Physics and analysis on noncompact manifolds[1]

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Theorem 1. *Let S be a smooth manifold equipped with a complete Riemannian metric g. Then the formally adjoint operators*

$$C_0^\infty\Omega_S^k \underset{\mathrm{d}_k^*}{\overset{\mathrm{d}_k}{\rightleftarrows}} C_0^\infty\Omega_S^{k+1}$$

have mutually adjoint closures

$$L^2\Omega_S^k \underset{\mathrm{d}_k^*}{\overset{\mathrm{d}_k}{\rightleftarrows}} L^2\Omega_S^{k+1} \ .$$

These closed operators satisfy

$$\operatorname{ran}\mathrm{d}_{k-1} \subseteq \ker\mathrm{d}_k, \qquad \operatorname{ran}\mathrm{d}_k^* \subseteq \ker\mathrm{d}_{k-1}^*$$

and there is a Hilbert-space direct-sum decomposition

$$L^2\Omega^k = \overline{\operatorname{ran} d_{k-1}} \oplus \ker\Delta_k \oplus \overline{\operatorname{ran}\delta_k}.$$

where the Laplacian on k-forms,

$$\Delta_k = \delta_k d_k + d_{k-1}\delta_{k-1},$$

is a nonnegative densely defined self-adjoint operator on $L^2\Omega^k$.

Theorem 2. *Let M be a $(3+1)$-dimensional static, globally hyperbolic space-time, with metric*

$$g_M = e^{2\Phi}(-\mathrm{d}t^2 + g).$$

Then, electromagnetism on M with gauge group $\mathbb{R}$ has a linear phase space

$$\mathbf{P} = \frac{\mathrm{dom}\{\mathrm{d}\colon L^2\Omega^1_S \to L^2\Omega^2_S\}}{\overline{\mathrm{ran}}\{\mathrm{d}\colon L^2\Omega^0_S \to L^2\Omega^1_S\}} \oplus \ker\{\mathrm{d}^*\colon L^2\Omega^1_S \to L^2\Omega^0_S\},$$

with continuous symplectic structure

$$\omega(X, X') = (E, A') - (E', A)$$

where $X = [A] \oplus E$ is a generic point of $\mathbf{P}$ and

$$(\alpha, \beta) = \int_S g(\alpha, \beta)\mathrm{vol}$$

is the canonical inner product induced on Ω^k_S by the optical metric g on S. The Hamiltonian is the continuous quadratic form

$$H[X] = \frac{1}{2}\big[(E, E) + (\mathrm{d}A, \mathrm{d}A)\big].$$

There phase space splits naturally into two sectors,

$$\mathbf{P} = \mathbf{P}_o \oplus \mathbf{P}_f,$$

and the direct summands

$$\mathbf{P}_f = \mathbf{P} \cap \ker\Delta \qquad \textit{and} \quad \mathbf{P}_o = \mathbf{P} \cap \mathrm{ran}\,\mathrm{d}^*_1$$

are preserved by time evolution. On $\mathbf{P}_o$, time evolution takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_o(t) \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} \cos(t\sqrt{\Delta}) & \sin(t\sqrt{\Delta}) \,/\, \sqrt{\Delta} \\ -\sqrt{\Delta}\,\sin(t\sqrt{\Delta}) & \cos(t\sqrt{\Delta}) \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}$$

and, on $\mathbf{P}_f$, it takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_f(t) \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}.$$

We define the twisted exterior derivative $D_k\colon C_0^\infty\Omega_S^k \to C_0^\infty\Omega_S^{k+1}$ by

$$D_k = e^{\frac{1}{2}(n-2p-1)\Phi}\mathrm{d}_k e^{-\frac{1}{2}(n-2p-1)\Phi}. \tag{1}$$

This operator has a formal adjoint

$$D_k^\dagger = e^{-\frac{1}{2}(n-2p-1)\Phi}\delta_{k+1}e^{\frac{1}{2}(n-2p-1)\Phi} \tag{2}$$

meaning that

$$(D_k^\dagger\alpha, \beta) = (\alpha, D_k\beta) \tag{3}$$

whenever $\alpha \in C_0^\infty\Omega^{k+1}$ and $\beta \in C_0^\infty\Omega^k$. In what follows we shall omit the subscript 'k' from the operators D_k and $D_k^\dagger$ when it is clear from context.

Theorem 3. *Let S be a smooth n-dimensional manifold equipped with a complete Riemannian metric g, and let Φ be a smooth real-valued function on S. Fix an integer $0 \le p \le n$. Then for any integer k, the operators*

$$C_0^\infty\Omega_S^k \underset{D_k^\dagger}{\overset{D_k}{\rightleftarrows}} C_0^\infty\Omega_S^{k+1}$$

defined in equations (1) and (2) have mutually adjoint closures, which we write as

$$L^2\Omega_S^k \underset{D_k^*}{\overset{D_k}{\rightleftarrows}} L^2\Omega_S^{k+1}$$

These closures satisfy

$$\operatorname{ran} D_{k-1} \subseteq \ker D_k, \qquad \operatorname{ran} D_k^* \subseteq \ker D_{k-1}^*,$$

and we obtain a direct sum decomposition

$$L^2\Omega^k = \overline{\operatorname{ran} D_{k-1}} \oplus \ker L_k \oplus \overline{\operatorname{ran} D_k^*}.$$

where the twisted Laplacian on k-forms,

$$L_k = D_k^* D_k + D_{k-1}D_{k-1}^*,$$

is a nonnegative densely defined self-adjoint operator on $L^2\Omega^k$.

Theorem 4. *Let M be a $(n+1)$-dimensional static, globally hyperbolic space-time, with metric*

$$g_M = e^{2\Phi}(-\mathrm{d}t^2 + g).$$

Then, p-form electromagnetism on M with gauge group $\mathbb{R}$ has a linear phase space

$$\mathbf{P} = \frac{\mathrm{dom}\{D_p \colon L^2\Omega_S^p \to L^2\Omega_S^{p+1}\}}{\overline{\mathrm{ran}}\{D_p \colon L^2\Omega_S^0 \to L^2\Omega_S^1\}} \oplus \ker\{D_p^* \colon L^2\Omega_S^1 \to L^2\Omega_S^0\},$$

with continuous symplectic structure

$$\omega(X, X') = (E, A') - (E', A)$$

where $X = [A] \oplus E$ is a generic point of $\mathbf{P}$ and

$$(\alpha, \beta) = \int_S g(\alpha, \beta)\mathrm{vol}$$

is the canonical inner product induced on Ω_S^k by the optical metric g on S. The Hamiltonian is the continuous quadratic form

$$H[X] = \frac{1}{2}\big[(E, E) + (D_p A, D_p A)\big].$$

There phase space splits naturally into two sectors,

$$\mathbf{P} = \mathbf{P}_o \oplus \mathbf{P}_f,$$

and the direct summands

$$\mathbf{P}_f = \mathbf{P} \cap \ker \Delta \qquad \textit{and} \quad \mathbf{P}_o = \mathbf{P} \cap \mathrm{ran}\, \mathrm{d}_1^*$$

are preserved by time evolution. On $\mathbf{P}_o$, time evolution takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_o(t) \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} \cos(t\sqrt{L_p}) & \sin(t\sqrt{L_p}) \,/ \sqrt{L_p} \\ -\sqrt{L_p}\, \sin(t\sqrt{L_p}) & \cos(t\sqrt{L_p}) \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}$$

and, on $\mathbf{P}_f$, it takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_f(t) \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}.$$

Proposition 5 (Gaffney). *If S is a complete oriented Riemannian manifold, then*

$$(\delta^* \alpha, \beta) = (\alpha, \mathrm{d}^* \beta)$$

whenever $\alpha \in \operatorname{dom} \delta^*$ and $\beta \in \operatorname{dom} \mathrm{d}^*$.

Proposition 6 (Kodaira decomposition). *If*

$$H \xrightarrow{\;S\;} H' \xrightarrow{\;T\;} H''$$

are densely defined closed operators and $\operatorname{ran} S \subseteq \ker T$, then

$$H' = \overline{\operatorname{ran} T^*} \oplus \ker(T^*T + SS^*) \oplus \overline{\operatorname{ran} S}.$$

Lemma 7. *The completion of* $C_0^\infty\Omega_S^k$ *with respect to the inner product* $(\ ,\ )$ *is* $L^2\Omega_S^k$.

Lemma 8. *A densely defined operator* T *is closable if, and only if,* T^* *is densely defined. In that case,* $\overline{T} = T^{**}$.

Lemma 9. *If*

$$H \xrightarrow{\ T\ } H'$$

is a densely defined operator, then

$$\ker T^* = (\operatorname{ran} T)^\perp \qquad \text{and} \qquad \ker T = (\operatorname{ran} T^*)^\perp \cap \operatorname{dom} T.$$

Lemma 10. *If*

$$H \xrightarrow{\ S\ } H' \xrightarrow{\ T\ } H''$$

are densely defined operators and $\operatorname{ran} S \subseteq \ker T$, *then*

$$\operatorname{ran} T^* \subseteq \ker S^*.$$

Lemma 11 (Chernoff). *If the metric* $c^{-2}g$ *makes* S *into a complete Riemannian manifold, the symmetric hyperbolic system* $\partial_t\alpha = T\alpha$ *with initial data in* $C_0^\infty E$ *has a unique solution on* $\mathbb{R} \times S$ *which is in* $C_0^\infty E$ *for all* $t \in \mathbb{R}$. *Moreover, if* T *is formally skew-adjoint* ($T + T^\dagger = 0$), *then* $-iT$ *and all its powers are essentially self-adjoint on* $C_0^\infty E$.

Lemma 12. *Let* H_1 *and* H_2 *be Hilbert spaces and let*

$$H_1 \underset{B}{\overset{A}{\rightleftarrows}} H_2$$

be densely defined operators that are formal adjoints of one another:

$$\langle A\phi, \psi\rangle_1 = \langle \phi, B\psi\rangle_2 \qquad \text{for all} \quad \phi \in \operatorname{dom} A, \psi \in \operatorname{dom} B.$$

Let $H = H_1 \oplus H_2$ *and let* S *be the densely defined operator*

$$\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$$

on H. *If* S *is essentially self-adjoint, then* A *and* B *have mutually adjoint closures.*

Lemma 13. *Suppose* S *is a complete Riemannian manifold and* Φ *a smooth real-valued function on* S. *Let*

$$T\colon L^2\Omega_S^k \oplus L^2\Omega_S^{k+1} \to L^2\Omega_S^k \oplus L^2\Omega_S^{k+1}$$

be the densely defined operator

$$\begin{pmatrix} 0 & iD_k^\dagger \\ iD_k & 0 \end{pmatrix}.$$

Then $-iT$ *and all its powers are essentially self-adjoint on* $C_0^\infty\Omega^k \oplus C_0^\infty\Omega^{k+1}$.

Corollary 14. *If S is a complete oriented Riemannian manifold, then*

$$\overline{\mathrm{d}} = \delta^* \qquad \text{and} \quad \overline{\delta} = \mathrm{d}^*.$$

Corollary 15. *If*

$$H \xrightarrow{\ S\ } H' \xrightarrow{\ T\ } H''$$

are densely defined closable operators and $\operatorname{ran} S \subseteq \ker T$, *then*

$$\operatorname{ran} \overline{S} \subseteq \ker \overline{T}.$$

Corollary 16. *Under the same hypothesis as Lemma 13, the operators*

$$C_0^\infty \Omega_S^k \underset{D_k^\dagger}{\overset{D_k}{\rightleftarrows}} C_0^\infty \Omega_S^{k+1}$$

have mutually adjoint closures, and the operators $D_k^\dagger D_k$ and $D_k D_{k-1}^\dagger$ are essentially self-adjoint on $C_0^\infty \Omega^k$.