow{\blacklozenge(\tau_1)} R^1_{(1)} \boxtimes R^1_{(2)} \boxtimes \Gamma(R^2) \\
(I \boxtimes (\bullet \boxtimes \bullet)) & I \boxtimes I \xrightarrow{\tau_2 \boxtimes \tau_2} \Gamma(r^1 \otimes R^1) \boxtimes R^2 \boxtimes r^2 \xrightarrow{\Gamma(\mu^1) \boxtimes \delta^2} \Gamma(R^1) \boxtimes R^2_{(1)} \boxtimes R^2_{(2)} = \\
& I \boxtimes I \xrightarrow{\blacklozenge(\tau_2)} \Gamma(R^1) \boxtimes R^2_{(1)} \boxtimes R^2_{(2)} \\
(I \boxtimes I) & I \boxtimes I \xrightarrow{(\Gamma \boxtimes 1)\tau_1} \Gamma(R^1) \boxtimes \Gamma(R^2) = I \boxtimes I \xrightarrow{(1 \boxtimes \Gamma)\tau_2} \Gamma(R^1) \boxtimes \Gamma(R^2)
\end{array}$$

We refer to these axioms by the same hieroglyphs as for the original axioms of a braided monoidal 2-category. Now we can define

5.7. Definition: A 2-quasitriangular structure for a 2-bialgebra $(\mathcal{C}, \otimes, I, \blacklozenge, \Gamma)$ consists of a quasi monoidal transformation $(R, \sigma) : \blacklozenge \Rightarrow T\blacklozenge$ and four invertible $\boxtimes$ -modifications

$$\begin{aligned}
\tilde{R}_1 &:= (\mu^1 \boxtimes \delta^2)_{R,r} : r^1 \otimes R^1 \boxtimes R^2 \boxtimes r^2 \Rightarrow R^1 \boxtimes R^2_{(1)} \boxtimes R^2_{(2)} \\
\tilde{R}_2 &:= (\delta^1 \boxtimes \mu^2)_{R,r} : r^1 \boxtimes R^1 \boxtimes r^2 \otimes R^2 \Rightarrow R^1_{(1)} \boxtimes R^1_{(2)} \boxtimes R^2 \\
\tau_1 &: I \Rightarrow R^1 \boxtimes \Gamma(R^2) \\
\tau_2 &: I \Rightarrow \Gamma(R^1) \boxtimes R^2
\end{aligned}$$

such that the coherence diagrams $(\bullet \boxtimes (\bullet \boxtimes \bullet \boxtimes \bullet)), ((\bullet \boxtimes \bullet \boxtimes \bullet) \boxtimes \bullet), ((\bullet \boxtimes \bullet) \boxtimes (\bullet \boxtimes \bullet)), (S^+ = S^-), (\bullet \boxtimes (I \boxtimes \bullet)), (\bullet \boxtimes (\bullet \boxtimes I)), ((\bullet \boxtimes I) \boxtimes \bullet), ((I \boxtimes \bullet) \boxtimes \bullet), ((\bullet \boxtimes \bullet) \boxtimes I), (I \boxtimes (\bullet \boxtimes \bullet))$ and $(I \boxtimes I)$ commute.

In the same way we can now figure out the axioms for morphisms of 2-quasitriangular structures. Let $(R, \sigma_R, \tilde{R}_1, \tilde{R}_2, \mathfrak{r}_{1R}, \mathfrak{r}_{2R})$ and $(T, \sigma_T, \tilde{T}_1, \tilde{T}_2, \mathfrak{r}_{1T}, \mathfrak{r}_{2T})$ be two quasitriangular structures on $\mathcal{C}$ and let $\mathfrak{a} : R^1 \boxtimes R^2 \Rightarrow T^1 \boxtimes T^2$ be a $\boxtimes$ -modification. This means the commutativity of

$$\begin{array}{ccc} R^1 \otimes C_{(1)} \boxtimes R^2 \otimes C_{(2)} & \xrightarrow{\sigma_R} & C_{(2)} \otimes R^1 \boxtimes C_{(1)} \otimes R^2 \\ \mathfrak{a} \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \blacklozenge(C) \downarrow & & \downarrow T \blacklozenge(C) \otimes_{\mathcal{C} \boxtimes \mathcal{C}} \mathfrak{a} \\ T^1 \otimes C_{(1)} \boxtimes T^2 \otimes C_{(2)} & \xrightarrow{\sigma_T} & C_{(2)} \otimes T^1 \boxtimes C_{(1)} \otimes T^2 \end{array}$$

Under which conditions does this modification induce a morphism of 2-braid structures? We have to make sure that the three conditions of definition 1.13 hold. The two conditions on the unit yield the following two diagrams.

$$\begin{array}{ccc} & I & \\ \mathfrak{r}_{1R} \swarrow & & \searrow \mathfrak{r}_{1T} \\ R^1 \boxtimes \Gamma(R^2) & \xrightarrow{(1 \boxtimes \Gamma)(\mathfrak{a})} & T^1 \boxtimes \Gamma(T^2) \end{array} \quad \begin{array}{ccc} & I & \\ \mathfrak{r}_{2R} \swarrow & & \searrow \mathfrak{r}_{2T} \\ \Gamma(R^1) \boxtimes R^2 & \xrightarrow{(\Gamma \boxtimes 1)(\mathfrak{a})} & \Gamma(T^1) \boxtimes T^2 \end{array}$$

The two coherence diagrams of the third condition yield:

$$\begin{array}{ccc} r^1 \otimes R^1 \boxtimes R^2 \boxtimes r^2 & \xrightarrow{\mu^1 \boxtimes \delta^2} & R^1 \boxtimes R^2_{(1)} \boxtimes R^2_{(2)} \\ \downarrow \mathfrak{a}_R & & \downarrow 1 \boxtimes \blacklozenge(\mathfrak{a}) \\ r^1 \otimes T^1 \boxtimes T^2 \boxtimes r^2 & & \\ \downarrow \mathfrak{a}_r & & \\ t^1 \otimes T^2 \boxtimes T^2 \boxtimes t^2 & \xrightarrow{\mu^1 \boxtimes \delta^2} & T^1 \boxtimes T^2_{(1)} \boxtimes T^2_{(2)} \end{array} \quad \begin{array}{ccc} r^1 \otimes R^1 \boxtimes r^2 \otimes R^2 & \xrightarrow{\delta^1 \boxtimes \mu^2} & R^1_{(1)} \boxtimes R^1_{(2)} \boxtimes R^2 \\ \downarrow \mathfrak{a}_R & & \downarrow \blacklozenge(\mathfrak{a}) \boxtimes 1 \\ r^1 \otimes T^1 \boxtimes r^2 \otimes T^2 & & \\ \downarrow \mathfrak{a}_r & & \\ t^1 \otimes T^1 \boxtimes t^2 \otimes T^2 & \xrightarrow{\delta^1 \boxtimes \mu^2} & T^1_{(1)} \boxtimes T^1_{(2)} \boxtimes T^2 \end{array}$$

The quasitriangular structures for a 2-bialgebra $\mathcal{C}$, together with their morphisms, constitute a category $\text{Quatri}(\mathcal{H})$. Now we can state our main theorem.

5.8. Theorem: *Given a strict 2-bialgebra $(\mathcal{C}, \otimes, I, \blacklozenge, \Gamma)$, there is a category equivalence between the category $\text{Quatri}(\mathcal{C})$ of quasitriangular structures for $\mathcal{C}$ and $2\text{-Braid}(\mathcal{C}\text{-Cat})$ the category of 2-braid structures for the monoidal 2-category $\mathcal{C}\text{-Cat}$.*

PROOF:

The result follows immediately from theorem 4.10 and theorem 4.11, since our axioms for 2-quasitriangular structures obviously induce the coherence axioms of 2-braid structures for the generated pseudo natural transformations. $\square$

Now we show that for a strict 2-bialgebra $\mathcal{H}$, whose underlying category is a finite 2-vector space, a 2-quasitriangular structure induces a bimonoidal functor $\mathcal{H}^* \rightarrow \mathcal{H}^{\text{sym}}$ and a morphism of 2-quasitriangular structures induces a bimonoidal transformation. We start with a finite 2-vector space $\mathcal{H}$ that carries a strict bimonoidal structure $(\mathcal{H}, \otimes, \blacklozenge, I, \Gamma)$. Let $(R^1 \boxtimes R^2, \sigma, \mu^1 \boxtimes \delta^2, \delta^1 \boxtimes \mu^2)$ be a 2-quasitriangular structure for $\mathcal{H}$. It is comfortable to identify $\mathcal{H}^*$ with the functor category $\mathfrak{C}_k(\mathcal{H}, \text{vec})$. We define a functor:

$$\begin{aligned} \mathcal{J}_{\mathcal{R}} : \mathcal{H}^* &\rightarrow \mathcal{H}^{\text{sym}} \\ \mathcal{F} &\mapsto R^1 \boxtimes \mathcal{F}(R^2) \end{aligned}$$

This functor carries a monoidal structure defined by:

$$\begin{aligned} \mathcal{J}_{\mathcal{R}}(\mathcal{F}) \otimes^{\text{sym}} \mathcal{J}_{\mathcal{R}}(\mathcal{G}) &= \mathcal{J}_{\mathcal{R}}(\mathcal{G}) \otimes \mathcal{J}_{\mathcal{R}}(\mathcal{F}) = r^1 \otimes R^1 \boxtimes \mathcal{F}(R^2) \boxtimes \mathcal{G}(r^2) \\ &\xrightarrow{\mu^1 \boxtimes \delta^2} R^1 \boxtimes \mathcal{F}(R^2_{(1)}) \boxtimes \mathcal{G}(R^2_{(2)}) = R^1 \boxtimes (\mathcal{F} \otimes \mathcal{G})(R^2) = \mathcal{J}_{\mathcal{R}}(\mathcal{F} \otimes \mathcal{G}) \end{aligned}$$

and

$$I \xrightarrow{\tau_1} R^1 \boxtimes \Gamma(R^2) = \mathcal{J}_{\mathcal{R}}(\Gamma) = \mathcal{J}_{\mathcal{R}}(I_{H^*})$$

We have to verify that this definition fulfills the associativity law and the unitary laws of a monoidal functor. We have $\mathcal{J}_{\mathcal{R}}(\mathcal{F}) \otimes^{\text{sym}} (\mathcal{J}_{\mathcal{R}}(\mathcal{G}) \otimes^{\text{sym}} \mathcal{J}_{\mathcal{R}}(\mathcal{K})) \cong (T^1 \otimes r^1) \otimes R^1 \boxtimes \mathcal{F}(R^2) \boxtimes \mathcal{G}(r^2) \boxtimes \mathcal{K}(T^2)$ and $\mathcal{J}_{\mathcal{R}}(\mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{K})) \cong R^1 \boxtimes \mathcal{F}(R^1_{(1)}) \boxtimes \mathcal{G}(R^1_{(2)}) \boxtimes \mathcal{K}(R^1_{(3)})$ and the coherence axiom for associativity has the following form.

$$\begin{array}{ccc} (T^1 \otimes r^1) \otimes R^1 \boxtimes \mathcal{F}(R^2) \boxtimes \mathcal{G}(r^2) \boxtimes \mathcal{K}(T^2) & \longrightarrow & T^1 \otimes (r^1 \otimes R^1) \boxtimes \mathcal{F}(R^2) \boxtimes \mathcal{G}(r^2) \boxtimes \mathcal{K}(T^2) \\ \downarrow (\mu^1 \boxtimes \delta^2)_{r,T} & & \downarrow (\mu^1 \boxtimes \delta^2)_{R,r} \\ r^1 \otimes R^1 \boxtimes \mathcal{F}(R^2) \boxtimes \mathcal{G}(r^2_{(1)}) \boxtimes \mathcal{K}(r^2_{(2)}) & & r^2 \otimes R^2 \boxtimes \mathcal{F}(R^2) \boxtimes \mathcal{G}(r^2_{(1)}) \boxtimes \mathcal{K}(r^2_{(2)}) \\ \downarrow \mu^1 \boxtimes \delta^2 & & \downarrow \mu^1 \boxtimes \delta^2 \\ R^1 \boxtimes \mathcal{F}(R^2_{(1)}) \boxtimes \mathcal{G}(R^2_{(2)}) \boxtimes \mathcal{K}(R^2_{(3)}) & \longrightarrow & R^1 \boxtimes \mathcal{F}(R^2_{(1)}) \boxtimes \mathcal{G}(R^2_{(2)}) \boxtimes \mathcal{K}(R^2_{(3)}) \end{array}$$

Obviously, this is included by the axiom $(\bullet \boxtimes (\bullet \boxtimes \bullet \boxtimes \bullet))$ for $\mathcal{R}$. For the unitary laws we observe that $\mathcal{J}_{\mathcal{R}}(\mathcal{F}) \otimes^{\text{sym}} I \cong I \otimes R^1 \boxtimes \mathcal{F}(R^2)$. Thus, the coherence axioms have the following form.

$$\begin{array}{ccccc} R^1 \otimes I \boxtimes \mathcal{F}(R^2) & \xrightarrow{\tau_1} & R^1 \otimes r^1 \boxtimes \Gamma(r^2) \boxtimes \mathcal{F}(R^2) & \xrightarrow{\mu^1 \boxtimes \delta^2} & R^1 \boxtimes \Gamma(R^2_{(1)}) \boxtimes \mathcal{F}(R^2_{(2)}) \\ & \searrow \rho & & \swarrow \lambda_c & \\ & & R^1 \boxtimes \mathcal{F}(R^2) & & \\ I \otimes R^1 \boxtimes \mathcal{F}(R^2) & \xrightarrow{\tau_1} & r^1 \otimes R^1 \boxtimes \mathcal{F}(R^2) \boxtimes \Gamma(r^2) & \xrightarrow{\mu^1 \boxtimes \delta^2} & R^1 \boxtimes \mathcal{F}(R^2_{(1)}) \boxtimes \Gamma(R^2_{(2)}) \\ & \searrow \lambda & & \swarrow \rho_c & \\ & & R^1 \boxtimes \mathcal{F}(R^2) & & \end{array}$$

These are included by the axioms $(\bullet \boxtimes (I \boxtimes \bullet))$ and $(\bullet \boxtimes (\bullet \boxtimes I))$, respectively.

For the comonoidal structure we define

$$\begin{aligned}
\mathcal{J}_{\mathcal{R}} \boxtimes \mathcal{J}_{\mathcal{R}}(\blacklozenge(\mathcal{F})) &= \mathcal{J}_{\mathcal{R}}(\mathcal{F}_{(1)}) \boxtimes \mathcal{J}_{\mathcal{R}}(\mathcal{F}_{(2)}) = r^1 \boxtimes R^1 \boxtimes \mathcal{F}_{(1)}(r^2) \boxtimes \mathcal{F}_{(2)}(R^2) \\
&= r^1 \boxtimes R^1 \boxtimes \mathcal{F}(r^1 \otimes R^1) \xrightarrow{\delta^1 \boxtimes \mu^2} R_{(1)}^1 \boxtimes R_{(2)}^1 \boxtimes \mathcal{F}(R^2) = \blacklozenge(R^1) \boxtimes \mathcal{F}(R^2) \\
&= \blacklozenge(\mathcal{J}_{\mathcal{R}}(\mathcal{F}))
\end{aligned}$$

and

$$\Gamma_{\mathcal{H}^*}(\mathcal{F}) = \mathcal{F}(I_{\mathcal{H}}) \xrightarrow{\tau_2} \Gamma(R^1) \boxtimes \mathcal{F}(R^2) = \Gamma(\mathcal{J}_{\mathcal{R}}(\mathcal{F}))$$

With a similar reasoning as before we can show that the coherence axiom for the coassociativity of a comonoidal functors is included by $((\bullet \boxtimes \bullet \boxtimes \bullet) \boxtimes \bullet)$ and the coherence axioms for the units are included by $((\bullet \boxtimes I) \boxtimes \bullet)$ and $((I \boxtimes \bullet) \boxtimes \bullet)$, respectively.

Next we prove that the four compatibility axioms of definition 3.12 for a bimonoidal functor are also satisfied. We start by rewriting the vertices of diagram 3.12(1) in a way that enables us to apply the monoidal and comonoidal structure morphisms. The first vertex of the diagram reads:

$$\begin{aligned}
&\mathcal{J}_{\mathcal{R}}(\mathcal{F}_{(1)}) \otimes^{\text{sym}} \mathcal{J}_{\mathcal{R}}(\mathcal{G}_{(1)}) \boxtimes \mathcal{J}_{\mathcal{R}}(\mathcal{F}_{(2)}) \otimes^{\text{sym}} \mathcal{J}_{\mathcal{R}}(\mathcal{G}_{(2)}) \\
&= r^1 \otimes R^1 \boxtimes \mathcal{F}_{(1)}(R^2) \boxtimes \mathcal{G}_{(1)}(r^2) \boxtimes t^1 \otimes T^1 \boxtimes \mathcal{F}_{(2)}(T^2) \boxtimes \mathcal{G}_{(2)}(t^2) \\
&= r^1 \otimes R^1 \boxtimes t^1 \otimes T^1 \boxtimes \mathcal{F}(R^2 \otimes T^2) \boxtimes \mathcal{G}(r^2 \otimes t^2)
\end{aligned} \tag{5.5}$$

The two upper vertices of diagram 3.12(1) agree since we are in a strict bimonoidal category and hence $\Xi = \text{id}$.

$$\begin{aligned}
&\mathcal{J}_{\mathcal{R}}(\mathcal{F}_{(1)} \otimes \mathcal{G}_{(1)}) \boxtimes \mathcal{J}_{\mathcal{R}}(\mathcal{F}_{(2)} \otimes \mathcal{G}_{(2)}) = \mathcal{J}_{\mathcal{R}}((\mathcal{F} \otimes \mathcal{G})_{(1)}) \boxtimes \mathcal{J}_{\mathcal{R}}((\mathcal{F} \otimes \mathcal{G})_{(2)}) \\
&= R^1 \boxtimes r^1 \boxtimes \mathcal{F}_{(1)}(R_{(1)}^2) \boxtimes \mathcal{G}_{(1)}(R_{(2)}^2) \boxtimes \mathcal{F}_{(2)}(r_{(1)}^2) \boxtimes \mathcal{G}_{(2)}(r_{(2)}^2) \\
&= R^1 \boxtimes r^1 \boxtimes \mathcal{F}((R^2 \otimes r^2)_{(1)}) \boxtimes \mathcal{G}((R^2 \otimes r^2)_{(2)})
\end{aligned} \tag{5.6}$$

For the same reason the two lower vertices agree.

$$\begin{aligned}
&\mathcal{J}_{\mathcal{R}}(\mathcal{F})_{(1)} \otimes^{\text{sym}} \mathcal{J}_{\mathcal{R}}(\mathcal{G})_{(1)} \boxtimes \mathcal{J}_{\mathcal{R}}(\mathcal{F})_{(2)} \otimes^{\text{sym}} \mathcal{J}_{\mathcal{R}}(\mathcal{G})_{(2)} \\
&= (\mathcal{J}_{\mathcal{R}}(\mathcal{F}) \otimes^{\text{sym}} \mathcal{J}_{\mathcal{R}}(\mathcal{G}))_{(1)} \boxtimes (\mathcal{J}_{\mathcal{R}}(\mathcal{F}) \otimes^{\text{sym}} \mathcal{J}_{\mathcal{R}}(\mathcal{G}))_{(2)} \\
&= (R^1 \otimes r^1)_{(1)} \boxtimes (R^1 \otimes r^1)_{(2)} \boxtimes \mathcal{F}(r^2) \boxtimes \mathcal{G}(R^2)
\end{aligned} \tag{5.7}$$

The final vertex reads:

$$\begin{aligned}
\mathcal{J}_{\mathcal{R}}(\mathcal{F} \otimes \mathcal{G})_{(1)} \boxtimes \mathcal{J}_{\mathcal{R}}(\mathcal{F} \otimes \mathcal{G})_{(2)} &= R_{(1)}^1 \boxtimes R_{(2)}^1 \boxtimes (\mathcal{F} \otimes \mathcal{G})(R^2) \\
&= R_{(1)}^1 \boxtimes R_{(2)}^1 \boxtimes \mathcal{F}(R_{(1)}^2) \boxtimes \mathcal{G}(R_{(2)}^2)
\end{aligned} \tag{5.8}$$

We can apply $\xi \boxtimes \xi = (\mu^1 \boxtimes \delta^2) \boxtimes (\mu^1 \boxtimes \delta^2)$ to get from (5.5) to (5.6) and then $\xi_{\mathcal{F} \otimes \mathcal{G}}^c = \delta^1 \boxtimes \blacklozenge(\mu^2)$ to get from (5.6) to the final vertex (5.8). This shows that the right-hand way around diagram $((\bullet \boxtimes \bullet) \boxtimes (\bullet \boxtimes \bullet))$ includes the upper way around 3.12(1). In the same way we can apply the

morphism $\xi^c \otimes_{C \boxtimes C} \xi^c = \delta^1 \boxtimes \mu^2 \otimes_{C \boxtimes C} \delta^1 \boxtimes \mu^2$ to get from (5.5) to (5.7) and then $\blacklozenge(\xi) = \blacklozenge(\mu^1) \boxtimes \delta^2$ to get to (5.8). This shows that the left-hand way around diagram $((\bullet \boxtimes \bullet) \boxtimes (\bullet \boxtimes \bullet))$ includes the lower way around 3.12(1) and hence we showed that $((\bullet \boxtimes \bullet) \boxtimes (\bullet \boxtimes \bullet))$ includes diagram 3.12(1). Analogously one can show that the axiom 3.12(2) is included by $((\bullet \boxtimes \bullet) \boxtimes I)$, axiom 3.12(3) is included by $(I \boxtimes (\bullet \boxtimes \bullet))$ and the last diagram 3.12(4) is included by $(I \boxtimes I)$.

Hence we have proved that any quasitriangular structure induces a bimonoidal functor. Note that the axiom $\mathbf{S}^+ = \mathbf{S}^-$ has no meaning for this functor. This fits the situation in one lower dimension, where we could only translate the axioms (QR2) - (QR5).

As a last step we show that any morphism $\mathfrak{a} : \mathcal{R}^1 \boxtimes \mathcal{R}^2 \Rightarrow T^1 \boxtimes T^2$ of quasitriangular structures induces a bimonoidal transformation. We define $\mathcal{J}_{\mathfrak{a}} : \mathcal{J}_{\mathcal{R}} \Rightarrow \mathcal{J}_{\mathcal{T}}$ by $\mathcal{J}_{\mathcal{R}}(\mathcal{F}) = R^1 \boxtimes \mathcal{F}(R^2) \xrightarrow{\mathfrak{a}} T^1 \boxtimes \mathcal{F}(T^2) = \mathcal{J}_{\mathcal{T}}(\mathcal{F})$. Plainly this defines a natural transformation. We only show that this transformation is monoidal, comonoidality being similar. To prove monoidality, the following two diagrams have to commute, which are included by the axioms of morphisms of 2-quasitriangular structures.

$$\begin{array}{ccc}
 r^1 \otimes R^1 \boxtimes \mathcal{F}(R^2) \boxtimes \mathcal{G}(r^2) & \xrightarrow{\xi_{\mathcal{R}} = \mu^1 \boxtimes \delta^2} & R^1 \boxtimes (\mathcal{F} \otimes \mathcal{G})(R^2) \\
 \downarrow \mathfrak{a} \boxtimes \mathfrak{a} & & \downarrow \mathfrak{a} \\
 t^1 \otimes T^1 \boxtimes \mathcal{F}(T^2) \boxtimes \mathcal{G}(t^2) & \xrightarrow{\xi_{\mathcal{T}} = \mu^1 \boxtimes \delta^2} & T^1 \boxtimes (\mathcal{F} \otimes \mathcal{G})(T^2)
 \end{array}
 \quad
 \begin{array}{ccc}
 & I & \\
 \swarrow \mathfrak{r}_{1R} & & \searrow \mathfrak{r}_{1T} \\
 R^1 \boxtimes \Gamma(R^2) & \xrightarrow{(1 \boxtimes \Gamma)(\mathfrak{a})} & T^1 \boxtimes \Gamma(T^2)
 \end{array}$$

We showed in detail how algebraic structure on the 2-category of representations of a Hopf category can be expressed in terms of the Hopf category. There is no need to stop at this point. In a next step we could examine braided 2-functors. To obtain those, a modification $f_R : R^1 \otimes X^1 \boxtimes R^2 \otimes X^2 \Rightarrow X^1 \otimes R^1 \boxtimes X^2 \otimes R^2$ is needed, fulfilling coherence axioms analogous to those in definition A.4. The condition for a monoidal pseudo natural transformation to be braided is then a generalized dual cocommutativity condition for the corresponding cogeбра. In case the 2-braiding is induced by a monoidal cobraiding σ , this is in fact the dual cocommutativity of definition 3.10. In the same way we could treat involutory structures, then. Rather than doing so we show how the theory can be used to work out examples of braided Gray monoids.

5.3 Categorification of the Drinfeld Double of a Finite Group

In this section we show how examples for braided monoidal 2-categories can be constructed from any finite group. Therefore we fix a finite group G for the rest of this chapter. As described in example 3.17, the category of G -graded vector spaces, $\mathcal{M}^{kG}$ is a Hopf category, called the ‘categorification’ of G . To this category we apply a higher dimensional analog of the Drinfeld Double construction. In [CY2], the resulting Hopf category $\mathfrak{D}(G)$ is called the categorification of the Drinfeld Double. This Hopf category can be described as follows. As a 2-vector space $\mathfrak{D}(G)$ has a basis $\{(g, \hat{x}) | g, x \in G\}$. The Hopf category structure functors are

defined by the same formulas as the Hopf algebra structure morphisms of the usual Drinfeld Double.

$$\begin{aligned}
(g, \hat{x}) \otimes (h, \hat{y}) &= \delta_{h^{-1}xh, y}(gh, \hat{x}) \\
I &= \oplus_x (e, \hat{x}) \\
\blacklozenge(g, \hat{x}) &= \oplus_k (g, \hat{k}) \boxtimes (g, \widehat{xk^{-1}}) \\
\Gamma(g, \hat{x}) &= \delta_{x, e} \\
\mathcal{S}(g, \hat{x}) &= (g^{-1}, \widehat{gx^{-1}g^{-1}})
\end{aligned}$$

Applying lemma 2.13, it is easily seen that $\mathfrak{D}(G)$ is a Hopf category that is strict as a monoidal category, as a comonoidal category and as a Hopf category, respectively. Note that as a comonoidal category $\mathfrak{D}(G)$ equals $\mathcal{M}^{kG} \boxtimes (\mathcal{M}^{kG})^*$. In the sequel we will do all calculations on the objects of the basis $(g, \hat{x})$ and write the tensor product in $\mathfrak{D}(G)$ as multiplication. We examine the quasitriangular structures living on $\mathfrak{D}(G)$. Since the Drinfeld Double of a finite group algebra is quasitriangular with R -matrix defined by

$$R^1 \boxtimes R^2 := \oplus_{g, x} (g, \hat{x}) \boxtimes (e, \hat{g}),$$

we can conclude that $(\mathfrak{D}(G), R^1 \boxtimes R^2)$ is a quasitriangular Hopf category with the morphisms $\sigma, \mu^1 \boxtimes \delta^2, \delta^1 \boxtimes \mu^2$ being the identities. Recall that the quasi monoidal transformation $(R^1 \boxtimes R^2, \text{id})$ corresponds to a 2-natural transformation when $\mathfrak{D}(G)$ is viewed as a one object 2-category. Recall also that by theorem 4.6 even such 2-natural transformations induce non trivial pseudo natural transformations. Yet, there may also be non identity solutions for the coherence morphisms. Let us consider $\mu^1 \boxtimes \delta^2$ first. We need an isomorphism

$$\begin{aligned}
\mu^1 \boxtimes \delta^2 : r^1 \otimes R^1 \boxtimes R^2 \boxtimes r^2 &= \oplus (h, \hat{y})(g, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}) \rightarrow \oplus (hg, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}) \\
&= R^1 \boxtimes R_{(1)}^2 \boxtimes R_{(2)}^2
\end{aligned}$$

The components on the left-hand side equal zero unless $y = gxg^{-1}$. Therefore, $\mu^1 \boxtimes \delta^2$ depends only on h, g and x . Since the components corresponding to different triples of group elements are different, we can write $\mu^1 \boxtimes \delta^2$ as a family of scalars

$$\alpha(h, g; x) : (h, \widehat{gxg^{-1}})(g, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}) \rightarrow (hg, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}).$$

In the same way the $\otimes$ -modification $\delta^1 \boxtimes \mu^2$ is given by

$$\begin{aligned}
\delta^1 \boxtimes \mu^2 : r^1 \boxtimes R^1 \boxtimes r^2 \otimes R^2 &= \oplus (h, \hat{y}) \boxtimes (g, \hat{x}) \boxtimes (e, \hat{h})(e, \hat{g}) \rightarrow \oplus (g, \hat{y}) \boxtimes (g, \hat{x}) \boxtimes (e, \hat{g}) \\
&= R_{(1)}^1 \boxtimes R_{(2)}^1 \boxtimes R^2
\end{aligned}$$

Here, the components on the left-hand side are zero unless $g = h$, whence $\delta^1 \boxtimes \mu^2$ is given by a family of scalars

$$\beta(g; x, y) : (g, \hat{y}) \boxtimes (g, \hat{x}) \boxtimes (e, \hat{g})(e, \hat{g}) \rightarrow (g, \hat{y}) \boxtimes (g, \hat{x}) \boxtimes (e, \hat{g}).$$

Hence, the two $\otimes$ -modifications $\mu^1 \boxtimes \delta^2$ and $\delta^1 \boxtimes \mu^2$ are given by maps $\alpha : G \times G \times G \rightarrow k^*$, $(h, g, x) \mapsto \alpha(h, g; x)$ and $\beta : G \times G \times G \rightarrow k^*$, $(h, g, x) \mapsto \beta(g; x, y)$, respectively.

Let $C = (a, \widehat{b})$ be an object in $\mathfrak{D}(G)$. We define the morphism σ that extends R to a quasi monoidal transformation. The source of σ reads:

$$\begin{aligned} R^1 \otimes C_{(1)} \boxtimes R^2 \otimes C_{(2)} &= \oplus_{g, x, k} (g, \widehat{x})(a, \widehat{k}) \boxtimes (e, \widehat{g})(a, \widehat{bk^{-1}}) \\ &= \oplus_g (g, \widehat{g^{-1}aba^{-1}})(a, \widehat{a^{-1}g^{-1}ab}) \boxtimes (e, \widehat{g})(a, \widehat{a^{-1}ga}) \\ &= \oplus_g (ga, \widehat{a^{-1}g^{-1}ab}) \boxtimes (a, \widehat{a^{-1}ga}) \end{aligned}$$

Here we made the substitutions $k = a^{-1}g^{-1}ab$ and $x = g^{-1}aba^{-1}$, all other components being zero. Of, course, the target of σ agrees with the source. We write the target in a way that its g th component agrees with the g th component in the source for each $g \in G$. A little calculation shows that therefore we have to set $x' = k$ and $k' = g' = a^{-1}ga$ in the following equation.

$$\begin{aligned} C_{(2)} \otimes R^1 \boxtimes C_{(1)} \otimes R^2 &= \oplus_{g', x', k'} (a, \widehat{bk'^{-1}})(g', \widehat{x'}) \boxtimes (a, \widehat{k'})(e, \widehat{g'}) \\ &= \oplus_g (a, \widehat{ba^{-1}g^{-1}a})(a^{-1}ga, \widehat{a^{-1}g^{-1}ab}) \boxtimes (a, \widehat{a^{-1}ga})(e, \widehat{a^{-1}ga}) \\ &= \oplus_g (ga, \widehat{a^{-1}g^{-1}ab}) \boxtimes (a, \widehat{a^{-1}ga}) \end{aligned}$$

Also here, components corresponding to different group elements g are different. The source and the target data of σ depend on three elements, whence σ is given by a family of scalars

$$\begin{aligned} \sigma(g, a, b) : (g, \widehat{g^{-1}aba^{-1}})(a, \widehat{a^{-1}g^{-1}ab}) \boxtimes (e, \widehat{g})(a, \widehat{a^{-1}ga}) \rightarrow \\ (a, \widehat{ba^{-1}g^{-1}a})(a^{-1}ga, \widehat{a^{-1}g^{-1}ab}) \boxtimes (a, \widehat{a^{-1}ga})(e, \widehat{a^{-1}ga}), \end{aligned}$$

whence it defines a map $\sigma : G \times G \times G \rightarrow k^*$, $(g, a, b) \mapsto \sigma(g, a, b)$.

For $(R^1 \boxtimes R^2, \sigma)$ to become a quasi monoidal transformation, we have to make sure that the coherence diagrams (D 5.1) and (D 5.2) commute. Let $C = (a, \widehat{b})$ and $D = (c, \widehat{d})$ be objects in $\mathfrak{D}(G)$. We consider the component $(g, \widehat{x})(a, \widehat{k})(c, \widehat{l}) \boxtimes (e, \widehat{g})(a, \widehat{bk^{-1}})(c, \widehat{dl^{-1}})$ in $R^1 \otimes C_{(1)} \otimes D_{(1)} \boxtimes R^2 \otimes C_{(2)} \otimes D_{(2)}$. This is different from zero only for $x = g^{-1}aba^{-1}$, $k = a^{-1}g^{-1}ab$ and $l = c^{-1}a^{-1}g^{-1}abc$. The upper way around the diagram (D 5.1) for this component has the following form. The underbracing is meant as a convenience for the reader to keep track on which tensorand a certain morphism acts.

$$\begin{aligned} &\underbrace{(g, \widehat{g^{-1}aba^{-1}})}_{R^1} \underbrace{(a, \widehat{a^{-1}g^{-1}ab})}_{C_{(1)}} \underbrace{(c, \widehat{c^{-1}a^{-1}g^{-1}abc})}_{D_{(1)}} \boxtimes \underbrace{(e, \widehat{g})}_{R^2} \underbrace{(a, \widehat{a^{-1}ga})}_{C_{(2)}} \underbrace{(c, \widehat{c^{-1}a^{-1}gac})}_{D_{(2)}} \\ &\quad \downarrow \sigma(g, a, b) \\ &\underbrace{(a, \widehat{ba^{-1}g^{-1}a})}_{R^1} \underbrace{(a^{-1}ga, \widehat{a^{-1}g^{-1}ab})}_{D_{(1)}} \underbrace{(c, \widehat{c^{-1}a^{-1}g^{-1}abc})}_{D_{(1)}} \boxtimes \underbrace{(a, \widehat{a^{-1}ga})}_{R^2} \underbrace{(e, \widehat{a^{-1}ga})}_{C_{(2)}} \underbrace{(c, \widehat{c^{-1}a^{-1}gac})}_{D_{(2)}} \\ &\quad \downarrow \sigma(a^{-1}ga, c, c^{-1}bc) \\ &\underbrace{(a, \widehat{ba^{-1}g^{-1}a})}_{R^1} \underbrace{(c, \widehat{c^{-1}ba^{-1}g^{-1}ac})}_{D_{(1)}} \underbrace{(c^{-1}a^{-1}gac, \widehat{c^{-1}a^{-1}gabc})}_{D_{(1)}} \boxtimes \underbrace{(a, \widehat{a^{-1}ga})}_{R^2} \underbrace{(c, \widehat{c^{-1}a^{-1}gac})}_{C_{(2)}} \underbrace{(e, \widehat{c^{-1}a^{-1}gac})}_{D_{(2)}} \end{aligned}$$

The last object is a component in $C_{(2)} \otimes D_{(2)} \otimes R^1 \boxtimes C_{(1)} \otimes D_{(1)} \otimes R^2$. Note that in our setup the morphisms Ξ are the identities. The second way around the diagram reads

$$\begin{array}{c} \underbrace{(g, \widehat{g^{-1}aba^{-1}})}_{R^1} \underbrace{(ac, \widehat{c^{-1}a^{-1}g^{-1}abc})}_{(CD)_{(1)}} \boxtimes \underbrace{(e, \widehat{g})}_{R^2} \underbrace{(ac, \widehat{c^{-1}a^{-1}gac})}_{(CD)_{(2)}} \\ \downarrow \sigma(g, ac, c^{-1}bc) \\ (ac, \widehat{c^{-1}ba^{-1}g^{-1}ac})(\widehat{c^{-1}a^{-1}gac}, \widehat{c^{-1}a^{-1}g^{-1}abc}) \boxtimes (ac, \widehat{c^{-1}a^{-1}gac})(e, \widehat{c^{-1}a^{-1}gac}) \end{array}$$

whence we get the formula

$$\sigma(g, a, b)\sigma(a^{-1}ga, c, c^{-1}bc) = \sigma(g, ac, c^{-1}bc), \quad (5.9)$$

or, if we substitute $d := c^{-1}bc$, we get $\sigma(g, a, cdc^{-1})\sigma(a^{-1}ga, c, d) = \sigma(g, ac, d)$. Note that diagram (D 5.2) yields the formula $\sigma(g, e, x) = 1$, which is a consequence of (5.9) if we set $a = c = e$.

To make $\otimes$ -modifications out of the isomorphisms $\mu^1 \boxtimes \delta^2$ and $\delta^1 \boxtimes \mu^2$, we have to require the diagrams (D 5.3) and (D 5.4), respectively. We consider only the diagram (D 5.4), the other one being similar. Again we consider just one component $(h, \widehat{y})(a, \widehat{k}) \boxtimes (g, \widehat{x})(a, \widehat{lk^{-1}}) \boxtimes (e, \widehat{h})(e, \widehat{g})(a, \widehat{bl^{-1}})$ of $r^1 \otimes C_{(1)} \boxtimes R^1 \otimes C_{(2)} \boxtimes r^2 \otimes R^2 \otimes C_{(3)}$. This component is zero unless $h = g$, $k = a^{-1}ya$, $l = a^{-1}xya$, $b = a^{-1}gxya$. The upper way around the diagram then reads:

$$\begin{array}{c} (g, \widehat{y})(a, \widehat{a^{-1}ya}) \boxtimes \underbrace{(g, \widehat{x})(a, \widehat{a^{-1}xa})}_{R^1} \boxtimes \underbrace{(e, \widehat{g})(e, \widehat{g})(a, \widehat{a^{-1}ga})}_{C_{(2)}} \\ \downarrow \sigma(g, a, a^{-1}gxa) \\ \underbrace{(g, \widehat{y})(a, \widehat{a^{-1}ya})}_{r^1} \boxtimes \underbrace{(a, \widehat{a^{-1}gag^{-1}a})(a^{-1}ga, \widehat{a^{-1}xa})}_{C_{(1)}} \boxtimes \underbrace{(e, \widehat{g})(e, \widehat{g})(a, \widehat{a^{-1}ga})}_{r^2} \boxtimes \underbrace{(e, \widehat{a^{-1}ga})}_{C_{(2)}} \\ \downarrow \sigma(g, a, a^{-1}gya) \\ (a, \widehat{a^{-1}gyg^{-1}a}) \underbrace{(a^{-1}ga, \widehat{a^{-1}ya})}_{r^1} \boxtimes (a, \widehat{a^{-1}gag^{-1}a}) \underbrace{(a^{-1}ga, \widehat{a^{-1}xa})}_{R^1} \boxtimes \underbrace{(a, \widehat{a^{-1}ga})}_{r^2} \boxtimes \underbrace{(e, \widehat{a^{-1}ga})}_{R^2} \\ \downarrow \beta(a^{-1}ga; a^{-1}xa, a^{-1}ya) \\ (a, \widehat{a^{-1}gyg^{-1}a})(a^{-1}ga, \widehat{a^{-1}ya}) \boxtimes (a, \widehat{a^{-1}gag^{-1}a})(a^{-1}ga, \widehat{a^{-1}xa}) \boxtimes (a, \widehat{a^{-1}ga})(e, \widehat{a^{-1}ga}) \end{array}$$

The lower way around the diagram reads:

$$\begin{array}{c} \underbrace{(g, \widehat{y})(a, \widehat{a^{-1}ya})}_{r^1} \boxtimes \underbrace{(g, \widehat{x})(a, \widehat{a^{-1}xa})}_{R^1} \boxtimes \underbrace{(e, \widehat{g})(e, \widehat{g})(a, \widehat{a^{-1}ga})}_{r^2 \otimes R^2} \\ \downarrow \beta(g; x, y) \\ \blacklozenge((g, \widehat{xy})(a, \widehat{a^{-1}xya})) \boxtimes (e, \widehat{g})(a, \widehat{a^{-1}ga}) \\ \downarrow \sigma(g, a, a^{-1}gxya) \\ \blacklozenge((a, \widehat{a^{-1}gxyg^{-1}a})(a^{-1}ga, \widehat{a^{-1}xya})) \boxtimes (a, \widehat{a^{-1}ga})(e, \widehat{a^{-1}ga}) \end{array}$$

Hence diagram (D 5.4) translates into the following formula

$$\sigma(g, a, a^{-1}gxa)\sigma(g, a, a^{-1}gya)\beta(a^{-1}ga; a^{-1}xa, a^{-1}ya) = \beta(g; x, y)\sigma(g, a, a^{-1}gxya) \quad (5.10)$$

Similarly, we get from diagram (D 5.3) the formula

$$\begin{aligned} \sigma(axa^{-1}, a, y^{-1}b)\sigma(aya^{-1}, a, bx^{-1})\alpha(y, x; x^{-1}y^{-1}b) = \\ \alpha(aya^{-1}, axa^{-1}; ax^{-1}y^{-1}ba^{-1})\sigma(ayxa^{-1}, a, b) \end{aligned} \quad (5.11)$$

Note that (5.11), (5.9) and (5.10) give relations how σ behaves on products in the first, second and third argument, respectively.

So far we have the necessary data and relations to get a pseudo natural transformation $\mathcal{R} : A \boxtimes B \Rightarrow B \boxtimes A$ and modifications $\tilde{R}_1 : (\mathcal{R}_{\mathcal{M}, \mathcal{N}} \boxtimes \mathcal{P})(\mathcal{N} \boxtimes \mathcal{R}_{\mathcal{M}, \mathcal{P}}) \Rightarrow \mathcal{R}_{\mathcal{M}, \mathcal{N} \boxtimes \mathcal{P}}$ and $\tilde{R}_2 : (\mathcal{M} \boxtimes \mathcal{R}_{\mathcal{N}, \mathcal{P}})(\mathcal{R}_{\mathcal{M}, \mathcal{P}} \boxtimes \mathcal{N}) \Rightarrow \mathcal{R}_{\mathcal{M} \boxtimes \mathcal{N}, \mathcal{P}}$ in the 2-category of representations of $\mathfrak{D}(G)$. To get a 2-braid structure we have to make sure that the diagrams $((\bullet \boxtimes \bullet \boxtimes \bullet) \boxtimes \bullet)$, $(\bullet \boxtimes (\bullet \boxtimes \bullet \boxtimes \bullet))$, $((\bullet \boxtimes \bullet) \boxtimes (\bullet \boxtimes \bullet))$ and $\mathbf{S}^+ = \mathbf{S}^-$ from definition 5.7 are satisfied. Note that $\mathcal{R}$ is an equivalence, whence we do not have to take care of the unit objects. For all axioms we figure out the starting vertex, all other vertices looking exactly the same. We pick one component from this object and apply successively the morphisms of the axiom to it. This way we obtain the formulas we are looking for. We first translate the axiom $(\bullet \boxtimes (\bullet \boxtimes \bullet \boxtimes \bullet))$. The starting vertex reads:

$$\begin{aligned} T^1 \otimes r^1 \otimes R^1 \boxtimes R^2 \boxtimes r^2 \boxtimes T^2 &= \oplus (k, \hat{z})(h, \hat{y})(g, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}) \boxtimes (e, \hat{k}) \\ &= \oplus (k, \widehat{hgxg^{-1}h^{-1}})(h, \widehat{gxxg^{-1}})(g, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}) \boxtimes (e, \hat{k}) \end{aligned}$$

In the last term we have written only the non zero components. For one such component we can write the upper way around the diagram the following way.

$$\begin{aligned} &(k, \widehat{hgxg^{-1}h^{-1}})(h, \widehat{gxxg^{-1}})(g, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}) \boxtimes (e, \hat{k}) \\ &\quad \downarrow \alpha(h, g; x) \\ &\underbrace{(k, \widehat{hgxg^{-1}h^{-1}})(hg, \hat{x})}_{r^1 \otimes R^1} \boxtimes \underbrace{(e, \hat{g}) \boxtimes (e, \hat{h})}_{\diamond(R^2)} \boxtimes \underbrace{(e, \hat{k})}_{r^2} \\ &\quad \downarrow \alpha(k, hg; x) \\ &(khg, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}) \boxtimes (e, \hat{k}) \end{aligned}$$

For the same component the lower way reads:

$$\begin{aligned} &\underbrace{(k, \widehat{hgxg^{-1}h^{-1}})(h, \widehat{gxxg^{-1}})(g, \hat{x})}_{r^1 \otimes R^1} \boxtimes (e, \hat{g}) \boxtimes \underbrace{(e, \hat{h}) \boxtimes (e, \hat{k})}_{\diamond(r^2)} \\ &\quad \downarrow \alpha(k, h; gxxg^{-1}) \\ &\underbrace{(kh, \widehat{gxxg^{-1}})(g, \hat{x})}_{r^1 \otimes R^1} \boxtimes \underbrace{(e, \hat{g}) \boxtimes (e, \hat{h}) \boxtimes (e, \hat{k})}_{R^2} \\ &\quad \downarrow \alpha(kh, g; x) \\ &(khg, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}) \boxtimes (e, \hat{k}) \end{aligned}$$

Thus we get for α the following formula.

$$\alpha(h, g; x)\alpha(k, hg; x) = \alpha(k, h; g x g^{-1})\alpha(kh, g; x) \quad (5.12)$$

In the diagram $((\bullet \boxtimes \bullet \boxtimes \bullet) \boxtimes \bullet)$, the starting vertex reads:

$$\begin{aligned} T^1 \boxtimes r^1 \boxtimes R^1 \boxtimes T^2 \otimes r^2 \otimes R^2 &= \oplus (k, \hat{z}) \boxtimes (h, \hat{y}) \boxtimes (g, \hat{x}) \boxtimes (e, \hat{k})(e, \hat{h})(e, \hat{g}) \\ &= \oplus (g, \hat{z}) \boxtimes (g, \hat{y}) \boxtimes (g, \hat{x}) \boxtimes (e, \hat{g}) \end{aligned}$$

We can write the diagram the following way.

$$\begin{array}{ccc} (g, \hat{z}) \boxtimes (g, \hat{y}) \boxtimes (g, \hat{x}) \boxtimes (e, \hat{g})(e, \hat{g})(e, \hat{g}) & \xrightarrow{\beta(g; x, y)} & (g, \hat{z}) \boxtimes (g, \hat{y}) \boxtimes (g, \hat{x}) \boxtimes (e, \hat{g})(e, \hat{g}) \\ \downarrow \beta(g; y, z) & & \downarrow \beta(g; xy, z) \\ (g, \hat{z}) \boxtimes (g, \hat{y}) \boxtimes (g, \hat{x}) \boxtimes (e, \hat{g})(e, \hat{g}) & \xrightarrow{\beta(g; x, yz)} & (g, \hat{z}) \boxtimes (g, \hat{y}) \boxtimes (g, \hat{x}) \boxtimes (e, \hat{g}) \end{array}$$

Therefore we get for β :

$$\beta(g; x, y)\beta(g; xy, z) = \beta(g; y, z)\beta(g; x, yz) \quad (5.13)$$

In the axiom $((\bullet \boxtimes \bullet) \boxtimes (\bullet \boxtimes \bullet))$ we start with the following object

$$\begin{aligned} t^1 T^1 \boxtimes r^1 R^1 \boxtimes T^2 R^2 \boxtimes t^2 r^2 &= \oplus (l, \hat{w})(h, \hat{y}) \boxtimes (k, \hat{z})(g, \hat{x}) \boxtimes (e, \hat{h})(e, \hat{g}) \boxtimes (e, \hat{l})(e, \hat{k}) \\ &= \oplus (h, \widehat{gyg^{-1}})(g, \hat{y}) \boxtimes (h, \widehat{gxg^{-1}})(g, \hat{x}) \boxtimes (e, \hat{g})(e, \hat{g}) \boxtimes (e, \hat{h})(e, \hat{h}) \end{aligned}$$

The left-hand way around the axiom corresponds to

$$\begin{array}{c} (h, \widehat{gyg^{-1}})(g, \hat{y}) \boxtimes (h, \widehat{gxg^{-1}})(g, \hat{x}) \boxtimes (e, \hat{g})(e, \hat{g}) \boxtimes (e, \hat{h})(e, \hat{h}) \\ \underbrace{\hspace{10em}}_{r^1} \quad \underbrace{\hspace{10em}}_{R^1} \quad \underbrace{\hspace{10em}}_{r^2 \otimes R^2} \\ \swarrow \beta(g; x, y) \\ (h, \widehat{gyg^{-1}})(g, \hat{y}) \boxtimes (h, \widehat{gxg^{-1}})(g, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h})(e, \hat{h}) \\ \underbrace{\hspace{10em}}_{T^1} \quad \underbrace{\hspace{10em}}_{r^1} \quad \underbrace{\hspace{10em}}_{T^2 \otimes r^2} \\ \downarrow \beta(h; , g x g^{-1}, g y g^{-1}) \\ (h, \widehat{gyg^{-1}})(g, \hat{y}) \boxtimes (h, \widehat{gxg^{-1}})(g, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}) \\ \underbrace{\hspace{10em}}_{\blacklozenge(r^1 \otimes R^1)} \quad \underbrace{\hspace{10em}}_{R^2} \quad \underbrace{\hspace{10em}}_{r^2} \\ \searrow \alpha(h, g; xy) \\ (hg, \hat{y}) \boxtimes (hg, \hat{x}) \boxtimes (e, \hat{g}) \boxtimes (e, \hat{h}) \end{array}$$

The right-hand way reads:

$$\begin{array}{c}
\underbrace{(h, \widehat{gyg^{-1}})(g, \widehat{y})}_{r^1 \otimes R^1} \boxtimes \underbrace{(h, \widehat{gxg^{-1}})(g, \widehat{x})}_{R^2} \boxtimes \underbrace{(e, \widehat{g})(e, \widehat{g})}_{r^2} \boxtimes \underbrace{(e, \widehat{h})(e, \widehat{h})}_{r^2} \\
\searrow \alpha(h, g; x) \\
\underbrace{(h, \widehat{gyg^{-1}})(g, \widehat{y})}_{T^1 \otimes r^1} \boxtimes (hg, \widehat{x}) \boxtimes \underbrace{(e, \widehat{g})(e, \widehat{g})}_{r^2} \boxtimes \underbrace{(e, \widehat{h})(e, \widehat{h})}_{T^2} \\
\downarrow \alpha(h, g; y) \\
\underbrace{(hg, \widehat{y})}_{r^1} \boxtimes \underbrace{(hg, \widehat{x})}_{R^1} \boxtimes \underbrace{(e, \widehat{g})(e, \widehat{g}) \boxtimes (e, \widehat{h})(e, \widehat{h})}_{\diamond(r^2 \otimes R^2)} \\
\swarrow \beta(hg; x, y) \\
(hg, \widehat{y}) \boxtimes (hg, \widehat{x}) \boxtimes (e, \widehat{g}) \boxtimes (e, \widehat{h})
\end{array}$$

Therefore we get the following formula.

$$\beta(g; x, y)\beta(h; gxg^{-1}, gyg^{-1})\beta^{-1}(hg; x, y) = \alpha(h, g; x)\alpha(h, g; y)\alpha^{-1}(h, g; xy) \quad (5.14)$$

The starting vertex for the axiom $\mathbf{S}^+ = \mathbf{S}^-$ reads

$$\begin{aligned}
r^1 R^1 \boxtimes T^1 R^2 \boxtimes T^2 r^2 &= \oplus (h, \widehat{y})(g, \widehat{x}) \boxtimes (k, \widehat{z})(e, \widehat{g}) \boxtimes (e, \widehat{k})(e, \widehat{h}) \\
&= \oplus (h, \widehat{gxg^{-1}})(g, \widehat{x}) \boxtimes (h, \widehat{g})(e, \widehat{g}) \boxtimes (e, \widehat{h})(e, \widehat{h})
\end{aligned}$$

Then for $\mathbf{S}^+ = \mathbf{S}^-$ we get the following diagram, in which the objects on which the morphisms on the left-hand side act are underlined and the objects on which the morphisms on the right-hand side act are overlined.

$$\begin{array}{ccc}
\underline{(h, \widehat{gxg^{-1}})(g, \widehat{x})} \boxtimes \underline{(h, \widehat{g})(e, \widehat{g})} \boxtimes \underline{(e, \widehat{h})(e, \widehat{h})} & & \overline{(h, \widehat{gxg^{-1}})(g, \widehat{x})} \boxtimes \overline{(h, \widehat{g})(e, \widehat{g})} \boxtimes \overline{(e, \widehat{h})(e, \widehat{h})} \\
\swarrow \alpha(h, g; x) & & \searrow \beta(h; gxg^{-1}, g) \\
(hg, \widehat{x}) \boxtimes \underline{(h, \widehat{g})(e, \widehat{g})} \boxtimes \underline{(e, \widehat{h})(e, \widehat{h})} & & (h, \widehat{gxg^{-1}})(g, \widehat{x}) \boxtimes \overline{(h, \widehat{g})(e, \widehat{g})} \boxtimes \overline{(e, \widehat{h})(e, \widehat{h})} \\
\downarrow \sigma(h, e, hg) & & \downarrow \sigma(hgh^{-1}, h, gx) \\
(hg, \widehat{x}) \boxtimes \underline{(e, \widehat{hgh^{-1}})(h, \widehat{g})} \boxtimes \underline{(e, \widehat{h})(e, \widehat{h})} & & \overline{(hgh^{-1}, \widehat{hgh^{-1}})(h, \widehat{x})} \boxtimes \overline{(e, \widehat{hgh^{-1}})(h, \widehat{g})} \boxtimes \overline{(e, \widehat{h})(e, \widehat{h})} \\
\swarrow \alpha(hgh^{-1}, h; x) & & \searrow \beta(h; g, x) \\
\underline{(hgh^{-1}, \widehat{hgh^{-1}})(h, \widehat{x})} \boxtimes \underline{(e, \widehat{hgh^{-1}})(h, \widehat{g})} \boxtimes \underline{(e, \widehat{h})(e, \widehat{h})} & & \overline{(hgh^{-1}, \widehat{hgh^{-1}})(h, \widehat{x})} \boxtimes \overline{(e, \widehat{hgh^{-1}})(h, \widehat{g})} \boxtimes \overline{(e, \widehat{h})(e, \widehat{h})}
\end{array}$$

The left-hand way around the diagram reads $\alpha(h, g; x)\sigma(h, e, hg)\alpha^{-1}(hgh^{-1}, h; x)$ and the right-hand way reads $\beta(h; gxg^{-1}, g)\sigma^{-1}(hgh^{-1}, h; gx)\beta^{-1}(h; g, x)$. Since $\sigma(x, e, y) = 1$ we get

$$\begin{aligned}
\sigma(g, a, b) &= \alpha^{-1}(a, a^{-1}ga; a^{-1}g^{-1}ab)\alpha(g, a; a^{-1}g^{-1}ab) \\
&\quad \beta(a; ba^{-1}g^{-1}a, a^{-1}ga)\beta^{-1}(a; a^{-1}ga, a^{-1}g^{-1}ab)
\end{aligned} \quad (5.15)$$

Therefore, axiom $\mathbf{S}^+ = \mathbf{S}^-$ says that σ is already determined by α and β . Using this equation, we get from (5.9), if we substitute $x = a^{-1}ga$, the following formula.

$$\begin{aligned} \sigma(axa^{-1}, a, b)\sigma(x, c, c^{-1}bc)\sigma^{-1}(axa^{-1}, ac, cbc) &= \\ \alpha^{-1}(a, x; x^{-1}b)\alpha(axa^{-1}, a; x^{-1}b)\beta(a; bx^{-1}, x)\beta^{-1}(a; x, x^{-1}b) & \\ \alpha^{-1}(c, c^{-1}xc; c^{-1}x^{-1}bc)\alpha(x, c; c^{-1}x^{-1}bc)\beta(c; c^{-1}bx^{-1}c, c^{-1}xc)\beta^{-1}(c; c^{-1}xc, c^{-1}a^{-1}g^{-1}bc) & \\ \alpha(ac, c^{-1}xc; c^{-1}x^{-1}bc)\alpha^{-1}(axa^{-1}, ac; c^{-1}x^{-1}bc)\beta^{-1}(ac; c^{-1}bx^{-1}c, c^{-1}xc)\beta(ac; c^{-1}xc, c^{-1}x^{-1}bc) & \\ = 1 & \end{aligned}$$

By (5.14) we can transform

$$\begin{aligned} \beta(c; c^{-1}bx^{-1}c, c^{-1}xc)\beta(a; bx^{-1}, x)\beta^{-1}(ac; c^{-1}bx^{-1}c, c^{-1}xc) &= \\ \alpha(a, c; c^{-1}bx^{-1}c)\alpha(a, c; c^{-1}xc)\alpha^{-1}(a, c; c^{-1}bc) & \\ \beta^{-1}(c; c^{-1}xc, c^{-1}x^{-1}bc)\beta^{-1}(a; x, x^{-1}b)\beta(ac; c^{-1}xc, c^{-1}x^{-1}bc) &= \\ \alpha^{-1}(a, c; c^{-1}x^{-1}bc)\alpha^{-1}(a, c; c^{-1}xc)\alpha(a, c; c^{-1}bc) & \end{aligned}$$

We can put this into our first equation and then reduce it to:

$$\begin{aligned} & \alpha^{-1}(a, x; x^{-1}b)\alpha(axa^{-1}, a; x^{-1}b)\alpha^{-1}(c, c^{-1}xc; c^{-1}x^{-1}bc)\alpha(x, c; c^{-1}x^{-1}bc) \\ & \underbrace{\alpha(ac, c^{-1}xc; c^{-1}x^{-1}bc)}_{\alpha^{-1}(axa^{-1}, ac; c^{-1}x^{-1}bc)} \underbrace{\alpha(a, c; c^{-1}bx^{-1}c)}_{\alpha^{-1}(a, c; c^{-1}x^{-1}bc)} \\ = & \alpha^{-1}(a, x; x^{-1}b)\alpha(axa^{-1}, a; x^{-1}b)\alpha^{-1}(c, c^{-1}xc; c^{-1}x^{-1}bc)\alpha(x, c; c^{-1}x^{-1}bc) \\ & \underbrace{\alpha(a, xc; c^{-1}x^{-1}bc)}_{\alpha^{-1}(axa^{-1}, a; x^{-1}b)} \underbrace{\alpha(c, c^{-1}xc; c^{-1}x^{-1}bc)}_{\alpha^{-1}(ax, c; c^{-1}x^{-1}bc)} \\ = & \underline{\alpha^{-1}(a, x; x^{-1}b)}\alpha(x, c; c^{-1}x^{-1}bc)\alpha(a, xc; c^{-1}x^{-1}bc)\underline{\alpha^{-1}(ax, c; c^{-1}x^{-1}bc)} \\ = & \underline{\alpha^{-1}(x, c; c^{-1}x^{-1}bc)}\alpha(x, c; c^{-1}x^{-1}bc)\alpha(a, xc; c^{-1}x^{-1}bc)\underline{\alpha^{-1}(a, xc; c^{-1}x^{-1}bc)} \\ = & 1 \end{aligned}$$

The first equality is an application of (5.12) to the underbraced and the underlined pair, respectively. The second is an obvious reducing and the third again an application of (5.12) to the underlined pair. Hence we see that with σ defined as required by (5.15) the equation (5.9) is automatically satisfied. A similar, but a bit more tedious calculation shows that the equations (5.11) and (5.10) are also satisfied. This leaves us only with the formulas (5.9), (5.10) and (5.11). We have proved the following theorem.

5.9. Theorem: *The quasitriangular structures on the categorification of the Drinfeld Double of a finite group G with the underlying object $R^1 \boxtimes R^2 = \oplus_{g,x}(g, \hat{x}) \boxtimes (e, \hat{g})$ are in bijection with the set of maps $\alpha, \beta : G \times G \times G \rightarrow k^*$ satisfying*

$$\begin{aligned} \beta(g; x, y)\beta(g; xy, z) &= \beta(g; y, z)\beta(g; x, yz) \\ \alpha(h, g; x)\alpha(k, hg; x) &= \alpha(k, h; gxg^{-1})\alpha(kh, g; x) \\ \beta(g; x, y)\beta(h; gxg^{-1}, gyg^{-1})\beta^{-1}(hg; x, y) &= \alpha(h, g; x)\alpha(h, g; y)\alpha^{-1}(h, g; xy) \end{aligned}$$

We can describe these formulas in terms of a double complex $(\mathcal{C}_{\bullet, \bullet}, d_1, d_2)$. Let $\mathcal{C}_{n, m} := \text{Map}(G^{n+m}, k^*)$. We denote an element in $\mathcal{C}_{n, m}$ by $f(x_1, \dots, x_n; x_{n+1}, \dots, x_{n+m})$. The differentials are the differentials from group cohomology with coefficients in a right module.

$$\begin{aligned} d_i(f)(x_1, \dots, x_{n+1}) &= f(x_2, \dots, x_{n+1}) \cdot \prod_{i=1}^{n-1} f^{(-1)^i}(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) \\ &\quad f^{(-1)^n}(x_1, \dots, x_n) \leftarrow x_{n+1} \end{aligned}$$

The horizontal differential in the n th row $d_1 : \mathcal{C}_{n, m} \rightarrow \mathcal{C}_{n, m+1}$ has coefficients in the multiplicative abelian group $\text{Map}(G^n, k^*)$, endowed with the trivial G -module structure. The vertical differential in the m th column $d_2 : \mathcal{C}_{n, m} \rightarrow \mathcal{C}_{n+1, m}$ has coefficients in $\text{Map}(G^m, k^*)$ and the (right) action is defined for $\phi \in \text{Map}(G^m, k^*)$ and $y \in G$ by

$$(\phi \leftarrow y)(g_1, \dots, g_m) := \phi(yg_1y^{-1}, \dots, yg_my^{-1})$$

We have $\beta \in \mathcal{C}_{1, 2}$, $\alpha \in \mathcal{C}_{2, 1}$ and $d_1(\beta) = 1$, $d_2(\alpha) = 1$ and $d_2(\beta) = d_1(\alpha)$, which means that the pair (α, β) , is a 2-cocycle in the total complex of $\mathcal{C}_{\bullet, \bullet}$.

Assume, we have two quasitriangular structures with the same underlying object $R^1 \boxtimes R^2$. This means we have two pairs (α, β) and (α', β') of 2-cocycles. How do the morphisms of quasitriangular structures look? We need a $\boxtimes$ -modification $\mathfrak{a} : R^1 \boxtimes R^2 \rightarrow R^1 \boxtimes R^2$, satisfying the coherence diagrams on page 77. Any such morphism is determined by a family of scalars $\iota(g; x) : \oplus_{g, x}(g, \hat{x}) \boxtimes (e, \hat{g}) \rightarrow \oplus_{g, x}(g, \hat{x}) \boxtimes (e, \hat{g})$ whence it is given by a map $\iota : G \times G \rightarrow k$. It is now easily worked out that the coherence diagram for being a $\boxtimes$ -modification reads:

$$\sigma'(g, a, b)\iota(g; g^{-1}aba^{-1}) = \iota(a^{-1}ga; a^{-1}g^{-1}ab)\sigma(g, a, b) \quad (5.16)$$

The two coherence axioms for being a morphism of 2-braid structures then read:

$$\begin{aligned} \iota(hg; x)\alpha(h, g; x) &= \alpha'(h, g; x)\iota(h; g x g^{-1})\iota(g; x) \\ \iota(g; xy)\beta(g; x, y) &= \beta'(g; xy)\iota(g; y)\iota(g; x) \end{aligned}$$

Obviously, all such maps are invertible unless they are constantly zero. Since $\iota = 0$ is not a morphism of quasitriangular structures, we can rewrite these formulas as:

$$\alpha(h, g; x) = \iota(g; x)\iota(h; g x g^{-1})\iota^{-1}(hg; x)\alpha'(h, g; x) \quad (5.17)$$

$$\beta(g; x, y) = \iota(g; x)\iota(g; y)\iota^{-1}(g; xy)\beta'(g; x, y) \quad (5.18)$$

Therefore we have $\alpha = d_2(\iota)\alpha'$ and $\beta = d_1(\iota)\beta'$. Since, it is readily shown, these two formulas already imply formula (5.16), we see that two quasitriangular structures are isomorphic, iff their corresponding 2-cocycles live in the same equivalence class. Since all morphisms of quasitriangular structures are isomorphisms, we have proved the following theorem.

5.10. Theorem: *The second cohomology group of the total complex of the double complex $(\mathcal{C}_{\bullet, \bullet}, d_1, d_2)$ parametrizes the equivalence classes of 2-quasitriangular structures with underlying object $R^1 \boxtimes R^2 = \oplus_{g, x}(g, \hat{x}) \boxtimes (e, \hat{g})$ for the categorification of the Drinfeld Double of a finite group G .*

5.4 Conclusion

We made some progress in understanding the connection between monoidal categories and 2-categories. However, a general reconstruction theorem is still out of reach. We are able to describe all possible 2-braid structures and, by using the same techniques, all associative structures on a Gray monoid that happens to be the 2-category of representations of a Hopf category. However, there exists a third interesting datum in a Gray monoid, namely the Gray morphism $\boxtimes_{f,g}$. This is a datum of the very pseudofunctor $\boxtimes$ itself. Since, by our techniques we can only lift the existing tensor product of $\mathfrak{C}_k$, we can not obtain Gray monoids with non trivial Gray morphisms. To achieve this we need a pseudofunctor $\boxtimes$ whose coherence morphism $\boxtimes_{-, -} = \xi$ is different from the identity only in compositions of the form $(1 \boxtimes f) \circ (g \boxtimes 1) \cong (g \boxtimes 1) \circ (1 \boxtimes f)$. Recall that a monoidal functor $\mathcal{F}$ always generates a 2-functor $\text{Pseud}(\mathcal{F}, \mathfrak{C}_k)$ rather than just a pseudofunctor. Since we defined $\blacklozenge$ to be a monoidal functor there is no possibility to achieve this weakening. There is no use rattling at the monoidal structure of $\blacklozenge$, which is responsible only for the coherence of a $\mathcal{C} \boxtimes \mathcal{C}$ -module $\mathcal{M}$ being a $\mathcal{C}$ -module. It seems that we have to weaken the notion functor itself, but in which direction, the author does not know.

The smoothness of our Drinfeld Double example, shows that the course taken up is a very promising one to pursue. The reason for the formulas 5.9, 5.11 and 5.10 coming from the coherence axioms of pseudo natural transformations and modifications to vanish when σ is defined as demanded by axiom $S^+ = S^-$ is not yet understood by the author. It would be tempting to work out the category of associators of ${}_{\mathfrak{D}(G)}\mathfrak{M}$, i.e. the 2-quasi bialgebra structures for $\mathfrak{D}(G)$ in the same way and then compare the results.

As was shown in [KV], each of the two ways around axiom $S^+ = S^-$ is a solution of the Zamolodchikov equation. In our example these two ways are maps $G \times G \times G \rightarrow k^*$ that read:

$$\alpha(h, g; x)\alpha^{-1}(hgh^{-1}, h; x) = \beta(h; gxg^{-1}, g)\sigma^{-1}(hgh^{-1}, h; gx)\beta^{-1}(h; g, x), \quad h, g, x \in G$$

This solution can be constructed out of any finite group with non trivial second cohomology. The author does not know how this is related with the known solutions.

The quantum double construction is not the only possible. Also a FRT-construction should be feasible. But therefore we have to deal with corepresentations of monoidal categories rather than representations.

Appendix

In this appendix we state all definitions of monoidal, braided and involutory morphisms of monoidal and braided monoidal 2-categories that have been left out in chapter 1. The definitions of monoidal morphisms are taken from [DS], with slightly modified notations. The definition of braided monoidal pseudofunctors first appeared in [BN], though without the unit axioms.

A.1. Definition: Let $\mathfrak{C}, \mathfrak{D}$ be Gray monoids. A weak monoidal pseudofunctor $\mathcal{F} : \mathfrak{C} \rightarrow \mathfrak{D}$ consists of a pseudofunctor $\mathcal{F} : \mathfrak{C} \rightarrow \mathfrak{D}$, two pseudo natural transformations, given on objects by

$$\Xi_{A,B}^{\mathcal{F}} : \mathcal{F}(A) \boxtimes \mathcal{F}(B) \rightrightarrows \mathcal{F}(A \boxtimes B) \quad \Upsilon^{\mathcal{F}} : I_{\mathfrak{D}} \rightarrow \mathcal{F}(I_{\mathfrak{C}})$$

and invertible modifications

$$(1) \quad \begin{array}{ccc} \mathcal{F}(A) \boxtimes \mathcal{F}(B) \boxtimes \mathcal{F}(C) & \xrightarrow{\mathcal{F}(A) \boxtimes \Xi_{B,C}^{\mathcal{F}}} & \mathcal{F}(A) \boxtimes \mathcal{F}(B \boxtimes C) \\ \Xi_{A,B}^{\mathcal{F}} \boxtimes \mathcal{F}(C) \downarrow & \omega_{A,B,C} \leftarrow & \downarrow \Xi_{A,B \boxtimes C}^{\mathcal{F}} \\ \mathcal{F}(A \boxtimes B) \boxtimes C & \xrightarrow{\Xi_{A \boxtimes B, C}^{\mathcal{F}}} & \mathcal{F}(A \boxtimes B \boxtimes C) \end{array}$$

$$(2) \quad \begin{array}{ccc} & \mathcal{F}(A) & \\ 1 \boxtimes \Upsilon \swarrow & \downarrow 1 & \searrow \Upsilon \boxtimes 1 \\ \mathcal{F}(A) \boxtimes \mathcal{F}(I) & \xrightarrow{\zeta_A^r} & \mathcal{F}(A) \\ \Xi_{A,I}^{\mathcal{F}} \searrow & & \swarrow \Xi_{I,A}^{\mathcal{F}} \\ \mathcal{F}(A) & \xleftarrow{\zeta_A^l} & \mathcal{F}(I) \boxtimes \mathcal{F}(A) \end{array}$$

such that the following diagrams commute

$$\begin{array}{ccccc}
& & \mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(CD) & \xrightarrow{1 \boxtimes \Xi^{\mathcal{F}}} & \mathcal{F}(A)\mathcal{F}(BCD) \\
& \nearrow 1 \boxtimes \Xi^{\mathcal{F}} & & \searrow \downarrow \omega_{A,B,C} \boxtimes D & \searrow \Xi^{\mathcal{F}} \\
\mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(C)\mathcal{F}(D) & & \downarrow \boxtimes \Xi^{\mathcal{F}}, \Xi^{\mathcal{F}} & \mathcal{F}(AB)\mathcal{F}(CD) & \xrightarrow{\quad} & \mathcal{F}(ABCD) \\
& \searrow \Xi^{\mathcal{F}} \boxtimes 1 & & \downarrow \omega_{A \boxtimes B, C, D} & \searrow \Xi^{\mathcal{F}} \\
& & \mathcal{F}(AB)\mathcal{F}(C)\mathcal{F}(D) & \xrightarrow{\Xi^{\mathcal{F}} \boxtimes 1} & \mathcal{F}(ABC)\mathcal{F}(D)
\end{array}$$

=

$$\begin{array}{ccccc}
& & \mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(CD) & \xrightarrow{1 \boxtimes \Xi^{\mathcal{F}}} & \mathcal{F}(A)\mathcal{F}(BCD) \\
& \nearrow 1 \boxtimes \Xi^{\mathcal{F}} & & \downarrow 1 \boxtimes \omega_{B,C,D} & \searrow \Xi^{\mathcal{F}} \\
\mathcal{F}(A)\mathcal{F}(B)\mathcal{F}(C)\mathcal{F}(D) & \xrightarrow{1 \boxtimes \Xi^{\mathcal{F}} \boxtimes 1} & \mathcal{F}(A)\mathcal{F}(BC)\mathcal{F}(D) & \downarrow \omega_{A,B} \boxtimes C, D & \searrow \Xi^{\mathcal{F}} \\
& \searrow \Xi^{\mathcal{F}} \boxtimes 1 & & \downarrow \omega_{A,B,C} \boxtimes 1 & \searrow \Xi^{\mathcal{F}} \\
& & \mathcal{F}(AB)\mathcal{F}(C)\mathcal{F}(D) & \xrightarrow{\Xi^{\mathcal{F}} \boxtimes 1} & \mathcal{F}(ABC)\mathcal{F}(D)
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{ccc}
\mathcal{F}(A)\mathcal{F}(B) & \xrightarrow{\Xi^{\mathcal{F}}} & \mathcal{F}(AB) \\
\uparrow 1 \boxtimes \Xi^{\mathcal{F}} & \searrow \downarrow \omega_{A,I,B} & \uparrow \Xi^{\mathcal{F}} \\
\mathcal{F}(A)\mathcal{F}(I)\mathcal{F}(B) & \xrightarrow{\Xi^{\mathcal{F}} \boxtimes 1} & \mathcal{F}(A)\mathcal{F}(B)
\end{array} & = & \begin{array}{ccc}
\mathcal{F}(A)\mathcal{F}(B) & \xrightarrow{\Xi^{\mathcal{F}}} & \mathcal{F}(AB) \\
\uparrow 1 \boxtimes \Xi^{\mathcal{F}} & \swarrow 1 & \downarrow \text{id} \\
\mathcal{F}(A)\mathcal{F}(I)\mathcal{F}(B) & \xrightarrow{\Xi^{\mathcal{F}} \boxtimes 1} & \mathcal{F}(A)\mathcal{F}(B)
\end{array}
\end{array}$$

A monoidal pseudofunctor is called strict, if Ξ is 2-natural and ω, ζ^r and ζ^l are the identities, respectively.

A.2. Definition: Let $\mathcal{F}, \mathcal{G} : \mathfrak{C} \rightarrow \mathfrak{D}$ be monoidal pseudofunctors between Gray monoids. A monoidal lax natural transformation consists of a lax natural transformation $\chi : \mathcal{F} \Rightarrow \mathcal{G}$, an invertible modification χ_2 and a 2-morphism χ^I , given by

$$\begin{array}{ccc}
\mathcal{F}(A) \boxtimes \mathcal{F}(B) & \xrightarrow{\Xi_{A,B}^{\mathcal{F}}} & \mathcal{F}(A \boxtimes B) \\
\downarrow \chi_A \boxtimes \chi_B & \searrow \chi_2; A, B, C \Rightarrow & \downarrow \chi_{A \boxtimes B} \\
\mathcal{G}(A) \boxtimes \mathcal{G}(B) & \xrightarrow{\Xi_{A \boxtimes B, C}^{\mathcal{G}}} & \mathcal{G}(A \boxtimes B)
\end{array}$$

$$\begin{array}{ccc}
& I_{\mathfrak{D}} & \\
\Upsilon_{\mathcal{F}} \swarrow & \chi^I \Rightarrow & \searrow \Upsilon_{\mathcal{G}} \\
\mathcal{F}(I_{\mathfrak{C}}) & \xrightarrow{\chi_I} & \mathcal{G}(I_{\mathfrak{C}})
\end{array}$$

The figure consists of two commutative diagrams, labeled (a) and (b), which illustrate the proof of the Frobenius law. Both diagrams share the same set of objects and arrows, but the right diagram (b) is a dualized version of the left diagram (a).

Diagram (a) - Left: This diagram represents the proof for the monoidal product F . The objects are arranged in a grid-like structure. The top row contains $F(A)F(BC)$ and $F(ABC)$. The middle row contains $F(A)F(B)F(C)$ and $F(AB)F(C)$. The bottom row contains $F(A)F(B)g(C)$ and $F(AB)g(C)$. The bottom-most row contains $g(A)g(B)g(C)$ and $g(ABC)$. Arrows are labeled with natural transformations: $1 \boxtimes \Xi^F$, Ξ^F , $\omega_{A,B,C}$, $\chi_{A \boxtimes B \boxtimes C}$, $\chi_{2A \boxtimes B, C}$, $\chi_{2A, B} \boxtimes 1$, Ξ^G , $\Xi^G \boxtimes 1$, $\chi_A \boxtimes \chi_B \boxtimes 1$, $\boxtimes \Xi^F, \chi$, and $1 \boxtimes 1 \boxtimes \chi_C$.

Diagram (b) - Right: This diagram represents the proof for the comonoidal product G . It is a dualized version of diagram (a). The objects are arranged in a grid-like structure. The top row contains $F(A)F(BC)$ and $F(ABC)$. The middle row contains $F(A)g(BC)$ and $F(A)g(B)g(C)$. The bottom row contains $g(A)g(BC)$ and $g(A)g(B)g(C)$. The bottom-most row contains $g(ABC)$ and $g(AB)g(C)$. Arrows are labeled with natural transformations: $1 \boxtimes \Xi^F$, Ξ^F , $\omega_{A,B,C}$, $\chi_{A \boxtimes B \boxtimes C}$, $\chi_{2A, B} \boxtimes 1$, $\chi_{2A \boxtimes B, C}$, $\chi_A \boxtimes 1 \boxtimes 1$, Ξ^G , $\Xi^G \boxtimes 1$, $\chi_A \boxtimes \chi_B \boxtimes 1$, $\boxtimes \Xi^G, 1$, and $1 \boxtimes \chi_2$.

$$\begin{array}{ccc}
1 \boxtimes \Upsilon^{\mathcal{F}} & \mathcal{F}(A) & 1 \\
\downarrow & \Rightarrow \zeta^l & \downarrow \\
\mathcal{F}(A) \mathcal{F}(I) & \xrightarrow{\quad} & \mathcal{F}(A) \\
\downarrow 1 \boxtimes \chi_I & \Downarrow \chi_{2; A, I} & \downarrow \chi_A \\
\mathcal{F}(A) \mathcal{G}(I) & & \mathcal{G}(A) \\
\downarrow \chi_A \boxtimes 1 & & \downarrow 1 \\
\mathcal{G}(A) \mathcal{G}(I) & \xrightarrow{\Xi^{\mathcal{G}}} & \mathcal{G}(A)
\end{array}
=
\begin{array}{ccc}
1 \boxtimes \Upsilon^{\mathcal{F}} & \mathcal{F}(A) & 1 \\
\downarrow & & \downarrow \\
\mathcal{F}(A) \mathcal{F}(I) & & \mathcal{F}(A) \\
\downarrow 1 \boxtimes \chi_I & \Downarrow 1 \boxtimes \chi^I & \downarrow \chi_A \\
\mathcal{F}(A) \mathcal{G}(I) & \Downarrow \boxtimes \Upsilon^{\mathcal{G}}, \chi_A & \mathcal{G}(A) \\
\downarrow \chi_A \boxtimes 1 & & \downarrow 1 \\
\mathcal{G}(A) \mathcal{G}(I) & \xrightarrow{\Xi^{\mathcal{G}}} & \mathcal{G}(A)
\end{array}$$

$$\begin{array}{ccc}
\Upsilon^{\mathcal{F}} \boxtimes 1 & \mathcal{F}(A) & 1 \\
\downarrow & \Rightarrow \zeta^{\mathcal{F}, r} & \downarrow \\
\mathcal{F}(I) \mathcal{F}(A) & \xrightarrow{\Xi^{\mathcal{F}}} & \mathcal{F}(A) \\
\downarrow \chi_I \boxtimes 1 & \Downarrow \chi_{2; I, A} & \downarrow \chi_A \\
\mathcal{G}(I) \mathcal{F}(A) & & \mathcal{G}(A) \\
\downarrow 1 \boxtimes \chi_A & \Downarrow \boxtimes \chi_I, \chi_A & \downarrow 1 \\
\mathcal{G}(I) \mathcal{G}(A) & \xrightarrow{\Xi_{A, I}^{\mathcal{G}}} & \mathcal{G}(A)
\end{array}
=
\begin{array}{ccc}
\Upsilon^{\mathcal{F}} \boxtimes 1 & \mathcal{F}(A) & 1 \\
\downarrow & & \downarrow \\
\mathcal{F}(I) \mathcal{F}(A) & \xrightarrow{\chi^I \boxtimes 1} & \mathcal{F}(A) \\
\downarrow \chi_I \boxtimes 1 & \Downarrow \Upsilon^{\mathcal{G}} \boxtimes 1 & \downarrow \chi_A \\
\mathcal{G}(I) \mathcal{F}(A) & & \mathcal{G}(A) \\
\downarrow 1 \boxtimes \chi_A & \Downarrow \boxtimes \chi_I, \chi_A & \downarrow 1 \\
\mathcal{G}(I) \mathcal{G}(A) & \xrightarrow{\Xi_{A, I}^{\mathcal{G}}} & \mathcal{G}(A)
\end{array}$$

A lax monoidal transformation is called *strict*, if χ_2 and χ^I are the identities.

A.3. Definition: Let $\chi, \psi : \mathcal{F} \Rightarrow \mathcal{G} : \mathfrak{C} \rightarrow \mathfrak{D}$ be monoidal lax natural transformations. A monoidal modification is a modification $\mathfrak{m} : \chi \Rightarrow \psi$, such that the following diagrams commute:

$$\begin{array}{ccc}
 \mathcal{F}(A) \boxtimes \mathcal{F}(B) & \xrightarrow{\Xi^{\mathcal{F}}} & \mathcal{F}(A \boxtimes B) \\
 \downarrow \chi_A \boxtimes \chi_B & \uparrow \psi_{2:A,B} & \downarrow \chi_{A \boxtimes B} \\
 \mathfrak{m}_A \boxtimes \mathfrak{m}_B & \xRightarrow{\psi_A \boxtimes \psi_B} & \mathfrak{m}_{A \boxtimes B} \\
 \downarrow \chi_A \boxtimes \chi_B & \uparrow \chi_{2:A,B} & \downarrow \chi_{A \boxtimes B} \\
 \mathcal{G}(A) \boxtimes \mathcal{G}(B) & \xrightarrow{\Xi^{\mathcal{G}}} & \mathcal{G}(A \boxtimes B)
 \end{array}$$

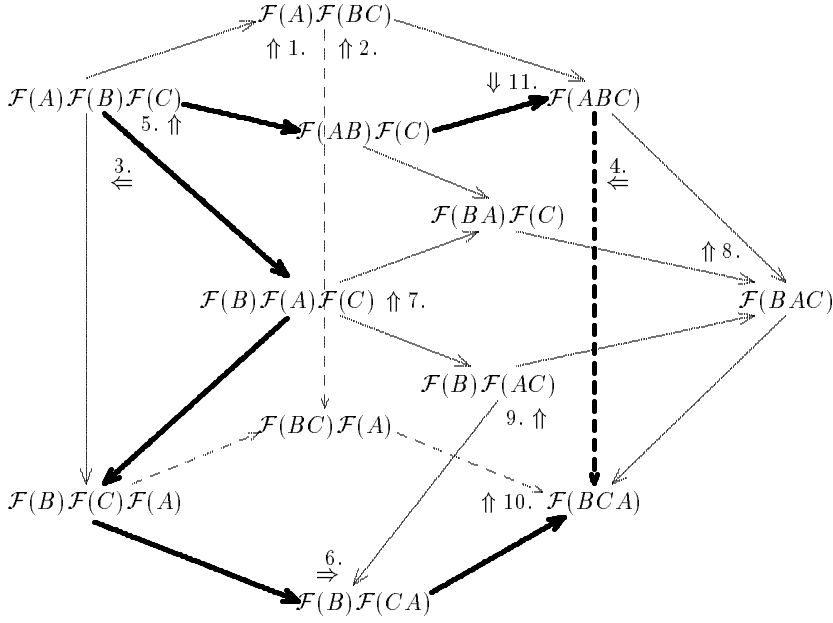
$$\begin{array}{ccc}
 I & \xrightarrow{\Upsilon^{\mathcal{F}}} & \mathcal{F}(I) \\
 & \uparrow \chi^I & \downarrow \chi \\
 I & \xrightarrow{\Upsilon^{\mathcal{G}}} & \mathcal{G}(I)
 \end{array}
 \begin{array}{c}
 \psi \\
 \Rightarrow \\
 \mathfrak{m}
 \end{array}
 =
 \begin{array}{ccc}
 I & \xrightarrow{\Upsilon^{\mathcal{F}}} & \mathcal{F}(I) \\
 & \uparrow \chi^I & \\
 I & \xrightarrow{\Upsilon^{\mathcal{G}}} & \mathcal{G}(I)
 \end{array}
 \begin{array}{c}
 \psi \\
 \Rightarrow \\
 \mathfrak{m}
 \end{array}$$

We denote the category of monoidal lax natural transformations between monoidal pseudofunctors $\mathcal{F}$ and $\mathcal{G}$ by $\text{PsNat}_{\boxtimes}(\mathcal{F}, \mathcal{G})$.

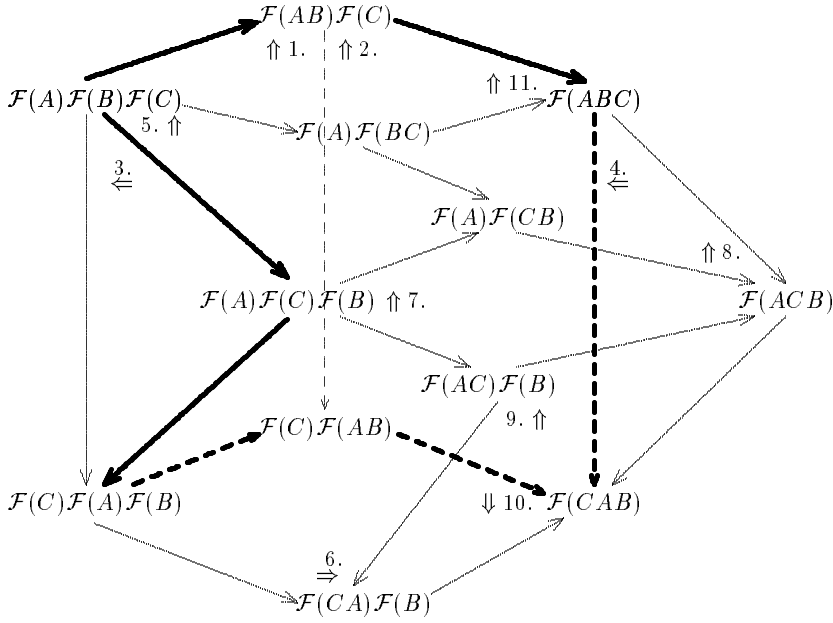
A.4. Definition: Let $\mathfrak{C}, \mathfrak{D}$ be braided Gray monoids. A braided monoidal pseudofunctor $(\mathcal{F}, \mathcal{F}_{\mathcal{R}})$ consists of a monoidal pseudofunctor $(\mathcal{F}, \Xi, \Upsilon, \omega, \zeta^l, \zeta^r) : \mathfrak{C} \rightarrow \mathfrak{D}$ and an invertible modification

$$\begin{array}{ccc}
 \mathcal{F}(A) \boxtimes \mathcal{F}(B) & \xrightarrow{\mathcal{R}_{\mathcal{F}(A), \mathcal{F}(B)}} & \mathcal{F}(B) \boxtimes \mathcal{F}(A) \\
 \Xi_{A,B}^{\mathcal{F}} \downarrow & \Downarrow \mathcal{F}_{\mathcal{R}_{A,B}} & \downarrow \Xi_{B,A} \\
 \mathcal{F}(A \boxtimes B) & \xrightarrow{\mathcal{F}(\mathcal{R}_{A,B})} & \mathcal{F}(B \boxtimes A)
 \end{array}$$

such that the following two diagrams commute.



$$\begin{array}{lll}
 1. \triangleq R_{(\mathcal{F}(A), \Xi)}^{-1} & 2. \triangleq \mathcal{F}_{R, A, B \boxtimes C} & 3. \triangleq \tilde{R}_{1(\mathcal{F}(A)|\mathcal{F}(B), \mathcal{F}(C))} & 4. \triangleq \mathcal{F}(\tilde{R}_{1(A|B, C)}) \\
 5. \triangleq \mathcal{F}_{R, A, B} \boxtimes \mathcal{F}(C) & 6. \triangleq \mathcal{F}(B) \boxtimes \mathcal{F}_{R, A, C} & 7. \triangleq \omega_{B, A, C} & 8. \triangleq \Xi_{R, A, B, C} \\
 9. \triangleq \Xi_{B, R, A, C} & 10. \triangleq \omega_{B, C, A} & 11. \triangleq \omega_{A, B, C}
 \end{array}$$



$$\begin{array}{lll}
 1. \triangleq R_{(\Xi, \mathcal{F}(C))}^{-1} & 2. \triangleq \mathcal{F}_{R, A \boxtimes B, C} & 3. \triangleq \tilde{R}_{2(\mathcal{F}(A), \mathcal{F}(B)|\mathcal{F}(C))} & 4. \triangleq \mathcal{F}(\tilde{R}_{2(A, B|C)}) \\
 5. \triangleq \mathcal{F}(A) \boxtimes \mathcal{F}_{R, B, C} & 6. \triangleq \mathcal{F}_{R, A, C} \boxtimes \mathcal{F}(B) & 7. \triangleq \omega_{A, C, B} & 8. \triangleq \Xi_{A, R, B, C} \\
 9. \triangleq \Xi_{R, A, B, B} & 10. \triangleq \omega_{C, A, B} & 11. \triangleq \omega_{A, B, C}
 \end{array}$$

The following two diagrams give the axioms for the units, where we identify along the identity 1-morphisms.

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & \text{id} & & \\
 & & \uparrow \zeta_A^r & & \\
 \mathcal{F}(A) \boxtimes I & \xrightarrow{1 \boxtimes \Upsilon^{\mathcal{F}}} & \mathcal{F}(A) \boxtimes \mathcal{F}(I) & \xrightarrow{\Xi^{\mathcal{F}}} & \mathcal{F}(A \boxtimes I) \\
 \downarrow \Downarrow \mathcal{R}_{\mathcal{F}(A), \zeta^{\mathcal{F}}} & & \downarrow \Downarrow \mathcal{F}_{\mathcal{R}} & & \downarrow \mathcal{F}(\mathcal{R}) \\
 \text{id} \curvearrowright \begin{array}{c} \downarrow \\ \mathcal{R} \end{array} \begin{array}{c} \downarrow \\ \Upsilon^{\mathcal{F}} \boxtimes 1 \end{array} \downarrow \Downarrow \zeta_A^l \curvearrowright \text{id} & & & & \\
 I \boxtimes \mathcal{F}(A) & \xrightarrow{\Upsilon^{\mathcal{F}} \boxtimes 1} & \mathcal{F}(I) \boxtimes \mathcal{F}(A) & \xrightarrow{\Xi^{\mathcal{F}}} & \mathcal{F}(I \boxtimes A) \\
 & & & & \downarrow \mathcal{F}(\mathcal{R}) \\
 & & & & \mathcal{F}(I \boxtimes A) \xleftarrow{\mathcal{F}(\mathcal{R})} \mathcal{F}(A \boxtimes I) \\
 & & & & \text{id} \curvearrowright \begin{array}{c} \downarrow \\ \mathcal{F}(\mathcal{R}) \end{array} \begin{array}{c} \downarrow \\ \mathcal{F}(\mathcal{R}) \end{array} \downarrow \Downarrow \zeta_A^l \curvearrowright \text{id}
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & \text{id} & & \\
 & & \uparrow \zeta_A^l & & \\
 I \boxtimes \mathcal{F}(A) & \xrightarrow{\Upsilon^{\mathcal{F}} \boxtimes 1} & \mathcal{F}(I) \boxtimes \mathcal{F}(A) & \xrightarrow{\Xi^{\mathcal{F}}} & \mathcal{F}(I \boxtimes A) \\
 \downarrow \Downarrow \mathcal{R}_{\zeta^{\mathcal{F}}, \mathcal{F}(A)} & & \downarrow \Downarrow \mathcal{F}_{\mathcal{R}} & & \downarrow \mathcal{F}(\mathcal{R}) \\
 \text{id} \curvearrowright \begin{array}{c} \downarrow \\ \mathcal{R} \end{array} \begin{array}{c} \downarrow \\ 1 \boxtimes \Upsilon^{\mathcal{F}} \end{array} \downarrow \Downarrow \zeta_A^r \curvearrowright \text{id} & & & & \\
 \mathcal{F}(A) \boxtimes I & \xrightarrow{1 \boxtimes \Upsilon^{\mathcal{F}}} & \mathcal{F}(A) \boxtimes \mathcal{F}(I) & \xrightarrow{\Xi^{\mathcal{F}}} & \mathcal{F}(A \boxtimes I) \\
 & & & & \downarrow \mathcal{F}(\mathcal{R}) \\
 & & & & \mathcal{F}(A \boxtimes I) \xleftarrow{\mathcal{F}(\mathcal{R})} \mathcal{F}(I \boxtimes A) \\
 & & & & \text{id} \curvearrowright \begin{array}{c} \downarrow \\ \mathcal{F}(\mathcal{R}) \end{array} \begin{array}{c} \downarrow \\ \mathcal{F}(\mathcal{R}) \end{array} \downarrow \Downarrow \zeta_A^r \curvearrowright \text{id}
 \end{array}
 \end{array}$$

A.5. Definition: Let $\chi : \mathcal{F} \Rightarrow \mathcal{G}$ be a monoidal pseudonatural transformation between braided pseudofunctors. Then χ is called *braided* if the following diagram commutes.

$$\begin{array}{ccccc}
 & & \mathcal{F}(A) \boxtimes \mathcal{F}(B) & \xrightarrow{\Xi_{A,B}^{\mathcal{F}}} & \mathcal{F}(A \boxtimes B) \\
 & \swarrow \mathcal{R}_{\mathcal{F}(A), \mathcal{F}(B)} & \uparrow \chi_{2;A,B} & & \searrow \mathcal{F}(\mathcal{R}_{A,B}) \\
 \mathcal{F}(B) \boxtimes \mathcal{F}(A) & \xrightarrow{\Xi_{B,A}^{\mathcal{F}}} & \mathcal{F}(B \boxtimes A) & & \downarrow \chi_{A \boxtimes B} \\
 \downarrow \chi_B \boxtimes \chi_A & & \downarrow \chi_{B \boxtimes A} & & \\
 \mathcal{G}(B) \boxtimes \mathcal{G}(A) & \xrightarrow{\Xi_{B,A}^{\mathcal{G}}} & \mathcal{G}(B \boxtimes A) & & \\
 \uparrow \tilde{R}_{\chi, \chi} & & \uparrow \chi_{2;B,A} & & \\
 \mathcal{G}(A) \boxtimes \mathcal{G}(B) & \xrightarrow{\Xi_{A,B}^{\mathcal{G}}} & \mathcal{G}(A \boxtimes B) & & \\
 \uparrow \chi_A \boxtimes \chi_B & & \uparrow \chi_{A \boxtimes B} & & \\
 \mathcal{F}(A) \boxtimes \mathcal{F}(B) & \xrightarrow{\Xi_{A,B}^{\mathcal{F}}} & \mathcal{F}(A \boxtimes B) & &
 \end{array}$$

Additional labels in the diagram include: $\mathcal{F}(\mathcal{R}_{A,B})$, $\chi_{A \boxtimes B}$, $\chi_B \boxtimes \chi_A$, $\chi_A \boxtimes \chi_B$, $\chi_{B \boxtimes A}$, $\chi_{A \boxtimes B}$, $\chi_{2;A,B}$, $\chi_{2;B,A}$, $\tilde{R}_{\chi, \chi}$, $\mathcal{R}_{\mathcal{F}(A), \mathcal{F}(B)}$, $\mathcal{F}(\mathcal{R}_{A,B})$, $\mathcal{G}(\mathcal{R}_{A,B})$, $\chi_{\mathcal{R}_{A,B}} \Rightarrow$.

A.6. Definition: Let $\mathcal{F} : \mathfrak{C} \rightarrow \mathfrak{D}$ be a braided functor between involutory 2-categories. $\mathcal{F}$ is called *involutory* if the following diagram commutes

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & & \Downarrow \mathfrak{I}_{A,B} & & \\
 \mathcal{F}(A) \boxtimes \mathcal{F}(B) & \xrightarrow{\mathcal{R}_{A,B}} & \mathcal{F}(B) \boxtimes \mathcal{F}(A) & \xrightarrow{\mathcal{R}} & \mathcal{F}(A) \boxtimes \mathcal{F}(B) \\
 \downarrow \Xi^{\mathcal{F}} & \downarrow \mathcal{F}_{\mathcal{R}} & \downarrow \Xi^{\mathcal{F}} & \downarrow \mathcal{F}_{\mathcal{R}} & \downarrow \Xi^{\mathcal{F}} \\
 \mathcal{F}(A \boxtimes B) & \xrightarrow{\mathcal{F}(\mathcal{R})} & \mathcal{F}(B \boxtimes A) & \xrightarrow{\mathcal{F}(\mathcal{R})} & \mathcal{F}(A \boxtimes B) \\
 & \uparrow \mathcal{F}(\mathfrak{I}_{A,B}) & & & \\
 & \text{id} & & &
 \end{array}$$

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