THE LANDAUER LIMIT AND THERMODYNAMICS OF BIOLOGICAL SYSTEMS

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**HEAT OF ERASING A BIT**

Thermodynamic cost to erase a bit - the minimal amount of entropy that must be expelled to the environment - is $\ln[2]$
HEAT OF ERASING A BIT

- Crucially, **DO** know precise pre-erasure value of bit
  - After all, *a computer is useless if don’t know its initial state*

Bennett, 2003: “If erasure is applied to random data, the operation may be thermodynamically reversible … but if it is applied to known data, it is thermodynamically irreversible.”
• Crucially, **DO** know precise pre-erasure value of bit
  - *After all, a computer is useless if don’t know its initial state*

• In fact, the prior distribution over the pre-erasure value – and in particular the entropy of that prior – is irrelevant

• Requires careful engineering to make this property hold

• “Local detailed balance” does not hold
• Example: *Adiabatic demagnetization*

• Exploited in modern engineering:
  
  o *Noisy error correction computing*
  
  o Real (not “pseudo”) random number generators
What is the thermodynamic cost for an arbitrary conditional distribution from $X = \{0, 1, 2, 3\}$ into itself?

E.g., what if

- 0 and 1 go to 0 (as in bit erasure);
  
  i.e., $P(0 \mid 0) = P(0 \mid 1) = 1$

- 2 goes to 0 with probability .8, stays the same otherwise;
  
  i.e., $P(0 \mid 2) = .8, P(2 \mid 2) = .2$

- 3 goes to 2 with probability .4, and to 0 with probability .6;
  
  i.e., $P(2 \mid 3) = .5, P(0 \mid 3) = .6$
\[ \mathbb{E}(\text{cost}) = \sum_{v_t, v_{t+1}} \pi(v_{t+1} \mid v_t) P(v_t) \ln \left[ \sum_{v_t'} \pi(v_{t+1} \mid v_t) \right] \]

\[ = I(W_{t+1}; V_{t+1}) - I(W_t; V_t) \]

- where \( v_t \) is the observable \( v \)'s value at time \( t \);
- \( \pi(\cdot \mid \cdot) \) is the conditional distribution of dynamics;
- \( I(\cdot; \cdot) \) is mutual information;
- \( W \) is unobserved degrees of freedom;
THERMODYNAMIC COST

\[ \mathbb{E}(\text{cost}) = \sum_{v_t, v_{t+1}} \pi(v_{t+1} \mid v_t) P(v_t) \ln \left[ \sum_{v_t'} \pi(v_{t+1} \mid v_t) \right] \]

- where \( v_t \) is the observable \( v \)'s value at time \( t \);
- \( \pi(\cdot \mid \cdot) \) is the conditional distribution of dynamics;

Example:

In a 2-to-1 map, \( \pi(0 \mid 0) = \pi(0 \mid 1) = 1 \),
so expected cost equals \( \ln[2] \)
Given the evolution kernel $\pi(., .)$, as one varies $P(v_t)$:

$$0 \leq \mathbb{E}(\text{cost}) + H(V_t) + H(V_{t+1}) \leq \log[|V|] - \max_{v_t} \left[ KL(\pi(V_{t+1} | a) \| \Pi^+(V_{t+1})) \right]$$

- where $v_t$ is the observable $v$’s value at time $t$;
- $H(.)$ is Shannon entropy;
- $KL(.) \| .)$ is KL divergence;
- $\Pi^+(v_{t+1}) = \sum_{v_t} \pi(v_{t+1} | v_t) / |V|$
Given the evolution kernel $\pi(. \mid .)$, as one varies $P(v_t)$:

$$0 \leq \mathbb{E}(\text{cost}) + H(V_t) + H(V_{t+1}) \leq \log(|V|) - \max_{v_t} \left[ KL(\pi(V_{t+1} \mid a) \mid\mid \Pi^+(V_{t+1})) \right]$$

- where $v_t$ is the observable $v$’s value at time $t$;
- $H(.)$ is Shannon entropy;
- $KL(. \parallel .)$ is KL divergence;
- $\Pi^+(v_{t+1}) = \sum_{v_t} \pi(v_{t+1} \mid v_t) / |V|$.

Example: In a 2-to-1 map both bounds are tight:

Thermodynamic cost is the drop in Shannon entropies over $V$.
For a $k$'th order Markov chain, thermodynamic cost during a single step is bounded below by

$$L \equiv H(V_t \mid V_{t+1}, \ldots, V_{t+k-1}) - H(V_{t+k} \mid V_{t+1}, \ldots, V_{t+k-1})$$

and above by

**Very messy expression**
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If $P(v)$ has reached stationarity, lower bound is

$$\mathbb{E}(\text{cost}) = I(V_t; V_{t+1}, \ldots, V_{t+k-1}) - I(V_{t+k}; V_{t+1}, \ldots, V_{t+k-1})$$

(Cf. Still et al. 2012)
THERMODYNAMIC COST

Tons of other fun results, including:

1) **Second law**: Thermodynamic cost is non-negative for any process that goes from distribution A to distribution B and then back to distribution B

2) Examples of coarse-graining (in the statistical physics sense) that *increase* thermodynamic cost

3) Examples of coarse-graining that *decrease* thermo. cost

4) Implications for optimal *compiler design*

5) Analysis of thermodynamic cost of *Hidden Markov Models*
**IMPLICATIONS FOR DESIGN OF BRAINS**

- $P(x_i)$ a dynamic process outside of a brain;
- Natural selection favors brains that:
  - (generate $v_i$’s that) predict future of $x$ accurately;
  - but …
  - not generate heat that needs to be dissipated;
  - not require free energy from environment (need to create all that heat)

Natural selection favors brains that:

1) Accurately predict future (*quantified with a fitness function*);
2) Using a prediction program with minimal thermo. cost
IMPLICATIONS FOR BIOCHEMISTRY

• Natural selection favors (phenotypes of) a prokaryote that:
  • (generate \(v_i\)’s that) maximize fitness;
    but …
  • not generate heat that needs to be dissipated;
  • not require free energy from environment (need to create all that heat)

Natural selection favors prokaryotes that:

1) Behave as well as possible (quantified with a fitness function);
2) While implementing behavior with minimal thermo. cost
\[ \mathbb{E}(cost) = \sum_{v_t, v_{t+1}} \pi(v_{t+1} \mid v_t) P(v_t) \ln \left[ \sum_{v_t'} \pi(v_{t+1} \mid v_t) \right] \]

- where \( v_t \) is the observable \( v \)'s value at time \( t \);
- \( \pi(. \mid .) \) is the conditional distribution of dynamics;

N.b., thermodynamic cost varies with \( t \):

For what kernels \( \pi(. \mid .) \) does thermo. cost increase with time?
\[ \mathbb{E}(cost) = \sum_{v_t, v_{t+1}} \pi(v_{t+1} \mid v_t) P(v_t) \ln \left( \sum_{v'_t} \pi(v_{t+1} \mid v_t) \right) \]

- where \( v_t \) is the observable \( v \)'s value at time \( t \);
- \( \pi(. \mid .) \) is the conditional distribution of dynamics;

Plug in \( \pi(. \mid .) \) of terrestrial biosphere:

Does thermo. cost of biosphere behavior increase with time?
COMPLEXITY DYNAMICS
OF BIOSPHERE

\[ \mathbb{E}(\text{cost}) = \sum_{v_t, v_{t+1}} \pi(v_{t+1} \mid v_t) P(v_t) \ln \left[ \sum_{v_t'} \pi(v_{t+1} \mid v_t) \right] \]

- where \( v_t \) is the observable \( v \)'s value at time \( t \);
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Plug in \( \pi(\cdot \mid \cdot) \) of terrestrial biosphere:

How far is thermo. cost of biosphere from upper bound of solar free energy flux?