Getting to the Bottom of Noether's Theorem

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The Philosophy and Physics of Noether's Theorems
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In the summers of 1926–1928, Alexandroff and Hopf lectured on topology in Göttingen. Noether attended and pointed out that ith Betti number is the rank of an abelian group

$$H_i(X) = \frac{\ker \partial_i}{\operatorname{im} \partial_{i+1}}$$

where

$$\partial_i \colon C_i(X) \to C_{i-1}(X)$$

is a map between 'chain groups'. She also noticed that a map of simplicial complexes induced a map of homology groups. *All this was new!*

Noether never published a single paper about these ideas, and they spread slowly.

> *"All these ideas were a very long time in the making because the people doing homology and homotopy theory were not algebraists and the algebraists didn't take any interest. The only person who took any interest was Emmy Noether."* — *Peter Hilton*

It’s the 100th anniversary of the paper in which Noether proved two theorems relating symmetries and conserved quantities: the first is commonly called “Noether’s theorem", while the second concerns what we now call gauge symmetries.

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

“Of course it would be sufficient if you asked Fräulein Noether to clarify this for me.” — Einstein to Hilbert, 1916

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For which types of physical theories do symmetries give conservation laws?

What are we assuming about the world, if we assume it's described by theories of this type?

It's hard to get to the bottom of these questions, but let's try.

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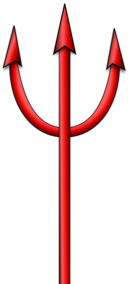
I will study this subject *algebraically*.



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A **Poisson algebra** is a real vector space A equipped with a multiplication making A into a commutative algebra:

$$a(bc) = (ab)c \qquad ab = ba$$

$$a(\beta b + \gamma c) = \beta ab + \gamma ac$$

together with a **Poisson bracket** making A into a Lie algebra:

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\} \qquad \{a, b\} = -\{b, a\}$$

$$\{a, \beta b + \gamma c\} = \beta\{a, b\} + \gamma\{a, c\}$$

and obeying the **Leibniz law**:

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

In classical mechanics a Poisson algebra A serves as our algebra of observables. Assume for any $a \in A$ there is a unique one-parameter group of maps

$$F_t^a \colon A \to A \qquad (t \in \mathbb{R})$$

obeying

$$\frac{d}{dt} F_t^a(b) = \{a, F_t^a(b)\}$$

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These assumption holds in all ‘nice’ cases. (Consult the demon of analysis.)

Then we can check that the maps F_t^a act as symmetries of our Poisson algebra:

$$F_t^a(\beta b + \gamma c) = \beta F_t^a(b) + \gamma F_t^a(c)$$

$$F_t^a(bc) = F_t^a(b) F_t^a(c)$$

$$F_t^a(\{b, c\}) = \left\{F_t^a(b), F_t^a(c)\right\}$$

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These are equivalent! And the proof is very pretty.

$$F_t^a(b) = b \qquad \forall t \in \mathbb{R}$$

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By the antisymmetry of the bracket this is true iff

$$\{b, a\} = 0$$

so running the argument backwards with a, b switched we get

$$F_t^b(a) = a \qquad \forall t \in \mathbb{R}$$

So, in Poisson mechanics, observables that *generate symmetries of the Hamiltonian* are the same as *conserved quantities*. And this follows from two main ideas:

- observables generate 1-parameter transformation groups via the bracket
- the bracket is antisymmetric, so:

$$\begin{array}{ll} \{a,b\} = 0 & (b \text{ is conserved by the transformations generated by } a) \\ & \Updownarrow \\ \{b,a\} = 0 & (a \text{ is conserved by the transformations generated by } b) \end{array}$$

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Any Lie group G has a Lie algebra L. L has a bracket that is antisymmetric.

Any $a \in L$ gives rise to a one-parameter group of maps

$$F_t^a \colon L \to L \qquad (t \in \mathbb{R})$$

obeying

$$\frac{d}{dt} F_t^a(b) = \left[a, F_t^a(b)\right]$$

and these are symmetries of our Lie algebra:

$$F_t^a(\beta b + \gamma c) = \beta F_t^a(b) + \gamma F_t^a(c)$$

$$F_t^a([b,c]) = \left[F_t^a(b), F_t^a(c)\right]$$

The part that's *not* just group theory is this: in a Poisson algebra, the *generators of 1-parameter groups of transformations* are also the *real-valued quantities we can measure*.

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It's not true in some theories!

It holds in ordinary 'complex' quantum mechanics, but it fails for real and quaternionic quantum mechanics, and also for stochastic processes:

- JB and Brendan Fong, A Noether theorem for Markov processes, arXiv:1203.2035.

A Poisson algebra combines the multiplication of *observables*:

$$a(bc) = (ab)c \qquad ab = ba$$

$$a(\beta b + \gamma c) = \beta ab + \gamma ac$$

with the bracket of *symmetry generators*:

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\} \qquad \{a, b\} = -\{b, a\}$$

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tied together by the Leibniz law:

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

It's a hybrid structure!

As we all know, the math simplifies if we take A to be a *noncommutative* but still associative algebra, and define $[a, b] = ab - ba$. Then we get the Lie algebra laws

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]] \qquad [a,b] = -[b,a]$$

$$[a, \beta b + \gamma c] = \beta[a,b] + \gamma[a,c]$$

and the Leibniz law

$$[a, bc] = [a,b]c + b[a,c]$$

for free! They follow from the laws of an associative algebra:

$$a(bc) = (ab)c$$

$$a(\beta b + \gamma c) = \beta ab + \gamma ac \qquad (\alpha a + \beta b)c = \alpha ac + \beta bc$$

This makes quantum mechanics much more tightly unified than classical mechanics.

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But there's a catch: not all operators on Hilbert space are observables!

- real-valued *observables* are *self-adjoint*: $a^* = a$
- *symmetry generators* are *skew-adjoint*: $a^* = -a$.

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We can turn self-adjoint a into skew-adjoint ia, which can generate symmetries:

$$F_t^a(b) = e^{ita}\, b\, e^{-ita}$$

This works fine, but it relies crucially on $i = \sqrt{-1}$.

Suppose $B(H)$ is the space of bounded operators on a real, complex or quaternionic Hilbert space H. Then

$$B(H) = O \oplus L$$

where

$$\begin{array}{rclcl} O & = & \{a \in B(H) : \; a^* = a\} & = & \textbf{observables} \\ L & = & \{a \in B(H) : \; a^* = -a\} & = & \textbf{symmetry generators} \end{array}$$

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$B(H)$ is an associative algebra but O and L are not. L is a real Lie algebra:

$$a, b \in L \implies [a, b] := ab - ba \in L$$

O is a real Jordan algebra:

$$a, b \in O \implies a \circ b := ab + ba \in O$$

The Lie algebra of symmetry generators L acts on the Jordan algebra of observables O:

$$a \in L, b \in O \implies [a, b] := ab - ba \in O$$

Any $a \in L$ gives rise to *two* one-parameter groups of maps

$$F^a_t \colon O \to O, \quad F^a_t \colon L \to L \qquad (t \in \mathbb{R})$$

both obeying

$$\frac{d}{dt} F^a_t(b) = [a, F^a_t(b)]$$

These maps preserve all the operations involving O and L.

However, only in the complex case do we have a bijection

$$\phi \colon O \xrightarrow{\sim} L$$

such that

$$[a, \phi(b)] = \phi([a, b]) \qquad \forall a \in L, b \in O$$

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In the complex case this map is

$$\phi(a) = ia.$$

In the real case we have no square root of -1. In the quaternionic case the different square roots of -1 fail to commute so if $a^* = a$ then usually $(ia)^* = a^* i^* = -ai \neq -ia$.

In short:

The key to Noether's theorem is the requirement that we can freely reinterpret *observables* as *symmetry generators*, and vice versa — in a way that's consistent with the action of symmetry generators on both observables and symmetry generators.

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The key to Noether's theorem is the requirement that we can freely reinterpret *observables* as *symmetry generators*, and vice versa — in a way that's consistent with the action of symmetry generators on both observables and symmetry generators.

In classical mechanics this is achieved by a hybrid structure: a Poisson algebra, whose elements are both observables and symmetry generators.

In an algebraic approach to quantum theory, this requirement singles out *complex* quantum mechanics. $i = \sqrt{-1}$ turns observables into symmetry generators, and vice versa.