Physics and analysis on noncompact manifolds\textsuperscript{1}

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Theorem 1. Let $S$ be a smooth manifold equipped with a complete Riemannian metric $g$. Then the formally adjoint operators

$$C^\infty_0 \Omega^k_S \overset{d_k}{\longrightarrow} C^\infty_0 \Omega^{k+1}_S$$

have mutually adjoint closures

$$L^2 \Omega^k_S \overset{d_k}{\longrightarrow} L^2 \Omega^{k+1}_S.$$ 

These closed operators satisfy

$$\text{ran } d_{k-1} \subseteq \ker d_k, \quad \text{ran } d_k^* \subseteq \ker d_{k-1}^*$$

and there is a Hilbert-space direct-sum decomposition

$$L^2 \Omega^k = \overline{\text{ran } d_{k-1}} \oplus \ker \Delta_k \oplus \overline{\text{ran } d_k}.$$

where the Laplacian on $k$-forms,

$$\Delta_k = \delta_k d_k + d_{k-1} \delta_{k-1},$$

is a nonnegative densely defined self-adjoint operator on $L^2 \Omega^k$. 
Theorem 2. Let $M$ be a $(3+1)$-dimensional static, globally hyperbolic space-time, with metric

$$g_M = e^{2\Phi}(-dt^2 + g).$$

Then, electromagnetism on $M$ with gauge group $\mathbb{R}$ has a linear phase space

$$P = \text{dom}(d: L^2\Omega^1_S \to L^2\Omega^2_S) \oplus \ker(d^*: L^2\Omega^1_S \to L^2\Omega^0_S),$$

with continuous symplectic structure

$$\omega(X, X') = (E, A') - (E', A)$$

where $X = [A] \oplus E$ is a generic point of $P$ and

$$(\alpha, \beta) = \int_S g(\alpha, \beta) \text{vol}$$

is the canonical inner product induced on $\Omega^k_S$ by the optical metric $g$ on $S$. The Hamiltonian is the continuous quadratic form

$$H[X] = \frac{1}{2}[(E, E) + (dA, dA)].$$

There phase space splits naturally into two sectors,

$$P = P_o \oplus P_f,$$

and the direct summands

$$P_f = P \cap \ker \Delta \quad \text{and} \quad P_o = P \cap \text{ran} \ d^*$$

are preserved by time evolution. On $P_o$, time evolution takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_o(t) \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} \cos(t\sqrt{\Delta}) & \sin(t\sqrt{\Delta}) / \sqrt{\Delta} \\ -\sqrt{\Delta} \sin(t\sqrt{\Delta}) & \cos(t\sqrt{\Delta}) \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}$$

and, on $P_f$, it takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_f(t) \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}.$$
We define the twisted exterior derivative $D_k: C_0^\infty \Omega^k_S \to C_0^\infty \Omega^{k+1}_S$ by

$$D_k = e^{\frac{1}{2}(n-2p-1)\Phi} d_k e^{\frac{1}{2}(n-2p-1)\Phi}. \quad (1)$$

This operator has a formal adjoint

$$D_k^\dagger = e^{-\frac{1}{2}(n-2p-1)\Phi} \delta_{k+1} e^{\frac{1}{2}(n-2p-1)\Phi} \quad (2)$$

meaning that

$$(D_k^\dagger \alpha, \beta) = (\alpha, D_k \beta) \quad (3)$$

whenever $\alpha \in C_0^\infty \Omega^{k+1}_S$ and $\beta \in C_0^\infty \Omega^k_S$. In what follows we shall omit the subscript ‘$k$’ from the operators $D_k$ and $D_k^\dagger$ when it is clear from context.

**Theorem 3.** Let $S$ be a smooth $n$-dimensional manifold equipped with a complete Riemannian metric $g$, and let $\Phi$ be a smooth real-valued function on $S$. Fix an integer $0 \leq p \leq n$. Then for any integer $k$, the operators

$$C_0^\infty \Omega^k_S \xrightarrow{D_k} C_0^\infty \Omega^{k+1}_S \xleftarrow{D_k^\dagger}$$

defined in equations (1) and (2) have mutually adjoint closures, which we write as

$$L^2 \Omega^k_S \xrightarrow{D_k} L^2 \Omega^{k+1}_S \xleftarrow{D_k^\dagger}$$

These closures satisfy

$$\text{ran } D_{k-1} \subseteq \ker D_k, \quad \text{ran } D_k^\dagger \subseteq \ker D_{k-1}^\dagger,$$

and we obtain a direct sum decomposition

$$L^2 \Omega^k = \text{ran } D_{k-1} \oplus \ker L_k \oplus \text{ran } D_k^\dagger,$$

where the twisted Laplacian on $k$-forms,

$$L_k = D_k^\dagger D_k + D_{k-1} D_k^\dagger,$$

is a nonnegative densely defined self-adjoint operator on $L^2 \Omega^k$. 

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**Theorem 4.** Let $M$ be a $(n+1)$-dimensional static, globally hyperbolic spacetime, with metric

$$g_M = e^{2\Phi}(-dt^2 + g).$$

Then, $p$-form electromagnetism on $M$ with gauge group $\mathbb{R}$ has a linear phase space

$$\mathbf{P} = \frac{\text{dom}(D_p; L^2\Omega^p_S \to L^2\Omega^{p+1}_S)}{\text{ran}(D_p; L^2\Omega^0_S \to L^2\Omega^1_S)} \oplus \ker(D^*_p; L^2\Omega^1_S \to L^2\Omega^0_S),$$

with continuous symplectic structure

$$\omega(X, X') = (E, A') - (E', A)$$

where $X = [A] \oplus E$ is a generic point of $\mathbf{P}$ and

$$(\alpha, \beta) = \int_S g(\alpha, \beta) \text{vol}$$

is the canonical inner product induced on $\Omega^k_S$ by the optical metric $g$ on $S$. The Hamiltonian is the continuous quadratic form

$$H[X] = \frac{1}{2} [(E, E) + (D_p A, D_p A)].$$

There phase space splits naturally into two sectors,

$$\mathbf{P} = \mathbf{P}_o \oplus \mathbf{P}_f,$$

and the direct summands

$$\mathbf{P}_f = \mathbf{P} \cap \ker \Delta \quad \text{and} \quad \mathbf{P}_o = \mathbf{P} \cap \text{ran} d^*_1$$

are preserved by time evolution. On $\mathbf{P}_o$, time evolution takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_o(t) \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} \cos(t\sqrt{L_p}) & \sin(t\sqrt{L_p})/\sqrt{L_p} \\ -\sqrt{L_p} \sin(t\sqrt{L_p}) & \cos(t\sqrt{L_p}) \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}$$

and, on $\mathbf{P}_f$, it takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_f(t) \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}.$$
Proposition 5 (Gaffney). If $S$ is a complete oriented Riemannian manifold, then

$$(\delta^* \alpha, \beta) = (\alpha, d^* \beta)$$

whenever $\alpha \in \text{dom} \delta^*$ and $\beta \in \text{dom} d^*$.

Proposition 6 (Kodaira decomposition). If

$$H \xrightarrow{S} H' \xrightarrow{T} H''$$

are densely defined closed operators and $\text{ran} S \subseteq \ker T$, then

$$H' = \overline{\text{ran} T^*} \oplus \ker (T^* T + SS^*) \oplus \text{ran} S.$$
Lemma 7. The completion of $C^\infty_0 \Omega^k_S$ with respect to the inner product $(\cdot, \cdot)$ is $L^2 \Omega^k_S$.

Lemma 8. A densely defined operator $T$ is closable if, and only if, $T^*$ is densely defined. In that case, $\overline{T} = T^{**}$.

Lemma 9. If $H \xrightarrow{T} H'$ is a densely defined operator, then
$$ \ker T^* = (\text{ran } T)^\perp \quad \text{and} \quad \ker T = (\text{ran } T^*)^\perp \cap \text{dom } T.$$  

Lemma 10. If $H \xrightarrow{S} H' \xrightarrow{T} H''$ are densely defined operators and $\text{ran } S \subseteq \ker T$, then $\text{ran } T^* \subseteq \ker S^*$.

Lemma 11 (Chernoff). If the metric $c^{-2}g$ makes $S$ into a complete Riemannian manifold, the symmetric hyperbolic system $\partial_t \alpha = T\alpha$ with initial data in $C^\infty_0 E$ has a unique solution on $\mathbb{R} \times S$ which is in $C^\infty_0 E$ for all $t \in \mathbb{R}$. Moreover, if $T$ is formally skew-adjoint ($T + T^\dagger = 0$), then $-iT$ and all its powers are essentially self-adjoint on $C^\infty_0 E$.

Lemma 12. Let $H_1$ and $H_2$ be Hilbert spaces and let
$$ H_1 \xrightarrow{A} H_2 $$
be densely defined operators that are formal adjoints of one another:
$$ \langle A\phi, \psi \rangle_1 = \langle \phi, B\psi \rangle_2 \quad \text{for all } \phi \in \text{dom } A, \psi \in \text{dom } B.$$  
Let $H = H_1 \oplus H_2$ and let $S$ be the densely defined operator
$$ \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} $$
on $H$. If $S$ is essentially self-adjoint, then $A$ and $B$ have mutually adjoint closures.

Lemma 13. Suppose $S$ is a complete Riemannian manifold and $\Phi$ a smooth real-valued function on $S$. Let $T: L^2 \Omega^k_S \oplus L^2 \Omega^{k+1}_S \to L^2 \Omega^k_S \oplus L^2 \Omega^{k+1}_S$ be the densely defined operator
$$ \begin{pmatrix} 0 & iD_k^\dagger \\ iD_k & 0 \end{pmatrix}. $$
Then $-iT$ and all its powers are essentially self-adjoint on $C^\infty_0 \Omega^k \oplus C^\infty_0 \Omega^{k+1}$. 

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Corollary 14. If $S$ is a complete oriented Riemannian manifold, then
\[ d = \delta^* \quad \text{and} \quad \overline{d} = d^*. \]

Corollary 15. If
\[ H \xrightarrow{S} H' \xrightarrow{T} H'' \]
are densely defined closable operators and $\text{ran} \; S \subseteq \ker T$, then
\[ \text{ran} \; S \subseteq \ker T. \]

Corollary 16. Under the same hypothesis as Lemma 13, the operators
\[ C_0^\infty \Omega^k \xrightarrow{D_k} C_0^\infty \Omega^{k+1} \]
have mutually adjoint closures, and the operators $D_k^\dagger D_k$ and $D_k D_k^{\dagger -1}$ are essentially self-adjoint on $C_0^\infty \Omega^k$. 

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