Higher-Dimensional Algebra III:
n-Categories and the Algebra of Opetopes

John C. Baez and James Dolan

Department of Mathematics, University of California
Riverside, California 92521
USA

email: baez@math.ucr.edu, jdolan@math.ucr.edu

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Abstract

We give a definition of weak n-categories based on the theory of operads. We work with operads having an arbitrary set S of types, or 'S-operads', and given such an operad O, we denote its set of operations by $\mathrm{elt}(O)$. Then for any S-operad O there is an $\mathrm{elt}(O)$-operad O^+ whose algebras are S-operads over O. Letting I be the initial operad with a one-element set of types, and defining $I^{0+} = I$, $I^{(i+1)+} = (I^{i+})^+$, we call the operations of $I^{(n-1)+}$ the 'n-dimensional opetopes'. Opetopes form a category, and presheaves on this category are called 'opetopic sets'. A weak n-category is defined as an opetopic set with certain properties, in a manner reminiscent of Street's simplicial approach to weak ω-categories. Similarly, starting from an arbitrary operad O instead of I, we define 'n-coherent O-algebras', which are n times categorified analogs of algebras of O. Examples include 'monoidal n-categories', 'stable n-categories', 'virtual n-functors' and 'representable n-prestacks'. We also describe how n-coherent O-algebra objects may be defined in any $(n+1)$-coherent O-algebra.

1 Introduction

A fundamental problem in higher-dimensional algebra is to set up a convenient theory of weak n-categories. Since there seems to be quite a bit of freedom in what such a theory could look like, we begin with a rough sketch of what is called for, and then summarize the ideas behind our approach.

As traditionally conceived, an n-category should be some sort of algebraic structure having objects or 0-morphisms, 1-morphisms between 0-morphisms, 2-morphisms between 1-morphisms, and so on up to n-morphisms. There should be various ways of composing j-morphisms, and these composition operations should satisfy various laws, such as associativity laws. In the so-called 'strict' n-categories, these laws are equations. While well-understood and tractable, strict n-categories are insufficiently general for many applications: what one usually encounters in nature are 'weak' n-categories, in which composition operations satisfy the appropriate laws *only up to equivalence.* Here the idea is that n-morphisms are equivalent precisely when they are equal, while for $j < n$ an equivalence between j-morphisms is recursively defined as a $(j+1)$-morphism from one to the other that is invertible up to equivalence.

What makes it difficult to define weak n-categories is that laws formulated as equivalences should satisfy laws of their own — so-called 'coherence laws' — so that one can manipulate them with some of the same facility as equations. Moreover, these coherence laws should also be equivalences satisfying their own coherence laws, again up to equivalence, and so on.

For example, a weak 1-category is just an ordinary category, defined by Eilenberg and MacLane [15] in their 1945 paper. In a category, composition of 1-morphisms is associative 'on the nose':

$$(fg)h = f(gh).$$

Weak 2-categories first appeared in the work of Bénabou [9] in 1967, under the name of 'bicategories'. In a bicategory, composition of 1-morphisms is associative only up to an invertible 2-morphism, the 'associator':

$$A_{f,g,h}\colon (fg)h \to f(gh).$$

The associator allows one to rebracket parenthesized composites of arbitrarily many 1-morphisms, but there may be many ways to use it to go from one parenthesization to another. For all these to be equal, the associator must satisfy a coherence law, the pentagon identity, which says that the following diagram commutes:

$$\begin{tikzcd}
((fg)h)i \arrow[r] \arrow[d] & (fg)(hi) \arrow[r] & f(g(hi)) \\
(f(gh))i \arrow[rr] & & f((gh)i) \arrow[u]
\end{tikzcd}$$

where all the arrows are 2-morphisms built using the associator. Weak 3-categories or 'tricategories' were defined by Gordon, Power and Street [18] in a paper that appeared in 1995. In a tricategory, the pentagon identity holds only up to an invertible 3-morphism, which satisfies a further coherence law of its own.

When one explicitly lists the coherence laws this way, the definition of weak n-category tends to grow ever more complicated with increasing n. To get around this, one must carefully study the origin of these coherence laws. So far, most of our insight into coherence laws has been won through homotopy theory, where it is common to impose equations *only up to homotopy*, with these homotopies satisfying coherence laws, again up to homotopy, and so on. For example, the pentagon identity and higher coherence laws for associativity first appeared in Stasheff's [28] work on the structure inherited by a space equipped with a homotopy equivalence to a space with an associative product. Subsequent work by Boardman and Vogt, May, Segal and others led to a systematic treatment of coherence laws in homotopy theory through the formalism of topological operads [1].

Underlying the connection between homotopy theory and n-category theory is a hypothesis made quite explicit by Grothendieck [19]: to any topological space one should be able to associate an n-category having points as objects, paths between points as 1-morphisms, certain paths of paths as 2-morphisms, and so on, with certain homotopy classes of n-fold paths as n-morphisms. This should be a special sort of weak n-category called a 'weak n-groupoid', in which all j-morphisms $(0 < j \leq n)$ are equivalences. Moreover, the process of assigning to each space its 'fundamental n-groupoid', as Grothendieck called it, should set up a complete correspondence between the theory of homotopy n-types (spaces whose homotopy groups vanish above the nth) and the theory of weak n-groupoids. This hypothesis explains why all the coherence laws for weak n-groupoids should be deducible from homotopy theory. It also suggests that weak n-categories will have features not found in homotopy theory, owing to the presence of j-morphisms that are not equivalences.

In addition, this hypothesis makes it clear in which contexts the laws governing composition of j-morphisms should hold only up to equivalence: namely, in those where *there is no preferred composite of j-morphisms; instead, the composite is best regarded as only unique up to equivalence.* In homotopy theory this arises from the arbitrary choice involved in parametrizing the composite of two paths. Because of this arbitrariness, composition of paths fails to be associative 'on the nose'. Instead, it is associative up to a homotopy, the associator, with this homotopy satisfying a coherence law, the pentagon identity, but again only up to homotopy, and so on.

While many ways around this problem have been explored, here we prefer to accept it as a fact of nature and develop a theory of weak n-categories in which composition of j-morphisms is not an operation in the traditional sense, but something a bit more subtle. Indeed, many forms of 'composition' in mathematics are of this sort, such as the disjoint union of sets or the tensor product of vector spaces. While one can artificially treat them as operations in the traditional sense, it is better to define them by universal properties. Uniqueness up to equivalence then follows automatically. Taking this as a hint, we shall define the composite of j-morphisms by a universal property.

Homotopy theory also makes it clear that when setting up a theory of n-categories, there is some choice involved in the shapes of ones j-morphisms — or in the language of topology, 'j-cells'. The traditional approach to n-categories is 'globular'. This means that for $j > 0$, each j-cell $f\colon x \to y$ has two $(j-1)$-cells called its 'source', $sf = x$, and 'target', $tf = y$, which for $j > 1$ satisfy

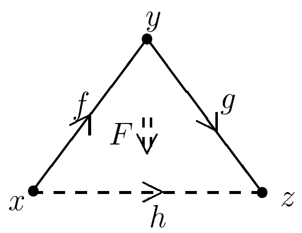
$$s(sf) = s(tf), \qquad t(sf) = t(tf)).$$

Thus a j-cell can be visualized as a 'globe', a j-dimensional ball whose boundary is divided into two $(j-1)$-dimensional hemispheres corresponding to its source and target. In homotopy theory, however, the simplicial approach is much more popular.

In a ‘simplicial set’, each j-cell f is shaped like a j-dimensional simplex, and has $j+1$ faces, certain $(j-1)$-cells $d_0 f, \ldots, d_j f$. In addition to these there are $(j+1)$-cells $i_0 f, \ldots, i_{j+1} f$ called ‘degeneracies’, and the face and degeneracy maps satisfy certain well-known relations.

In the simplicial approach, weak n-groupoids are described using ‘Kan complexes’. It is worth recalling these here, because they begin to illustrate how composite j-morphisms can be defined by a universal property. A ‘j-dimensional horn’ in a simplicial set is, roughly speaking, a configuration in which all but one of the faces of a j-simplex have been filled in by $(j-1)$-cells in a consistent way. A simplicial set for which any horn can be extended to a j-cell is called a ‘Kan complex’. Kan complexes serve to describe arbitrary homotopy types. Algebraically, we may think of them as a simplicial version of ‘weak ω-groupoids’, since they can have nontrivial j-cells for arbitrarily large j. A Kan complex represents a homotopy n-type, or in other words a weak n-groupoid, if for $j > n+1$ any configuration in which all the faces of a j-simplex have been filled in by $(j-1)$-cells in a consistent way can be uniquely extended to a j-cell.

Consider for example the case $j = 2$. Suppose, as shown in Figure 1, that two faces of a 2-simplex have been filled in by 1-cells f and g such that $d_1 f = d_0 g = y$. Then in a Kan complex we can extend this horn to a 2-cell F, which has as its third face a 1-cell h.

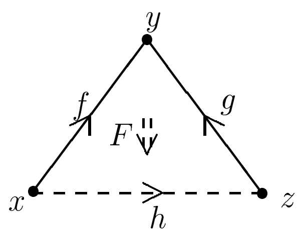


1. Extending a horn to a cell

In this situation, we may think of h as ‘a composite’ of f and g, and F as a ‘process of composing’ f and g. There is not a unique preferred composite. However, it automatically follows from the definition of Kan complex that any two composites are equivalent. Here two j-cells with all the same faces are said to be ‘equivalent’ if there is a $(j+1)$-cell having these j-cells as two of its faces, the rest being degenerate.

Kan complexes serve as a highly efficient formalism in which to do homotopy theory [25]. In particular, there is no need to explicitly list coherence laws! They are all implicit in the fact that every horn can be extended to a cell, and they all become explicit — in their simplicial forms — if one makes composition into an operation of the traditional sort by arbitrarily choosing an extension of every horn. It is tempting, therefore, to develop a simplicial approach to weak n-categories.

This was done by Street [29], who actually dealt with weak ω-categories. Like Kan complexes, these are simplicial sets. However, only certain ‘admissible’ horns, having the correct sort of orientation, are required to have extensions. For example, we do not require the horn shown in Figure 2 to have an extension, since the missing face would correspond to a composite of f and an inverse of g, which we expect to exist in a weak n-groupoid, but not in a weak n-category.



2. A horn that need not have an extension in a n-category

Second, for those horns that are required to have extensions, we require the existence of a 'universal' extension. The point is that, unlike ω-groupoid case, we cannot think of every $(j+1)$-cell as a process of composing all but one of its faces to obtain the remaining face. Instead, Street's weak ω-categories are equipped with a distinguished set of 'universal' cells which we can think of this way. These satisfy some axioms: there are no universal 0-cells, all universal 1-cells are degenerate, and all degenerate cells are universal. Last, and most importantly, any composite of universal cells is universal.

Our definition of weak n-categories resembles Street's, but with two major differences. First, while simplices are convenient in algebraic topology, they are not well adapted to the 'unidirectional' or 'noninvertible' character of the j-morphisms in n-category theory, as is clear from the rather technical combinatorics involved in orienting the faces of a simplex and defining admissible horns. This raises the possibility that a more convenient theory could be set up with j-cells of some other shapes — shapes motivated more by the inner logic of n-category theory than by traditional concerns of algebraic topology. In our approach we use certain shapes called 'opetopes'. (*Nota bene*: The first two syllables of 'opetope' are pronounced exactly as in the word 'operation'.)

Opetopes arise naturally from the theory of operads. Roughly speaking, an 'operad' is an algebraic gadget consisting of a collection of abstract operations closed under composition. These operations may have any finite number of arguments, and we work with operads in which the arguments are 'typed' or many-sorted. Any such operad is determined by: 1) its types, 2) its operations, and 3) its 'reduction laws', or equations stating that some composite of operations equals a given operation. This description of an operad is like a presentation in terms of generators and relations. An operad also has 'algebras' in which its operations are represented as actual functions. From the viewpoint of mathematical logic, an operad is a kind of theory, and its algebras are models of that theory. As always, it is useful to study operads both syntactically, in terms of their presentations, and semantically, in terms of their algebras.

We define the 'slice operad' O^+ of an operad O in such a way that an algebra of O^+ is precisely an operad over O, i.e., an operad with the same set of types as O, equipped with an operad homomorphism to O. Syntactically, it turns out that:

1. The types of O^+ are the operations of O.
2. The operations of O^+ are the reduction laws of O.
3. The reduction laws of O^+ are the ways of combining reduction laws of O to give other reduction laws.

This gets at the heart of the process of 'categorification', in which laws are promoted to operations and these operations satisfy new coherence laws of their own. Here the coherence laws arise simply from the ways of combining the the old laws.

The simplest operad of all is the initial operad, I. Syntactically speaking, this is the operad with only one type and only one operation, the identity. Semantically, I is the operad whose algebras are just sets, without any extra structure at all. Starting with I and iterating the slice operad construction $j-1$ times, we obtain an operad whose operations we call 'j-dimensional opetopes.' A 0-dimensional opetope is just a point, and a 1-dimensional opetope is just an oriented interval. For $j > 1$, a j-dimensional opetope may have any number of 'infaces' but only one 'outface'. Thanks to the above syntactic description of the slice operad construction, it turns out that a j-dimensional opetope corresponds simply to a way of pasting together its infaces — certain $(j-1)$-dimensional opetopes — to obtain its outface.

A weak n-category will be an 'opetopic set' with certain extra properties, similar to those defining a Kan complex, but a bit more complicated. The analog of an admissible horn is a 'niche', which is a configuration in which all the infaces of an opetope have been filled in with cells, but not the outface. We require that every niche can be extended to a universal cell, and regard the outface of such a universal cell as 'a composite' of its infaces. We also require composites of universal cells to be universal.

Here we must note the second major difference between our approach and Street's. It turns out that if one works with n-categories instead of ω-categories, one need not (and should not) arbitrarily designate certain cells as universal; instead, universality becomes a property. In our framework an n-category typically has j-cells for arbitrarily large j, but they act like 'equations' for $j > n$, so every j-cell is defined to be universal for $j > n$. Universality for j-cells of lower dimension is defined in a recursive manner. The basic idea is that a given cell occupying some niche is universal if any other occupant factors through that one, up to equivalence. Here the notion of 'equivalence' must also be recursively defined.

A brief outline of our paper is as follows. In Section 2 we introduce some necessary material on operads. In Section 3 we describe the slice operad construction, opetopes and opetopic sets. In Section 4 we define weak n-categories and begin to study them, along with the more general 'n-coherent O-algebras', which are the n-categorical analogs of operad algebras. In the Conclusions we compare other approaches to weak n-categories and discuss the all-important question of when two approaches can be considered equivalent.

Henceforth by 'n-category' we always mean 'weak n-category', as defined in this paper. For more background on n-category theory and why it should be interesting, see the previous papers in this series, which we refer to as HDA0 [4], HDA1 [6], and HDA2 [2]. As in those papers, we use the ordering in which the composite of morphisms $f: x \to y$ and $g: y \to z$ is written as fg, but when dealing with operads we write the composite of a k-ary operation f with the operations $g_1, \ldots, g_k$ as $f \cdot (g_1, \ldots, g_k)$.

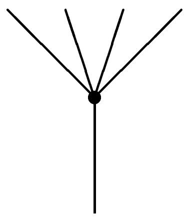
2 Operads

It turns out to be convenient to describe weak n-categories using the theory of operads. Operads are a formalism for dealing with algebraic structures having operations of arbitrary finite arity satisfying arbitrary 'reduction laws', that is, equational laws saying that some composite of operations equals some operation. For the benefit of the reader unfamiliar with operads, we begin in Section 2.1 by recalling the traditional sort of operad [24]. We call these 'untyped' operads because they are suited to the case when the inputs and output of every operation are of the same type.

Then, with the help of some generalities about monoid objects in Section 2.2, we introduce the more general 'typed' operads needed for this paper in Section 2.3. While all we really need are operads with an arbitrary *set* of types, we find it somewhat illuminating to define operads with an arbitrary small *category* of types. In Section 2.4 we show how a functor $F: C \to D$ gives a way to turn operads with type category D into operads with type category C, and in Section 2.5 we conclude with another basic operad construction, the 'slice operad of an operad algebra'. Some of the material in these sections makes for tough going, so it may be helpful at points to consult our introduction to n-categories [3].

2.1 Untyped operads

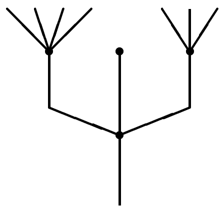
An *untyped operad* O has, for each $k \geq 0$, a set O_k of *k-ary operations.* We may visualize an element of O_k as a tree as in Figure 3. This tree has one black dot or *node* representing the operation itself, k lines or *edges* coming in from above representing the inputs of the operation, and one edge going out from below representing the output.



3. An element of O_k for $k = 4$

We may compose these trees by attaching the output edges of k of them to the input edges of a tree with k inputs, as shown in Figure 4. (In the resulting tree some of the edges may be drawn as broken lines for convenience.) More precisely, for any integers $i_1, \ldots, i_k \geq 0$ there is a function

$$\begin{array}{rcl} O_k \times O_{i_1} \times \cdots \times O_{i_k} & \to & O_{i_1+\cdots+i_k}. \\ (f, g_1, \ldots, g_k) & \mapsto & f \cdot (g_1, \ldots, g_k) \end{array}$$



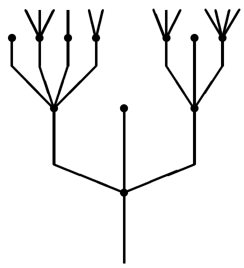
4. Composition in an operad

We require composition to be 'associative', in the sense that

$$f \cdot (g_1 \cdot (h_{11}, \ldots, h_{1i_1}), \ldots, g_k \cdot (h_{k1}, \ldots, h_{ki_k})) =$$

$$(f \cdot (g_1, \ldots g_k)) \ \cdot \ (h_{11}, \ldots, h_{1i_1}, \ldots \ldots, h_{k1}, \ldots, h_{ki_k})$$

whenever both sides are well-defined. This makes composites such as those shown in Figure 5 unambiguous.



5. Associativity for composition in an operad

We also require the existence of an 'unit' $1 \in O_1$ such that

$$1 \cdot (f) = f, \qquad f \cdot (1, \ldots, 1) = f$$

for all $f \in O_k$.

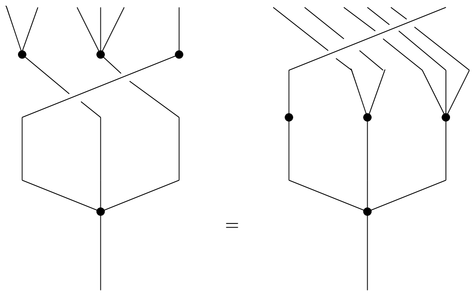
What we have so far is an *planar untyped operad*. For a full-fledged untyped operad, we also assume there is a right action of the symmetric group S_k on O_k for all k, for which the following compatibility conditions hold. First, for any $f \in O_k$, $\sigma \in S_k$, and $g_j \in O_{i_j}$ for $1 \leq j \leq k$, we have

$$(f\sigma) \cdot (g_{\sigma(1)}, \ldots, g_{\sigma(k)}) = (f \cdot (g_1, \ldots, g_k))\, \rho(\sigma),$$

where

$$\rho \colon S_k \to S_{i_1 + \cdots + i_k}$$

is the obvious homomorphism. We illustrate this condition in Figure 6.



6. Compatibility condition for symmetric group actions

Second, for any $f \in O_k$, and $g_j \in O_{i_j}$, $\sigma_j \in S_{i_j}$ for $1 \leq j \leq k$, have

$$f \cdot (g_1\sigma_1, \ldots, g_k\sigma_k) = (f \cdot (g_1, \ldots, g_k))\, \rho'(\sigma_1, \ldots, \sigma_k),$$

where

$$\rho' \colon S_{i_1} \times \cdots \times S_{i_k} \to S_{i_1 + \cdots + i_k}$$

is the obvious homomorphism.

Operads are mainly interesting for their algebras. Given an untyped operad O as above, one defines an *O-algebra* to be a set A on which the operations of O are rendered concrete. In other words, there are maps

$$\alpha \colon O_k \to \hom(A^k, A)$$

sending the identity operation $1 \in O_1$ to the identity function from A to itself, and sending composites to composites:

$$\alpha(f \cdot (g_1, \ldots, g_k)) = \alpha(f) \circ (\alpha(g_1) \times \cdots \times \alpha(g_k)).$$

We also require that the maps α satisfy

$$\alpha(f\sigma) = \alpha(f)\sigma,$$

where $f \in O_k$ and $\sigma \in S_k$ acts on $\hom(A^k, A)$ on the right by permuting the factors in A^k. We omit this requirement if O is merely planar.

2.2 Monoid objects and their actions

An untyped operad O for which only O_1 is nonempty is just a monoid, so we may think of a monoid as a kind of operad. Interestingly, however, there is a rather different way to think of any operad as a kind of monoid, or more precisely, a 'monoid object'. By the internalization principle discussed in HDA0, we can generalize the definition of 'monoid' from the category of sets to any sufficiently similar category — in fact, any monoidal category. If M is a strict monoidal category, a *monoid object* in M is an object $m \in M$ equipped with a product $\mu\colon m \otimes m \to m$ and unit $\iota\colon 1 \to m$, such that the following diagrams commute:

$$\begin{tikzcd}
m \otimes m \otimes m \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"'] & m \otimes m \arrow[d, "\mu"] \\
m \otimes m \arrow[r, "\mu"] & m
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes m \arrow[rr, "\iota\otimes 1"] \arrow[dr, "1"'] & & m \otimes m \arrow[dl, "\mu"] \\
& m &
\end{tikzcd}$$

$$\begin{tikzcd}
m \otimes 1 \arrow[rr, "1\otimes\iota"] \arrow[dr, "1"'] & & m \otimes m \arrow[dl, "\mu"] \\
& m &
\end{tikzcd}$$

These represent associativity and the left and right unit laws, respectively. When the monoidal category M is not strict, one simply inserts the natural isomorphisms $(m \otimes m) \otimes m \cong m \otimes (m \otimes m)$ and $1 \otimes m \cong m \cong m \otimes 1$ where needed.

One may then define an action of the monoid object m on any object in M. More generally, one may define an action of m on any object in a category on which M acts. Recall that an *action* of M on a category C is a monoidal functor $A\colon M \to \mathrm{end}(C)$, where the monoidal category $\mathrm{end}(C)$ has endofunctors on C as objects and natural transformations between these as morphisms. Equivalently, we may think of the action A as a functor $A\colon M \times C \to C$ satisfying certain conditions. Here it is convenient to write $A(m, c)$ simply as $m \otimes c$.

Suppose that $m \in M$ is an monoid object and $A\colon M \times C \to C$ is an action. If A is a strict monoidal functor, we define an *action* of m in C *riding* the action A to be a morphism

$$\alpha\colon m \otimes c \to c$$

in C making the following diagrams commute:

$$\begin{tikzcd}
m \otimes m \otimes c \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\alpha"'] & m \otimes c \arrow[d, "\alpha"] \\
m \otimes c \arrow[r, "\alpha"] & c
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes c \arrow[rr, "i\otimes 1"] \arrow[dr, "1"'] & & m \otimes c \arrow[dl, "\alpha"] \\
& c &
\end{tikzcd}$$

When A is not strict, one simply inserts the natural isomorphisms $(m \otimes m) \otimes c \cong m \otimes (m \otimes c)$ and $1 \otimes c \cong c$ where needed.

Given a monoid object $m \in M$ and an action A of M on C, we define the *category of actions* of m in C *riding* the action A as follows. The objects of this category are actions of m in C riding A, and given two such actions

$$\alpha\colon m \otimes c \to c, \qquad \alpha'\colon m \otimes c' \to c',$$

we define a morphism from α to α' to be a morphism $f\colon c \to c'$ such that the following diagram commutes:

$$\begin{tikzcd}
m \otimes c \arrow[r, "\alpha"] \arrow[d, "1\otimes f"'] & c \arrow[d, "f"] \\
m \otimes c' \arrow[r, "\alpha'"] & c'
\end{tikzcd}$$

It is worth noting an interesting pattern. We may generalize the notion of 'monoid' to that of a monoid object m in any monoidal category M, but the notion of 'monoidal category' is itself a categorification of the notion of 'monoid'. Similarly, we may define an action of the monoid object $m \in M$ on $c \in C$ whenever the monoidal category M acts on C.

We see here two instances of the following principle: *certain algebraic structures can be defined in any category equipped with a categorified version of the same structure.* Another instance was mentioned in HDA0: we may define a commutative monoid object in any symmetric monoidal category. We name this principle the *microcosm principle,* after the theory, common in pre-modern correlative cosmologies, that every feature of the microcosm (e.g. the human soul) corresponds to some feature of the macrocosm. Of course, the above formulation of the microcosm principle is rather vague; we give a precise version in Section 4.3.

Even without a precise formulation, the microcosm principle can serve as a useful guide when seeking the most general way to internalize certain algebraic structures. For example, we may apply the microcosm principle to morphisms between monoid objects. Suppose we are given a monoidal functor $F\colon M \to M'$ between monoidal categories. Then given monoid objects $m \in M$ and $m' \in M'$, we define a morphism $f\colon m \to m'$ *riding* F to be a morphism $f\colon F(m) \to m'$ in M' making the following diagrams commute:

$$\begin{tikzcd}
F(m \otimes m) \arrow[d, "F(\mu)"'] & F(m) \otimes F(m) \arrow[l] \arrow[r, "f \otimes f"] & m' \otimes m' \arrow[d, "\mu'"] \\
F(m) \arrow[rr, "f"] & & m'
\end{tikzcd}$$

$$\begin{tikzcd}
F(1_M) \arrow[d, "F(\iota)"'] & 1_{M'} \arrow[l] \arrow[d, "\iota'"] \\
F(m) \arrow[r, "f"] & m'
\end{tikzcd}$$

Here μ, ι are the product and unit for m, while μ', ι' are the product and unit for m'. If F is a strict monoidal functor, the unlabelled arrows $F(m) \otimes F(m) \to F(m \otimes m)$ and $1_{M'} \to F(1_M)$ are identity morphisms. If F is a weak monoidal functor, these arrows are isomorphisms supplied by the definition of weak monoidal functor. However, in Sections 2.4 and 2.5 we will need the case where F is merely a lax monoidal functor (which Eilenberg and Kelly [14] call simply a monoidal functor). Then these arrows are morphisms, not necessarily isomorphisms, supplied by the definition of a lax monoidal functor.

We call a morphism of monoid objects riding an identity functor a 'homomorphism'. In other words, given monoid objects m, m' in a monoidal category M, we define a *homomorphism* $f\colon m \to m'$ to be a morphism in M for which the following diagrams commute:

$$\begin{tikzcd}
m \otimes m \arrow[r, "f \otimes f"] \arrow[d, "\mu"'] & m' \otimes m' \arrow[d, "\mu'"] \\
m \arrow[r, "f"] & m'
\end{tikzcd}$$

$$\begin{tikzcd}
& 1 \arrow[dl, "\iota"'] \arrow[d, "\iota'"] \\
m \arrow[r, "f"] & m'
\end{tikzcd}$$

Whenever $F\colon M \to M'$ is a lax monoidal functor and m is a monoid object in M, $F(m)$ becomes a monoid object in M' in a natural way, and a morphism of monoid objects $f\colon m \to m'$ riding F can also be thought of as a homomorphism from $F(m)$ to m'.

In the next section, we define an operad to be a monoid object in a certain monoidal category of 'signatures'. An algebra of the operad will then be an action of this monoid object, riding a certain action of the category of signatures. We will use the concepts of morphisms and homomorphisms between monoid objects to define morphisms and homomorphisms between operads.

2.3 Typed operads

To define weak n-categories, we need operads for which the inputs and output of each operation are 'many-sorted', or 'typed'. In what follows, we first define these operads and their algebras, and then give a rather lengthy explanation of our definitions.

Definition 1. *For a category C, let the category* $\mathrm{fam}(C)$ *of C-*families *be the category where an object is a finite list of objects of C, and where a morphism from $(x_1, \ldots, x_j)$ to $(y_1, \ldots, y_k)$ is a bijection $b\colon \{1, \ldots, j\} \to \{1, \ldots, k\}$ together with, for each i, a morphism from x_i to $y_{b(i)}$, with composition of morphisms given by the obvious rule.*

Definition 2. *For a category C, let* $\mathrm{svf}(C)$ *be the category of set-valued functors on C, that is, the category whose objects are functors from C to* Set *and whose morphisms are natural transformations between these.*

Notice that $\mathrm{fam}(C)$ is a symmetric monoidal category, where the tensor product of the families $(x_1, \ldots, x_j)$ and $(y_1, \ldots, y_k)$ is $(x_1, \ldots, x_j, y_1, \ldots, y_k)$. In fact, $\mathrm{fam}(C)$ is the free symmetric monoidal category on C in an appropriate sense. In a similar sense, the category $\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}})$ of set-valued contravariant functors on $\mathrm{fam}(C)$ is the 'free symmetric 2-rig on C', where a *2-rig* is a symmetric monoidal cocomplete category for which the monoidal structure preserves small colimits in each argument. By this universal property, the monoidal category $\mathrm{end}(\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}}))$ of endomorphisms preserving small colimits and the symmetric monoidal structure is equivalent to $\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}} \times C)$.

Definition 3. *Given a category C, we define the category* $\mathrm{prof}(C)$ *of C-*profiles *to be* $\mathrm{fam}(C)^{\mathrm{op}} \times C$*.*

Definition 4. *Given a category C, we define the category* $\mathrm{sig}(C)$ *of C-*signatures *to be* $\mathrm{svf}(\mathrm{prof}(C))$*.*

By the above remarks $\mathrm{sig}(C)$ is a monoidal category. Note that $\mathrm{sig}(C)$ has an action on $\mathrm{svf}(C)$, which we call the *tautologous action*. Thus we may make the following definitions:

Definition 5. *Given a small category C, we define a C-*operad *be a monoid object in* $\mathrm{sig}(C)$*, and define* $\mathrm{op}(C)$*, the category of C-operads, to be the category of monoid objects in* $\mathrm{sig}(C)$*.*

Definition 6. *Given a C-operad O, we say C is the category of* types *of O, and write* $C = \mathrm{type}(O)$*.*

Definition 7. *For a C-operad O, we define the category O-*alg *of O-*algebras *to be the category of actions of O in* $\mathrm{svf}(C)$ *riding the tautologous action of* $\mathrm{sig}(C)$ *on* $\mathrm{svf}(C)$*.*

To get a feel for these definitions, let us see how in a special case they reduce to the definitions of untyped operads and their algebras.

Example 8. *Untyped operads as C-operads with $C = 1$.* Here we take C to be the terminal category 1, the category with one object x and one morphism 1_x. We denote the objects of $\mathrm{fam}(C)$ as $1, x, x^2, \ldots$. Note that $\hom(x^j, x^k)$ is the empty set unless $j = k$, in which case it is the symmetric group S_k. An object A of $\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}})$ assigns to each object x^k of $\mathrm{fam}(C)$ a set A_k equipped with an S_k-action. It is illuminating to write A as a formal power series:

$$A = A_0 + A_1 x + A_2 x^2 + \cdots.$$

Then the coproduct in $\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}})$ corresponds to addition of formal power series, where we add coefficients by taking their disjoint union. Similarly, the monoidal structure corresponds to multiplication of formal power series, but where we multiply the coefficients as follows: to multiply a set with S_j-action and a set with S_k-action, we take the Cartesian product with its natural $S_j \times S_k$-action and then induce an action of S_{j+k} along the obvious inclusion $S_j \times S_k \hookrightarrow S_{j+k}$. (Here by 'inducing an action' we mean the left adjoint of restricting an action to a subgroup.)

An endomorphism P of $\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}})$ preserving small colimits and the symmetric monoidal structure will be determined by its action on the generating object x (by which we mean the formal power series A with A_1 being a one-element set and A_i empty for $i \neq 1$). We have

$$P(x) = P_0 + P_1 x + P_2 x^2 + \cdots,$$

so P is determined by the sets P_k. Note that each set P_k is equipped with an action of S_k, by the functoriality of P. Conversely, any collection of sets P_k with S_k-actions determines such an endomorphism P. We may also think of P as the object of $\mathrm{sig}(C)$ assigning to each C-profile (x^k, x) a set P_k. We call the elements of P_k the 'k-ary operations' of P.

A C-operad is a monoid object in $\mathrm{sig}(C)$. To understand what this amounts to, we must understand the monoidal structure in $\mathrm{sig}(C)$. This corresponds to the composition of endomorphisms of $\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}})$, or in other words, composition of formal power series. Given C-signatures P and Q, their composite is given by

$$(P \circ Q)(x) = P(Q(x)).$$

Thus we have

$$\begin{array}{rcl}
(P \circ Q)_0 & = & P_0 + P_1 Q_0 + P_2 Q_0^2 + \cdots \\
(P \circ Q)_1 & = & P_1 Q_1 + P_2(Q_0 Q_1 + Q_1 Q_0) + \cdots \\
(P \circ Q)_2 & = & P_1 Q_2 + P_2(Q_0 Q_2 + Q_1^2 + Q_2 Q_0) + \cdots
\end{array}$$

and so on, where we add and multiply the coefficients as before. Note that an element of $(P \circ Q)_j$ consists of an element of P_k, for arbitrary $k \geq 0$, together with a choice of elements of the sets $Q_{i_1}, \ldots, Q_{i_k}$, where $i_1 + \cdots + i_k = j$.

Given a C-operad O, the product $\mu\colon O \circ O \to O$ gives a collection of functions from $(O \circ O)_j$ to O_j. This amounts to a collection of functions

$$O_k \times O_{i_1} \times \cdots \times O_{i_k} \to O_{i_1 + \cdots + i_k}.$$

We leave it to the reader to check that in this special case $C = 1$, O being a monoid object in the category of C-signatures is precisely equivalent to the conditions in the definition of an operad given at the beginning of Section 2. In particular, the associativity and unit laws there correspond to the associativity and unit laws required of a monoid object, while the conditions involving the symmetric groups correspond to the fact that the product $\mu: O \circ O \to O$ is a symmetric monoidal natural transformation between symmetric monoidal functors.

In this case, the tautologous action of $\mathrm{sig}(C)$ on $\mathrm{svf}(C)$ works as follows. The category $\mathrm{svf}(C)$ is just Set, so suppose we are given a C-signature P and a set A. Then P acts on A to give the set

$$P(A) = P_0 + P_1 A + P_2 A^2 + \cdots$$

If O is a C-operad, an O-algebra is a set A together with an action of O on A, that is, a function $\alpha: O(A) \to A$ satisfying certain conditions. Alteratively, as in the definition of the algebra of an untyped operad, we can think of this action as a collection of functions $\alpha: O_k \to \hom(A^k, A)$. We leave it for the reader to check that the conditions α must satisfy to be an action are just the conditions given in Section 2.

Our use of formal power series above appears already in the generating function approach to combinatorics [13] and its categorical interpretation in terms of 'species' by Joyal [21]. As shown in Figure 7, what is at work here is the analogy between ordinary set-theoretic linear algebra and categorified linear algebra.

commutative rig k	symmetric 2-rig Set
set S	category C
$k\langle S\rangle$ or $\hom(S,k)$	$\mathrm{svf}(C^{\mathrm{op}})$
FS	$\mathrm{fam}(C)$
$k[S]$ or $k[[S]]$	$\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}})$
$\mathrm{end}(k[S])$ or $\mathrm{end}(k[[S]])$	$\mathrm{end}(\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}})) = \mathrm{sig}(C)$

7. Set-theoretic linear algebra versus categorified linear algebra

Recall that a *rig* is a set with two monoid structures $+$ and $\cdot$, where $+$ is commutative and $\cdot$ distributes over $+$. A 2-rig, as defined earlier, is a categorified analog of a rig. In set-theoretic linear algebra we may work over any commutative rig k, while in categorified linear algebra we may work over any symmetric 2-rig. The free commutative rig on one element is $\mathbb{N}$, while the free symmetric 2-rig on one object is Set. For simplicity, in Figure 7 we only consider categorified linear algebra over Set, although other symmetric 2-rigs are also interesting. It is most common in set-theoretic linear algebra to work over a field or commutative ring, but working over $\mathbb{N}$ is important in combinatorics, and heightens the analogy to categorified linear algebra over Set.

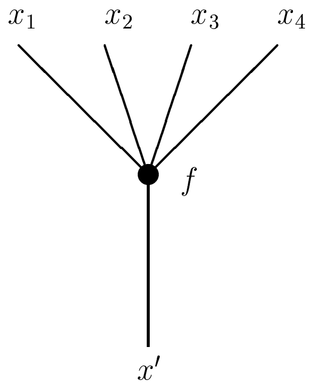
Given a set S one may form the free k-module $k\langle S\rangle$ on S. Similarly, given a category C one may form the free cocomplete category $\mathrm{svf}(C^{\mathrm{op}})$ on C; note that a cocomplete category is automatically a Set-module in the sense of Kapranov and Voevodsky [22]. One may also form the free commutative monoid FS on the set S. The free commutative k-algebra on S is then $k\langle FS\rangle$, usually denoted by $k[S]$. Similarly, one may form the free symmetric monoidal category $\mathrm{fam}(C)$ on the category

C. The free symmetric 2-rig on C is then $\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}})$. The monoidal category $\mathrm{end}(\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}})) = \mathrm{sig}(C)$ is thus a categorified version of the monoid $\mathrm{end}(k[S])$.

There are some rough spots in this analogy. In particular, while we can pull back k-valued functions along any function $f\colon S \to T$, obtaining a k-linear map $f^*\colon k\langle T\rangle \to k\langle S\rangle$, we cannot in general push them forwards. In contrast to this, not only can we pull back set-valued functors along any functor $F\colon C \to D$, obtaining a functor $F^*\colon \mathrm{svf}(D) \to \mathrm{svf}(C)$, we can also push them forward using the left adjoint $F_*\colon \mathrm{svf}(C) \to \mathrm{svf}(D)$. Both F^* and F_* preserve small colimits. In short, while the free k-module on a set transforms only contravariantly under functions, the free cocomplete category on a category transforms both covariantly and contravariantly under functors. This plays an important role in Section 2.4.

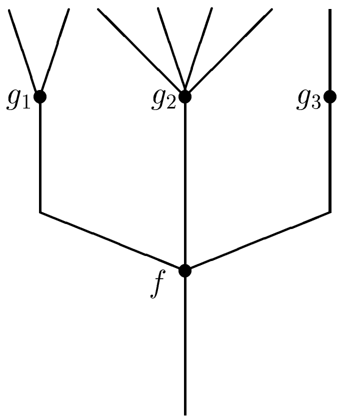
There is a kind of substitute for the free k-module on a set that transforms covariantly: the k-module $\hom(S, k)$ of functions from S to k. In some ways $\mathrm{svf}(\mathrm{fam}(C^{\mathrm{op}}))$ resembles $\hom(FS, k) = k[[S]]$ more than $k[S]$, which explains the importance of formal power series in the generating function approach to combinatorics. Of course, $k\langle S\rangle$ and $\hom(S, k)$ are isomorphic when S is finite; the categorified situation works more smoothly because cocomplete categories are closed under arbitrary colimits, while k-modules are only closed under finite linear combinations.

To conclude this section, let us unpack our abstract definitions of general C-operads and their algebras to obtain equivalent 'nuts-and-bolts' descriptions along more traditional lines. First we introduce some handy notation. Given an object $(x_1, \ldots, x_k) \in \mathrm{fam}(C)$ and an object $x \in C$, we write the corresponding C-profile as $(x_1, \ldots, x_k, x')$. A C-signature P assigns to this C-profile a set $P(x_1, \ldots, x_k, x')$ which we call the set of *operations* of P with profile $(x_1, \ldots, x_k, x')$. As in Figure 8, we may visualize such an operation as a gadget with k inputs of types $x_1, \ldots, x_k$ and one output of type x'. Given an operation with this profile, we call $x_1, \ldots, x_k$ its *input types* and x' its *output type*, and the tuple $(x_1, \ldots, x_k)$ its *arity*. (In the untyped case we sometimes call the integer k the arity.)



8. An operation f with profile (x_1, x_2, x_3, x_4, x')

Since the tensor product of objects in $\mathrm{sig}(C)$ is given by composing endomorphisms of $\mathrm{svf}(\mathrm{fam}(C)^{\mathrm{op}})$, we may write the monoidal structure in $\mathrm{sig}(C)$ as $\circ$. One may check that given C-signatures P and Q, and an operation f of P and operations $g_1, \ldots, g_k$ of Q for which the arity of f is the product of the arities of the g_i, we obtain an operation of $P \circ Q$. We denote this operation by $f \circ (g_1, \ldots, g_k)$. The output type of $f \circ (g_1, \ldots, g_k)$ is the output type of f, while its arity type is the product of the arities of $g_1, \ldots, g_k$. We may visualize $f \circ (g_1, \ldots, g_k)$ as in Figure 9.



9. An operation $f \circ (g_1, \ldots, g_k)$ of $P \circ Q$, where $k = 3$

Now suppose that O is a C-operad. Then it is a monoid object in $\mathrm{sig}(C)$, and the product $\mu\colon O \circ O \to O$ sends each operation $f \circ (g_1, \ldots, g_k)$ in $O \circ O$ to an operation in O which we denote by $f \cdot (g_1, \ldots, g_k)$. One may check that the associativity of the product μ implies an associativity law like that for untyped operads. Also, the unit $\iota\colon 1 \to O$ gives O an operation ι_f of profile (x, x') for every morphism $f\colon x \to x'$ in C. One may also check that the unit law and compatibility with symmetric group actions hold as in an untyped operad. With a little more work, one can verify:

Proposition 9. *For any small category C, a C-operad O gives:*

1. *for any C-profile $(x_1, \ldots, x_k, x')$, a set $O(x_1, \ldots, x_k, x')$*

2. *for any $f \in O(x_1, \ldots, x_k, x')$ and any $g_1 \in O(x_{11}, \ldots, x_{1i_1}, x_1), \ldots,$ $g_k \in O(x_{k1}, \ldots, x_{ki_k}, x_k)$, an element*

$$f \cdot (g_1, \ldots, g_k) \in O(x_{11}, \ldots, x_{1i_1}, \ldots\ldots, x_{k1}, \ldots, x_{ki_k}, x')$$

3. *for each morphism $f\colon x \to x'$ in C, an element $\iota(f) \in O(x, x')$*

4. *for any permutation $\sigma \in S_k$, a map*

$$\begin{array}{rcl} \sigma\colon O(x_1, \ldots, x_k, x') & \to & O(x_{\sigma(1)}, \ldots, x_{\sigma(k)}, x') \\ f & \mapsto & f\sigma \end{array}$$

such that:

(a) *whenever both sides make sense,*

$$f \cdot (g_1 \cdot (h_{11}, \ldots, h_{1i_1}), \ldots, g_k \cdot (h_{k1}, \ldots, h_{ki_k})) =$$

$$(f \cdot (g_1, \ldots g_k)) \;\cdot\; (h_{11}, \ldots, h_{1i_1}, \ldots\ldots, h_{k1}, \ldots, h_{ki_k})$$

(b) *for any* $f \in O(x_1, \ldots, x_k, x')$*,*

$$f = \iota(1_{x'}) \cdot f = f \cdot (1_{x_1}, \ldots, 1_{x_k})$$

(c) *for any* $f \in O(x_1, \ldots, x_k, x')$ *and* $\sigma, \sigma' \in S_k$*,*

$$f(\sigma\sigma') = (f\sigma)\sigma'$$

(d) *for any* $f \in O(x_1, \ldots, x_k, x')$*,* $\sigma \in S_k$*, and* $g_1 \in O(x_{11}, \ldots, x_{1i_1}, x_1), \ldots,$ $g_k \in O(x_{k1}, \ldots, x_{ki_k}, x_k)$*,*

$$(f\sigma) \cdot (g_{\sigma(1)}, \ldots, g_{\sigma(k)}) = (f \cdot (g_1, \ldots, g_k))\, \rho(\sigma),$$

where $\rho\colon S_k \to S_{i_1+\cdots+i_k}$ *is the obvious homomorphism.*

(e) *for any* $f \in O(x_1, \ldots, x_k, x')$*,* $g_1 \in O(x_{11}, \ldots, x_{1i_1}, x_1), \ldots,$ $g_k \in O(x_{k1}, \ldots, x_{ki_k}, x_k)$*, and* $\sigma_1 \in S_{i_1}, \ldots, \sigma_k \in S_{i_k}$*,*

$$(f \cdot (g_1\sigma_1, \ldots, g_k\sigma_k)) = (f \cdot (g_1, \ldots, g_k))\, \rho'(\sigma_1, \ldots, \sigma_k),$$

where $\rho'\colon S_{i_1} \times \cdots \times S_{i_k} \to S_{i_1+\cdots+i_k}$ *is the obvious homomorphism.*

Conversely, such data determine a unique C*-operad.*

We can give a similar description of the algebras of a C-operad O. An O-algebra is an action $\alpha\colon O(A) \to A$, but we usually denote it simply as A. Given an O-algebra A and an object $x \in C$, we call $A(x)$ the set of *elements of type* x of A. For any C-profile $(x_1, \ldots, x_k, x')$, the action α gives a function

$$O(x_1, \ldots, x_k, x') \times A(x_1) \times \cdots \times A(x_k) \to A(x')$$

which we write as

$$(f, a_1, \ldots, a_k) \mapsto f(a_1, \ldots, a_k).$$

Alternatively, we sometimes write this as a function

$$O(x_1, \ldots, x_k, x') \to \hom(A(x_1) \times \cdots \times A(x_k), A(x'))$$

which by abuse of language we also call α. One may then verify the following:

Proposition 10. *For any C-operad O, an O-algebra A gives:*

1. *for any object $x \in C$, a set $A(x)$.*
2. *for any C-profile $(x_1, \ldots, x_k, x')$, a function*

$$\alpha\colon O(x_1, \ldots, x_k, x') \to \hom(A(x_1) \times \cdots \times A(x_k), A(x'))$$

such that:

(a) *whenever both sides make sense,*

$$\alpha(f \cdot (g_1, \ldots, g_k)) = \alpha(f) \circ (\alpha(g_1) \times \cdots \times \alpha(g_k))$$

(b) *for any $x \in C$, $\alpha(\iota(1_x))$ acts as the identity on $A(x)$*

(c) *for any $f \in O(x_1, \ldots, x_k, x')$ and $\sigma \in S_k$,*

$$\alpha(f\sigma) = \alpha(f)\sigma,$$

where $\sigma \in S_k$ acts on $\hom(A(x_1) \times \cdots \times A(x_k), A)$ on the right by permuting the factors.

Conversely, such data determine a unique O-algebra.

Starting in Section 3 we will restrict attention to operads whose type category has only identity morphisms. Such a category is said to be discrete. Since the category Set is isomorphic to the category having small discrete categories as objects and functors as morphisms, we need not worry much about the difference between small discrete categories and sets. Thus we may easily extend the terminology above to define S-profiles, S-signatures, S-operads, and so on when S is a set. For example, we define an *S-operad* to be an operad whose type category is the discrete category with S as its set of objects.

2.4 Pullback operads

Given a functor $F\colon C \to D$ and a D-operad O, we now construct a certain C-operad, the 'pullback' F^*O. First recall that D-signatures can be regarded as set-valued functors on $\mathrm{prof}(D) = \mathrm{fam}(D)^{\mathrm{op}} \times D$, and likewise for C-signatures. Thus we may pull back D-signatures to C-signatures along F, giving a functor

$$F^*\colon \mathrm{sig}(D) \to \mathrm{sig}(C).$$

The proposition below makes F^* into a lax monoidal functor. As in Section 2.2, for any D-operad O, the pullback F^*O then becomes a C-operad.

Proposition 11. *For any functor $F: C \to D$, $F^*: \mathrm{sig}(D) \to \mathrm{sig}(C)$ can be given the structure of a lax monoidal functor.*

Proof - Note that $F: C \to D$ induces a pullback functor

$$F^\sharp: \mathrm{svf}(D^{\mathrm{op}}) \to \mathrm{svf}(C^{\mathrm{op}}),$$

preserving small colimits, and also, because $\mathrm{svf}(C^{\mathrm{op}})$ is the free cocomplete category on C, a functor

$$F_\sharp: \mathrm{svf}(C^{\mathrm{op}}) \to \mathrm{svf}(D^{\mathrm{op}})$$

preserving small colimits. In fact, $F^\sharp$ is right adjoint to $F_\sharp$. By the universal property of $\mathrm{svf}(\mathrm{fam}(C^{\mathrm{op}}))$ and $\mathrm{svf}(\mathrm{fam}(D^{\mathrm{op}}))$, the functors $F^\sharp$ and $F_\sharp$ induce morphisms of symmetric monoidal cocomplete categories:

$$R: \mathrm{svf}(\mathrm{fam}(D^{\mathrm{op}})) \to \mathrm{svf}(\mathrm{fam}(C^{\mathrm{op}}))$$

and

$$L: \mathrm{svf}(\mathrm{fam}(C^{\mathrm{op}})) \to \mathrm{svf}(\mathrm{fam}(D^{\mathrm{op}}))$$

with the former being right adjoint to the latter.

Now recall that the category $\mathrm{sig}(D)$ is equivalent, as a monoidal category, to the category $\mathrm{end}(\mathrm{svf}(\mathrm{fam}(D)^{\mathrm{op}}))$. Thus we may identify $\mathrm{sig}(D)$ with this latter category, which is strictly monoidal. A D-signature S is then an endomorphism

$$S: \mathrm{svf}(\mathrm{fam}(D^{\mathrm{op}})) \to \mathrm{svf}(\mathrm{fam}(D^{\mathrm{op}})),$$

and the composite

$$R \circ S \circ L: \mathrm{svf}(\mathrm{fam}(C^{\mathrm{op}})) \to \mathrm{svf}(\mathrm{fam}(C^{\mathrm{op}}))$$

is a C-signature. This composition process extends to a functor from $\mathrm{sig}(D)$ to $\mathrm{sig}(C)$, which one may check is equivalent to F^*.

To make F^* into a lax monoidal functor it thus suffices to find a natural transformation $\Phi_{S,T}: F^*(S) \circ F^*(T) \to F^*(S \circ T)$ making the following diagram commute for any D-signatures S, T, U:

$$\begin{tikzcd}
F^*(S) \circ F^*(T) \circ F^*(U) \arrow[r, "\Phi_{S,T} \circ 1"] \arrow[d, "1 \circ \Phi_{T,U}"'] & (S \circ T) \circ F^*(U) \arrow[d, "F^*_{S \circ T, U}"] \\
F^*(S) \circ F^*(T \circ U) \arrow[r, "\Phi_{S, T \circ U}"] & F^*(S \circ T \circ U)
\end{tikzcd} \tag{1}$$

together with a morphism $\phi\colon 1_{\mathrm{sig}(D)} \to F^*(1_{\mathrm{sig}(C)})$ making the following diagrams commute for any D-signature S:

$$\begin{tikzcd}
1 \circ F^*(S) \arrow[r, "1"] \arrow[d, "\phi\circ 1"'] & F^*(S) \arrow[d, "1"] \\
F^*(1) \circ F^*(S) \arrow[r, "\Phi_{1,S}"] & F^*(1 \circ S)
\end{tikzcd} \tag{2}$$

$$\begin{tikzcd}
F^*(S) \circ 1 \arrow[r, "1"] \arrow[d, "1\circ\phi"'] & F^*(S) \arrow[d, "1"] \\
F^*(S) \circ F^*(1) \arrow[r, "\Phi_{S,1}"] & F^*(S \circ 1)
\end{tikzcd} \tag{3}$$

Since R is the right adjoint of L, there is a natural transformation $\epsilon\colon L \circ R \Rightarrow 1$, the counit of the adjunction. Since

$$F^*(S) \circ F^*(T) = R \circ S \circ L \circ R \circ T \circ L$$

while

$$F^*(S \circ T) = R \circ S \circ T \circ L,$$

we may use ϵ to define

$$\Phi_{S,T} = 1_{R\circ S} \circ \epsilon \circ 1_{T\circ L}\colon R \circ S \circ L \circ R \circ T \circ L \Rightarrow R \circ S \circ T \circ L.$$

The commutativity of (1) is then easy to check. Similarly, the unit $\iota\colon 1 \Rightarrow R \circ L$ of the adjunction gives a morphism $\phi\colon 1 \to f^*(1) = R \circ L$. The commutativity of (2) and (3) then follows from the triangle identities for an adjunction, which say that

$$R \xrightarrow{\iota\circ 1} R \circ L \circ R \xrightarrow{1\circ\epsilon} R$$

and

$$L \xrightarrow{1\circ\iota} L \circ R \circ L \xrightarrow{\epsilon\circ 1} L$$

are identity morphisms. $\square$

Here we note another interesting wrinkle in the analogy between set-theoretic linear algebra and categorified linear algebra. A function $f: S \to T$ from the finite set S to the finite set T induces a function $f^*: \mathrm{end}(k[T]) \to \mathrm{end}(k[S])$, using the isomorphism $\mathrm{end}(k[S]) \cong k\langle FS \times S\rangle$. However, in contrast to Proposition 11, this is not a monoid homomorphism.

The same thing happens in the simpler context of matrix algebras. For any finite set S, the set $k\langle S \times S\rangle$ becomes a monoid under matrix multiplication. Similarly, for any category C, $\mathrm{svf}(C^{\mathrm{op}} \times C)$ becomes a monoidal category, called the category of *distributors* from C to C. Given a function $f: S \to T$ between finite sets, the pullback $f^*: k\langle T \times T\rangle \to k\langle S \times S\rangle$ is only a monoid homomorphism when f is one-to-one. However, for any functor $F: C \to D$, the pullback $F^*: \mathrm{svf}(D^{\mathrm{op}} \times D) \to \mathrm{svf}(C^{\mathrm{op}} \times C)$ is a lax monoidal functor. In fact, this follows from Proposition 11, using the fact that a distributor may be regarded as a signature with only unary operations.

2.5 The slice operad of an algebra

Given an O-algebra A, the slice operad A^+ is an operad whose algebras are O-algebras over A, that is, equipped with a homomorphism to A. We give an explicit construction of the slice operad and then prove it has this property.

Recall that given a category C and an object $A \in \mathrm{svf}(C)$, the category $\mathrm{elt}(A)$ of elements of A has pairs (x, y) with $x \in C$ and $y \in A(x)$ as objects, and morphisms $f: x \to x'$ with $A(f)(y) = y'$ as morphisms from (x, y) to (x', y'). Composition of morphisms is defined in the obvious manner. In this situation there is a functor $p: \mathrm{elt}(A) \to C$ with $p(x, y) = x$ and $p(f) = f$.

Now suppose that O is a C-operad and A is an O-algebra. Then A is an object of $\mathrm{svf}(C)$, so as in the previous section we may form the pullback p^*O, which is an $\mathrm{elt}(A)$-operad. Thus the following makes sense:

Definition 12. *For a C-operad O and an O-algebra A, the* slice operad of A, *written A^+, is the sub-operad of p^*O for which an operation g of p^*O of profile $(a_1, \ldots, a_k, a')$ is included if and only if it satisfies $g(a_1, \ldots, a_k) = a'$.*

Proposition 13. *Suppose O is a C-operad and A is an O-algebra. Then A^+-*alg *is equivalent to the category of O-algebras over A. That is, an A^+-algebra is an O-algebra B equipped with an O-algebra homomorphism $f_B: B \to A$, and a morphism between A^+-algebras is an O-algebra morphism $g: B \to B'$ for which the following diagram commutes:*

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "f_B"'] & & B' \arrow[dl, "f_{B'}"] \\ & A & \end{tikzcd}$$

Proof - One may check this explicitly. Alternatively, since the operations of A^+ are certain operations of O, we obtain a forgetful functor from O-alg to A^+-alg. This has a left adjoint $L\colon A^+\text{-alg} \to O\text{-alg}$ sending the terminal object of A^+-alg to $A \in O$-alg. This gives a functor from A^+-alg to the category of O-algebras equipped with a homomorphism to A, which one may check is an equivalence. $\square$

3 Opetopes and Opetopic Sets

We now begin to address the crucial issue of categorification: the process whereby, in passing from an n-categorical context to an $(n+1)$-categorical context, laws are promoted to operations and these new operations satisfy new laws of their own. Our approach to this issue relies heavily on operads.

In all that follows, we restrict attention to operads having a set of types, in the manner explained at the end of Section 2.3. Note that any such operad is determined by:

1. its types
2. its operations
3. its reduction laws

where by 'reduction laws' we mean all equations stating that a given composite of operations, possibly with their arguments permuted, equals a given operation. (Here we include unary and nullary composites.) Our approach to categorification relies on a construction that yields for any operad O a new operad O^+ having operations corresponding to the reduction laws of O. This construction works roughly as follows. In Section 3.1, we show that S-operads are themselves the algebras of a certain operad. This allows us to apply the slice operad construction to S-operads, obtaining for each S-operad O a new operad O^+ whose algebras are S-operads over O. It turns out that:

1. The types of O^+ are the operations of O.
2. The operations of O^+ are the reduction laws of O.
3. The reduction laws of O^+ are the ways of combining reduction laws of O to give other reduction laws of O.

We give numerous examples of this construction in Section 3.2. In Section 3.3 we introduce the n-dimensional O-opetopes, which are the operations in the nth iterated slice operad O^{n+}, and we describe a notation for them involving lists of labelled trees which we call 'metatrees'. We pay special attention to the I-opetopes, or simply 'opetopes', because they serve as the basic shapes for cells in our approach to n-category theory. In Section 3.4 we give a description of the algebras of O^+ for any S-operad O. Finally, in Section 3.5, we describe 'opetopic sets'; n-categories are opetopic sets with certain properties.

3.1 The operad for operads

Given a small category C, we denote by $|C|$ the set of objects of C. We now show that for any set S, S-operads are the algebras of a certain $|\mathrm{prof}(S)|$-operad. More precisely, recall from Section 2.3 that the category of S-operads, $\mathrm{op}(S)$, is the category of monoid objects in $\mathrm{sig}(S)$. Then we have:

Theorem 14. *For any set S, there is a $|\mathrm{prof}(S)|$-operad whose category of algebras is equivalent to* $\mathrm{op}(S)$.

Proof - We construct a $|\mathrm{prof}(S)|$-operad X whose category of algebras is equivalent to $\mathrm{op}(S)$. The basic idea is that the operations of X are the ways of composing operations in S-operads, while possibly permuting their arguments.

Note that any S-operad has an underlying S-signature, giving us a functor

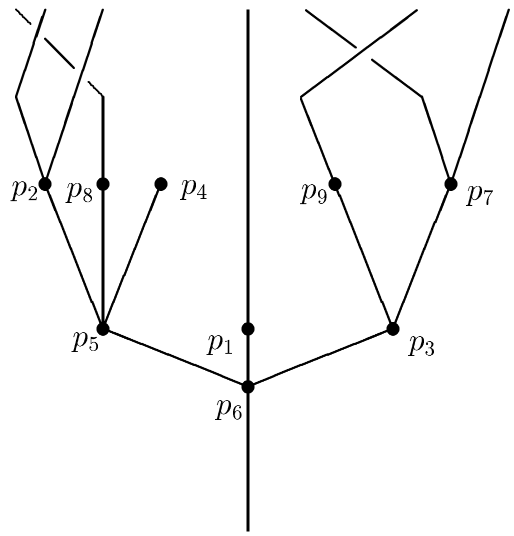
$$R\colon \mathrm{op}(S) \to \mathrm{sig}(S).$$

This functor has a left adjoint

$$L\colon \mathrm{sig}(S) \to \mathrm{op}(S)$$

assigning to each S-signature the free operad on that S-signature. Let T_S denote the terminal S-operad, and let F equal $L(R(T_S))$, the free S-operad on the underlying S-signature of T_S. Note that the terminal S-operad has one operation for each S-profile, so we may identify S-profiles with operations of T_S. We may think of F as the S-operad freely generated by all these operations.

The operations of F are in one-to-one correspondence with certain labelled trees called *T_S-trees*. A typical T_S-tree is shown in Figure 10. An T_S-tree is, first of all, a *combed tree*; it is planar except at the very top, where we allow an arbitrary permutation of the edges. Second, each node is labelled with an operation of T_S, or in other words, an S-profile. A node labelled by the S-profile $(x_1, \ldots, x_k, x')$ must have k edges coming into it from above. Moreover, we require that it be possible to label every edge with an element of S in such a way that for any node labelled by the S-profile $(x_1, \ldots, x_k, x')$, the edges coming into that node from above are labelled by the elements $x_1, \ldots, x_k$ in that order from left right, while the edge coming out of it from below is labelled by the element x'.



10. A T_S-tree

In this graphical notation, we compose operations in F by combining trees essentially as in Section 2.1, and then 'combing' the resulting tree so that all the permutations of edges occur at the very top.

Now let us turn to the $|\mathrm{prof}(S)|$-operad X. The operations of X are given as follows. Suppose that $p_1, \ldots, p_k$ are S-profiles. Then X has one operation f of arity $(p_1, \ldots, p_k)$ for each operation $\overline{f}$ of F that can be written as a composite of the operations $p_1, \ldots, p_k$. Given such an operation f of X, we define its output type to be the profile of $\overline{f}$.

Alternatively, we may describe the operations of X using T_S-trees. The operad X has one operation of arity $(p_1, \ldots, p_k)$ for each T_S-tree with nodes labelled by the S-profiles $p_1, \ldots, p_k$, each p_i labelling exactly one node. This description makes it a bit easier to visualize how each operation of X is a way of composing operations in an S-operad. For example, let f be the operation of X of arity $(p_1, \ldots, p_9)$ corresponding to the T_S-tree in Figure 10. Suppose that O is any S-operad having operations o_i with profiles p_i. Then we can compose the o_i and permute their arguments, following the pattern given by the T_S-tree, to obtain the operation

$$(o_6 \cdot (o_5 \cdot (o_2, o_8, o_4), o_1, o_3 \cdot (o_9, o_7)))\sigma$$

where σ is the permutation at the top of the T_S-tree, namely

$$(1, 2, 3, 4, 5, 6, 7) \mapsto (3, 1, 2, 4, 6, 5, 7).$$

In general, suppose that $f \in X(p_1, \ldots, p_k, p')$ and O is a C-operad. Given operations o_i of O of type p_i, we may compose them and permute their arguments in the manner described by the T_S-tree for f to obtain an operation of type p', which we denote by $\alpha(f)(o_1, \ldots, o_k)$. Thus we obtain a map

$$\alpha\colon X(p_1, \ldots, p_k, p') \to \hom(O(p_1) \times \cdots \times O(p_k), O(p'))$$

where $O(p)$ denotes the set of operations of O of type p.

Composition of operations of X is defined as follows. Suppose X has operations f and $g_1, \ldots, g_k$ of profiles for which the composite $f \cdot (g_1, \ldots, g_k)$ should be well-defined. Let $\overline{g}_i$ denote the operations of F corresponding to the operations g_i. Then we define $f \cdot (g_1, \ldots, g_k)$ by

$$\overline{f \cdot (g_1, \ldots, g_k)} = \alpha(f)(\overline{g}_1, \ldots, \overline{g}_k).$$

We finish giving X the structure of a $|\mathrm{prof}(S)|$-operad with the help of Proposition 9. First, the only morphisms in $|\mathrm{prof}(S)|$ are identity morphisms, so for any S-profile p we need an operation $\iota(1_p) \in X(p, p)$. We take this to be the unique operation with that profile corresponding to the operation p of F. Second, for any operation $f \in X(p_1, \ldots, p_k, p')$ and $\sigma \in S_k$ we need an operation $f\sigma \in X(p_{\sigma(1)}, \ldots, p_{\sigma(k)}, p')$. We define $f\sigma$ to be the unique operation of arity $(p_{\sigma(1)}, \ldots, p_{\sigma(k)})$ corresponding to the operation $\overline{f}$ of F. One may then check that X is a $|\mathrm{prof}(S)|$-operad by verifying conditions a) - d) of Proposition 9; we leave this to the reader.

Any S-operad O becomes an X-algebra with the help of Proposition 10. We have already defined the sets $O(p)$ for any S-profile p and the action

$$\alpha\colon X(p_1, \ldots, p_k, p') \to \hom(O(p_1) \times \cdots \times O(p_k), O(p')),$$

so one must only verify conditions a) - c). We leave this to the reader as well. Finally, it is straightforward to check that any X-algebra is naturally a S-operad, and that a homomorphism of X-algebras is the same as a homomorphism of S-operads. $\square$

In fact, there is also a $\mathrm{prof}(C)$-operad for C-operads for any small category C. This played an important role in an earlier version of our approach [5], but for various reasons we now prefer in what follows to work only with operads having a set, rather than a category, of types.

3.2 The slice operad of an operad

Definition 15. *Given a S-operad O, let the* slice operad *of O, denoted O^+, be the* $\mathrm{elt}(O)$*-operad whose algebras are S-operads over O, i.e., equipped with a C-operad homomorphism to O.*

The existence of O^+ is guaranteed by Proposition 13 and Theorem 14. The point is that since S-operads are the algebras of a certain operad, we can apply the slice operad construction to S-operads.

Since O^+ is an $\mathrm{elt}(O)$-operad, it follows that the types of O^+ are the operations of O. Also, by examining the proof of Theorem 14 one may check that the operations of O^+ are the reduction laws of O, and the reduction laws of O^+ are the ways of combining reduction laws of O to obtain new reduction laws. This will become clearer in the next section.

To get a feel for this important construction, let us consider some examples:

Example 16. *The initial untyped operad I as the operad for sets.* Since S-operads form a category we may speak of initial and terminal S-operads. In the case $S = 1$, the initial S-operad I is the untyped operad whose only operation is the identity. In other words, I is the untyped operad with only one unary operation and no operations of higher arity. Its algebras are simply sets, so we say that I is the operad for sets.

Example 17. *I^+ as the operad for monoids.* Note that I^+ is an $\mathrm{elt}(I)$-operad, but $\mathrm{elt}(I) = 1$, so I^+ is an untyped operad. By definition, it is the operad for untyped operads over I. An untyped operad admits a homomorphism to I only if all its operations are unary, in which case it has a unique homomorphism to I. An operad with only unary operations is just a monoid, so I^+ is the operad for monoids. The operad I^+ has $k!$ operations of arity k, corresponding to all the elements of S_k, or in other words, the different orderings in which one can multiply k elements of a monoid. The symmetric group S_k acts on these operations in an obvious way.

In the next example we consider an iterated slice operad. Note that in Definition 15 above, $\mathrm{elt}(O)$ is a small discrete category, or in other words just the *set* of operations of O, since in applying Theorem 13 we are treating O as a set-valued functor on the discrete category $|\mathrm{prof}(S)|$. Thus if O is an operad with a set of types, so is O^+, so we may iterate the slice operad construction.

Example 18. *I^{++} as the operad for planar untyped operads.* By definition, I^{++} is the $\mathrm{elt}(I^+)$-operad for untyped operads over I^+. An untyped operad O admits a homomorphism to I^+ if and only if S_k acts freely on the set O_k of k-ary operations of O. A homomorphism $f\colon O \to I^+$ is then determined by the sets $P_k = f^{-1}(g) \subseteq O_k$, where g is the k-ary operation of I^+ corresponding to the identity element of S_k as in Example 17. One can check that the sets P_k equipped with the composition operation of O form a planar untyped operad P, and conversely, any planar untyped operad comes from an untyped operad over I^+ in this manner, unique up to isomorphism. Thus I^{++} is the operad for planar untyped operads.

Example 19. *The terminal untyped operad T as the operad for commutative monoids.* In the case $S = 1$, the terminal S-operad T has one operation of each arity. An algebra A of T is thus a commutative monoid, with the unique k-ary operation of T acting as the map

$$\begin{array}{rcl} A^k & \to & A \\ (a_1, \ldots, a_k) & \mapsto & a_1 \cdots a_k. \end{array}$$

Example 20. *T^+ as the operad for untyped operads.* A T^+-algebra is an untyped operad over T. Since T is terminal, a T^+-algebra is just an untyped operad, so T^+ is the $\mathrm{elt}(T)$-operad for operads.

More generally, for any set S there is a terminal S-operad T_S, having one operation of each profile. Alternatively, T_S is the pullback of the operad T along the unique functor from S to the terminal category 1. The slice operad T_S^+ is the operad for S-operads. In fact, $\mathrm{elt}(T_S)$ is isomorphic to $|\mathrm{prof}(S)|$, and T_S^+ is the $|\mathrm{prof}(S)|$-operad for S-operads constructed in Theorem 14.

At this point a comment is in order about why we base our approach on operads rather than planar operads. To bootstrap our way up to the definition of n-categories, we want a simple sort of algebraic theory that is powerful enough for theories of this sort to be themselves models of a theory of this sort. Theorem 14 says that operads have this property. Planar operads are simpler than operads, but planar operads are not sufficiently powerful: there is, for example, no planar operad for planar untyped operads.

More precisely, for any small category C we define a *planar C-operad* to be a monoid object in the category of endomorphisms of the free 2-rig on C. Taking $C = 1$ we recover the usual definition of planar untyped operad. Following Example 18, one may check that a planar C-operad is the same as C-operad O equipped with a 'planar structure': a morphism $f\colon O \to F^*(I^+)$, where $F^*\colon \mathrm{sig}(1) \to \mathrm{sig}(C)$ comes from the unique functor $F\colon C \to 1$. To give the operad for planar untyped operads a planar structure, one would need such a morphism from I^{++} to $F^*(I^+)$. One may check that no such morphism exists.

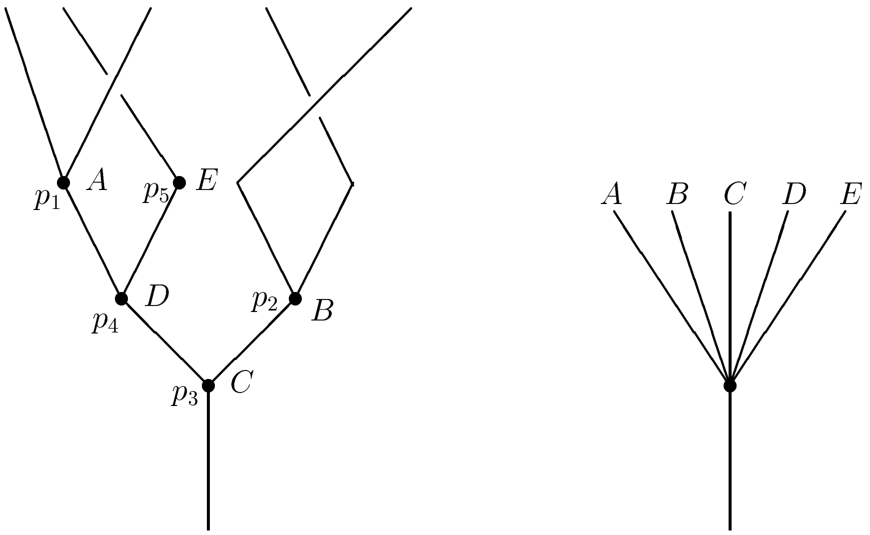
3.3 Opetopes

Opetopes arise when we iterate the slice operad constuction:

Definition 21. *Given an S-operad O, we define O^{0+} to be O, and define $O^{(n+1)+} = (O^{n+})^+$ for $n \geq 1$.*

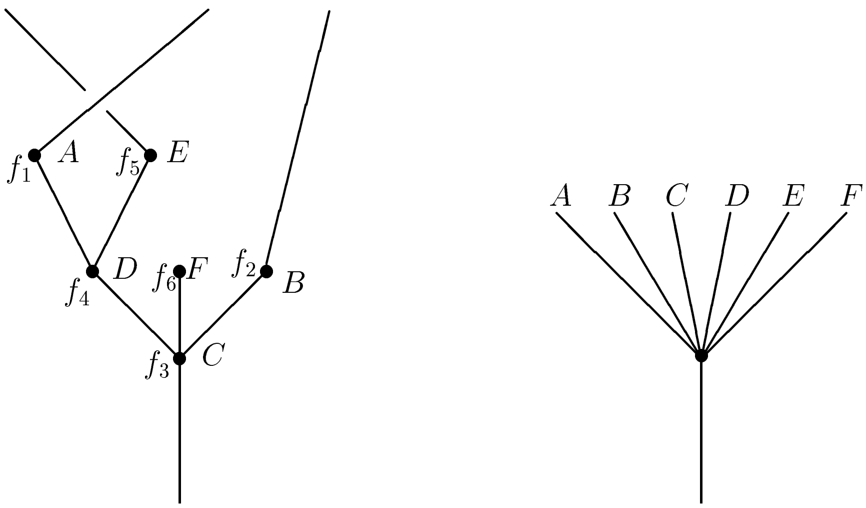
Definition 22. *Given an S-operad O, we define an n-*dimensional O-opetope *to be a type of O^{n+}. We define an n-*dimensional opetope *to be a type of I^{n+}, where I is the initial untyped operad.*

Recall that in Theorem 14 we constructed an operad for S-operads, and in Example 20 we saw that this was just T_S^+, the slice operad of the terminal S-operad. The proof of Theorem 14 thus amounts to a description of the operations of T_S^+ in terms of 'T_S-trees': trees with nodes labelled by S-profiles in a consistent way. A T_S-tree is not quite enough to specify a unique operation of T_S^+. Rather, for any ordering $p_1, \ldots, p_k$ of the S-profiles labelling the nodes of an T_S-tree, there is a unique operation of T_S^+ of arity $(p_1, \ldots, p_k)$ corresponding to that T_S-tree. We can keep track of this ordering by labelling the nodes of the T_S-tree with additional distinct symbols $A, B, C, \ldots$, and drawing a second tree with one node having n edges coming into it from above labelled by these symbols in the desired order. This second tree must be planar; also, we use each symbol exactly once as a label on this second tree. An example is shown in Figure 11. Note that we use arbitrary symbols $A, B, C, \ldots$ rather than the S-profiles themselves to label the second tree, because the S-profiles might not be distinct.



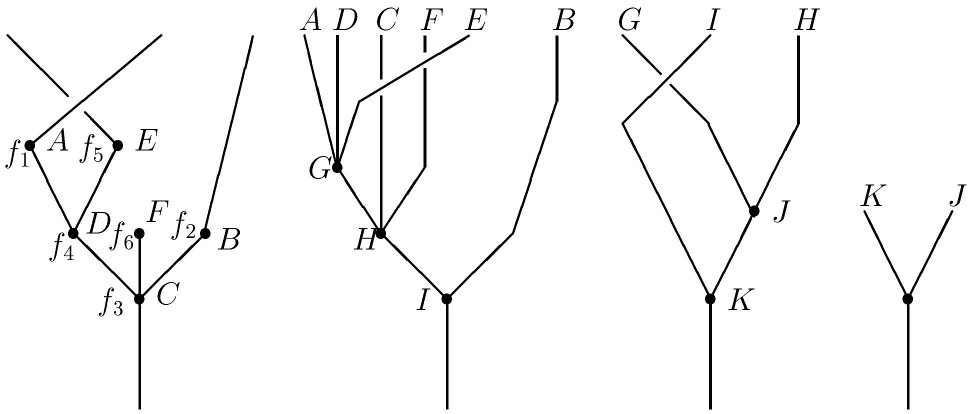
11. An operation of the operad for S-operads

It is easy to extend this notation to describe the operations of O^+ for any S-operad O. Recalling the definition of slice operads given in Section 2.5, it is clear that an operation of O^+ can be specified as in Figure 12. The first tree is an arbitrary O-*tree*. This is a combed tree with nodes labelled by operations of O. We require that a node labelled by a k-ary operation have k edges coming into it from above. Moreover, we require that it be possible to label every edge with an element of S in such a way that for any node labelled by an operation with profile $(x_1, \ldots, x_k, x')$, the edges coming into that node from above are labelled by the elements $x_1, \ldots, x_k$ in that order, while the edge coming out of it from below is labelled by the element x'. As before, we also label each node of this first tree with a distinct symbol A, B, C, etc.. Also as before, the second tree is planar and has only one node, with n edges coming into that node from above, labelled by the same symbols $A, B, C, \ldots$ in any order. These specify the order of the input types of the operation of O^+ we are describing.



12. An operation of O^+

More generally, for any $n > 1$ one can specify any n-dimensional O-opetope by means of an *n-dimensional O-metatree*, as in Figure 13.



13. A 3-dimensional O-metatree

This is a list of n labelled trees, the last of which is a planar tree with only one node, while the rest are combed trees. The first tree is an arbitrary O-tree. For $1 \leq i < n$, every node of the ith tree is labelled with a distinct symbol, and the same symbols also label all the edges at the very top of the $(i+1)$st tree, each symbol labelling exactly one edge. In addition, each edge of the $(i+1)$st tree must correspond to a subtree of the ith tree in such a way that:

1. The edge at the very top of the $(i+1)$st tree labelled by a given symbol corresponds to the subtree of the ith tree whose one and only node is labelled by the same symbol.

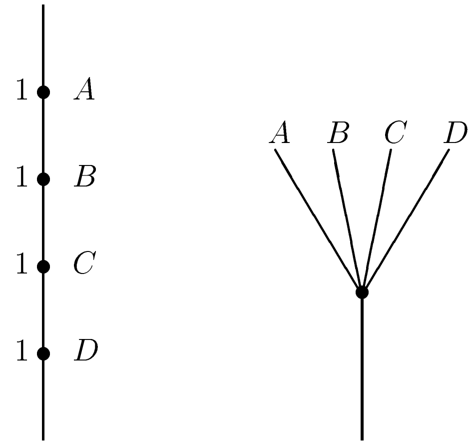
2. The edge of the $(i+1)$st tree coming out of a given node from below corresponds to the subtree that is the union of the subtrees corresponding to the edges coming into that node from above.

3. The edge at the very bottom of the $(i+1)$st tree corresponds to the whole ith tree.

Special care must be taken when the node of the last tree has no edges coming into it from above. This can only occur when all the previous trees are empty. This sort of metatree describes a nullary operation of $O^{(n-1)+}$ whose output type is an identity operation 1_x of $O^{(n-2)+}$. To specify which identity operation, we need to label the edge coming out of the node of the last tree from below with the operation 1_x.

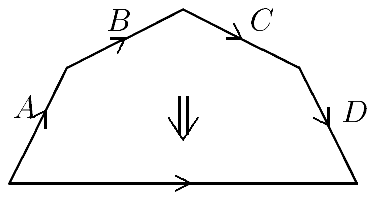
We conclude this section with some examples which begin to explain the role opetopes play in n-category theory.

Example 23. *Metatree notation for operations of I^+.* Let I be the initial untyped operad as in Example 16. Since the only operation in I is the unary operation 1, a metatree for a typical operation of I^+ looks like that in Figure 14. As we expect from Example 17, I^+ has $n!$ operations of arity n.



14. An operation of I^+

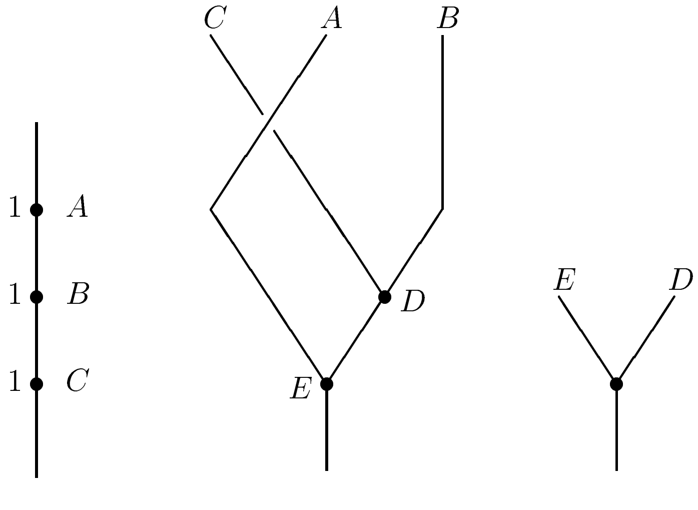
The term 'opetope' is explained by the fact that one can associate to the n-dimensional opetopes certain labelled n-dimensional combinatorial polytopes, or generalizations thereof. In particular, the operations of I^+ are the 2-dimensional opetopes, and the k-ary operations of I^+ correspond to polygons with k labelled 'infaces' and one 'outface'. For example, the 4-ary operation in Figure 14 corresponds to the polygon shown in Figure 15, with four labelled infaces and one outface.



15. 2-dimensional opetope represented as a polytope

The degenerate cases $k = 0$ and $k = 1$ are a bit of a nuisance because one cannot represent 'unigons' and 'bigons' as convex geometrical polytopes. Nonetheless, one can still draw them if one allows curved edges, and these drawings are widely used in 2-categorical commutative diagrams. In fact, the bigon is the only basic shape of 2-cell in the traditional globular approach to n-category theory; to achieve the effect of 2-cells with other shapes one resorts to pasting theorems [12, 20, 26]. In the opetopic approach the the basic shapes of cells are the opetopes, which may have any number of infaces but always exactly one outface. For example, we use a 2-cell shaped like the opetope in Figure 15 to represent an operation having the 1-cells A, B, C, and D as inputs and the outface 1-cell as its output. In particular, we use a 'universal' 2-cell of this sort — as defined below in Section 4.1 — to represent a process of composing the 1-cells A, B, C, and D. The outface is then called a 'composite' of these 1-cells.

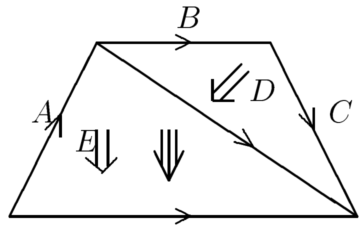
Example 24. *Metatree notation for operations of I^{++}.* A metatree for a typical operation of I^{++} is shown in Figure 16.



16. An operation of I^{++}

The operations of I^{++} are the 3-dimensional opetopes, and we can associate to them certain 3-dimensional combinatorial polytopes or generalizations thereof. For example, the operation of Figure 16 corresponds to the polytope shown in Figure 17, having two triangular 'infaces' labelled D and E on top, and having the quadrilateral on the bottom as ‘outface’. Note that while this is a combinatorial polytope, it cannot be realized as a convex geometrical polytope. As in the 2-dimensional case, there are also ‘degenerate’ 3-dimensional opetopes that cannot be realized as combinatorial polytopes in the strict sense. Also note that Figure 17 does not record all the information needed to uniquely specify an operation of I^{++}, because it does not keep track of the permutations in the metatree of Figure 16. Because of these problems we find it better to describe opetopes using metatrees. Nonetheless, the polytopes may help the reader relate our approach to other work on n-categories.

In the opetopic approach to n-categories, we use a universal 3-cell shaped like that in Figure 17 to represent the process of composing the 2-cells D and E in the indicated manner to obtain a 2-cell shaped like the outface. More generally, an n-dimensional opetope always has some number of $(n-1)$-dimensional opetopes as infaces, pasted together in a manner described by a tree, together with a single $(n-1)$-dimensional opetope as outface. A universal n-cell of this shape then describes a process of composing $(n-1)$-cells shaped like the infaces to obtain an $(n-1)$-cell shaped like the outface.



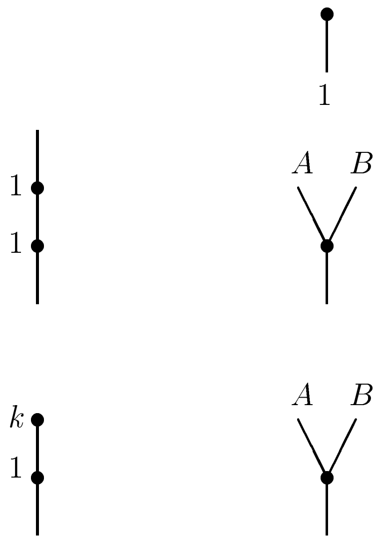
17. A 3-dimensional opetope represented as a polytope

3.4 Algebras of slice operads

The following examples lead up to a concrete description, for any S-operad O, of the algebras of O^+.

Example 25. *The free operad on one nullary operation, K, as the operad for pointed sets.* Let K be the untyped operad with one nullary operation k, one unary operation 1 (the identity operation), and no other operations. A K-algebra is simply a pointed set.

Example 26. *K^+ as the operad for monoid actions.* Since K has two operations, K^+ has two types, k and 1. The operations of K^+ include the three operations shown in metatree notation in Figure 18: a nullary operation with output type 1, a binary operation with profile $(1, 1, 1)$, and a binary operation with profile $(k, 1, k)$. All the operations of K^+ are generated from these three by composition. A K^+-algebra A thus consists of a set $A(1)$ and a set $A(k)$ together with a special element $i \in A(1)$, a map $m: A(1) \times A(1) \to A(1)$ and a map $a: A(k) \times A(1) \to A(1)$ satisfying certain laws. One may check that these laws say precisely that $A(1)$ is a monoid with a right action on $A(k)$.



18. Three operations of K^+

Example 27. *The free operad on one unary operation, F_1, as the operad for functions.* Let F_1 be the operad with two types, say x and x', and three operations: a unary operation f with profile (x, x'), and the two identity operations, which we call 1_x and $1_{x'}$. An F_1-algebra is simply a function.

Example 28. *F_1^+ as the operad for monoid bi-actions.* Since F_1 has three operations, F_1^+ has three types: f, 1_x, and $1_{x'}$. Following Example 26, one may check that an F_1^+-algebra A consists of two monoids $A(1_x)$ and $A(1_{x'})$ together with a set $A(f)$ equipped with an an action of $A(1_x)^{\mathrm{op}} \times A(1_{x'})$.

Example 29. *The free operad on one k-ary operation, F_k, as the operad for k-ary multi-functions.* Generalizing from the previous examples, we let F_k be the operad with $k+1$ types, say $x_1, \ldots, x_k, x'$, and one k-ary operation f with profile $(x_1, \ldots, x_k, x')$, together with the operations required by the definition of an operad: the $k+1$ identity operations, which we call $1_{x_1}, \ldots, 1_{x_k}, 1_{x'}$, and the k-ary operations obtained from f by the action of the permutation group S_k. An F_k-algebra A is a collection of sets $A_1, \ldots, A_k, A'$ and a function from $A_1 \times \cdots \times A_k$ to A'. We call this a *k-ary multi-function.*

Example 30. *F_k^+ as the operad for $(k, 1)$ monoid multi-actions.* An F_k^+-algebra consists of $k+1$ monoids $A_1, \ldots, A_k, A'$ and a set equipped with an action of

$$A_1^{\mathrm{op}} \times \cdots \times A_k^{\mathrm{op}} \times A'.$$

We call this a $(k, 1)$ *multi-action* of the monoids in question, since it can be thought of as k right actions and one left action, all of which commute.

Since every S-operad O may be presented as the quotient of a free operad on some set of operations, Example 30 suggests the following general picture of O^+-algebras. Given an S-operad O, let us say that an operation f of O^{n+} is *degenerate* if $n = 0$ and f is an identity operation, or if $n > 0$ and f is either an identity operation, a nullary operation, or an operation with one or more degenerate operations as input types. For example, all the operations of I^{n+} are degenerate in this sense.

Theorem 31. *For any S-operad O, an O^+-algebra A consists of:*

1. *for each type x of O, a monoid $A(x)$*
2. *for each nondegenerate operation g of O with profile $(x_1, \ldots, x_k, x')$, a set $A(g)$ equipped with a $(k, 1)$ multi-action of the monoids $A(x_1), \ldots, A(x_k), A(x')$*
3. *for each nondegenerate reduction law of O — that is, for each nondegenerate operation G of O^+ with profile $(g_1, \ldots, g_k, g')$ — a morphism*
$$A(G)\colon G(A(g_1), \ldots, A(g_k)) \to A(g')$$
of multi-actions
4. *for each nondegenerate way of combining reduction laws of O to obtain another reduction law — that is, for each nondegenerate operation $\mathcal{G}$ of O^{++} with profile $(G_1, \ldots, G_k, G')$ — an equation*
$$\mathcal{G}(A(G_1), \ldots, A(G_k)) = A(G').$$

Proof - First, points 3 and 4 require a bit of clarification. An operation G of O^+ with profile $(g_1, \ldots, g_k, g')$ corresponds to an O-metatree, and this metatree gives a recipe for tensoring the multi-actions on $A(g_1), \ldots, A(g_k)$ in a tree-like pattern, obtaining a set we denote by $G(A(g_1), \ldots, A(g_k))$, equipped with a multi-action of the same monoids that act on $A(g')$. Similarly, an operation G of O^{++} with profile $(G_1, \ldots, G_k, G')$ corresponds to a metatree that specifies how to compose the morphisms $A(G_1), \ldots, A(G_k)$ in a tree-like pattern, obtaining a morphism with the same source and target as $A(G')$, which we denote by $\mathcal{G}(A(G_1), \ldots, A(G_k))$.

Next, suppose A is an O^+-algebra. We have seen that A consists of:

(a) for each type g of O^+, a set $A(g)$

(b) for each operation G of O^+ with profile $(g_1, \ldots, g_k, g')$, a function
$$A(G)\colon A(g_1) \times \ldots \times A(g_k) \to A(g')$$

(c) for each reduction law of O^+ — that is, for each operation $\mathcal{G}$ of O^{++} with profile $(G_1, \ldots, G_k, G')$ — an equation
$$\mathcal{G}(A(G_1), \ldots, A(G_k)) = A(G')$$

where again we use metatree notation to compose the functions $A(G_1), \ldots, A(G_n)$ in the tree-like pattern specified by the operation $\mathcal{G}$ of O^{++}. In what follows we show how (a)-(c) give 1-4; by examining our argument one can check that the converse holds as well.

Recall first that the types of O^+ are the operations of O. These are either identity operations or nondegenerate operations. Item (a) applied to any identity operation 1_x of O gives a set which we denote as $A(x)$. Item (a) applied to any nondegenerate operation g of O gives a set $A(g)$.

Recall next that the operations of O^+ are the reduction laws of O. Any operation of O^+ is either an identity operation, a nullary operation, an operation with an identity operation of O as an input type, or a nondegenerate operation. We consider these cases in turn.

Item (b) applied to any identity operation 1_g of O^+ gives a function from $A(g)$ to itself. However, (c) applied to the nullary operation of O^{++} with 1_g as output type implies that this function is the identity.

There is one nullary operation of O^+ with output type 1_x for each type x of O. Item (b) applied to this operation equips the set $A(x)$ with a distinguished element.

There are many operations of O^+ having an identity operation of O as an input type, but they are all composites of nondegenerate operations with operations of the following three kinds, so by (c) it suffices to consider only these three kinds. First, there are identity operations 1_{1_x} of O, which we have already treated. Second, there is the binary operation of O^+ with profile $(1_x, 1_x, 1_x)$. By (b) it follows that $A(x)$ is equipped with a binary product, and (c) then implies that $A(x)$ is a monoid with this product and its distinguished element. Third, there are the operations of composing an operation g of O with profile $(x_1, \ldots, x_k, x')$ with the identity operations $1_{x_1}, \ldots, 1_{x_k}, 1_{x'}$. By (b) and (c) it follows that $A(g)$ is equipped with a $(k, 1)$ multi-action of the monoids $A(x_1), \ldots, A(x_k), A(x')$.

Item (b) applied to any nondegenerate operation G of O^+ with profile $(g_1, \ldots, g_k, g')$ gives a function

$$A(G)\colon A(g_1) \times \ldots \times A(g_k) \to A(g')$$

and (c) implies that this function defines a morphism of multi-actions

$$A(G)\colon G(A(g_1), \ldots, A(g_k)) \to A(g').$$

Recall finally that the operations of O^{++}, or reduction laws of O^+, are ways of combining reduction laws of O to give other reduction laws of O. Applying (c) to an operation G of O^{++} with profile $(G_1, \ldots, G_k, G')$ we obtain an equation

$$\mathcal{G}(A(G_1), \ldots, A(G_k)) = A(G').$$

One can check that the equations coming from nondegenerate operations $\mathcal{G}$ imply those coming from degenerate operations. $\square$

3.5 Opetopic sets

In topology it is common to take simplices as the basic shapes for cells. There is a category with simplices as objects and face and degeneracy maps as morphisms. Presheaves on this category — i.e., set-valued functors on the opposite category — are called 'simplicial sets'. In our approach to n-category theory we take opetopes as the basic shapes for cells. Opetopes form a category, and presheaves on this category are called 'opetopic sets'.

Here, however, we give a recursive definition of opetopic sets that does not rely on the category of opetopes. For this it is convenient to introduce some notation.

Definition 32. *Given a set S, a set over S is a set Y equipped with a function to S. Given an S-operad O and a set Y over S, we define O_Y to be the pullback operad F^*O, where F is the function from Y to S.*

We then define opetopic sets as follows:

Definition 33. *Given an S-operad O, an O-*opetopic set* X is defined recursively as a set $X(0)$ over S together with a $(O_{X(0)})^+$-opetopic set.*

If we work out the implications of this definition, we see that if O is an S-operad, an O-opetopic set X consists of an set $X(n)$ over $S(n)$ for each integer $n \geq 0$, where

$$S(0) = S, \qquad S(n+1) = \mathrm{elt}(O(n)_{X(n)}),$$

and $O(n)$ is the $S(n)$-operad given by

$$O(0) = O, \qquad O(n+1) = (O(n)_{X(n)})^+.$$

Note also that

$$S(n) = \mathrm{type}(O(n)).$$

Definition 34. *Let O be an S-operad and X an O-opetopic set. We define an n-*dimensional cell *(or n-*cell*) of X to be an element of $X(n)$. We define an n-*dimensional frame *in X to be an element of $S(n)$. For $n \geq 1$, we define an n-*dimensional opening *in X to be an operation of $O(n-1)$.*

Since $X(n)$ is a set over $S(n)$, there is a map from n-dimensional cells to n-dimensional frames, and for any cell of X we may speak of the frame *of* that cell. Also, for $n \geq 1$, the tautologous morphism from the pullback of an operad to the operad itself gives a map from operations of $O(n-1)_{X(n-1)}$, which are n-dimensional frames, to operations of $O(n-1)$, which are n-dimensional openings. Thus for $n \geq 1$ we may speak of any frame s of X as being *in* some opening o, and given any cell x with frame s, we also say that x is *in* o.

Let o be an n-dimensional opening in X. We define an *o-cell* to be a cell x in o. The frame of x is an operation of $O(n)_{X(n)}$, and has profile $(a_1, \ldots, a_k, b)$ for some $(n-1)$-dimensional cells $a_1, \ldots, a_k, b$. It is convenient to use the following schematic picture of x:

$$(a_1, \ldots, a_k) \xrightarrow{\quad x \quad} b$$

We call $a_1, \ldots, a_k$ the *infaces* of x, and b the *outface* of x.

Similarly, we define an *o-frame* to be a frame in o, and depict an o-frame with profile $(a_1, \ldots, a_k, b)$ as follows:

$$(a_1, \ldots, a_k) \xrightarrow{\quad ? \quad} b$$

An 'o-niche' is like an o-frame with the outface missing. Suppose that the opening o has profile $(s_1, \ldots, s_k, t)$. We define an *o-niche* to be a tuple $(a_1, \ldots, a_k)$ of $(n-1)$-dimensional cells with a_i having s_i as its frame. We depict this o-niche as follows:

$$(a_1, \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

The concept of niche serves as our substitute for the concept of a horn in a simplicial set.

Similarly, a 'punctured o-niche' is like an o-frame with the outface and one inface missing. We define a *punctured o-niche* to be a tuple $(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k)$ of cells with a_i having s_i as its frame, and depict this as:

$$(a_1, \ldots, a_{j-1}, ?, a_{j+1}, \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

In the case where one of these configurations (o-frame, o-niche, or punctured o-niche) can be extended to an actual o-cell, the o-cell is called an *occupant* of the configuration. Occupants of the same frame (resp. niche) are called *frame-competitors* (resp. *niche-competitors*).

To make O-opetopic sets into a category we need to define morphisms between them. Roughly speaking, a morphism $\phi\colon X \to X'$ between O-opetopic sets is a function sending each cell $x \in X(n)$ to a cell $\phi(x) \in X'(n)$ of the same shape, such that ϕ of any face of x is the corresponding face of $\phi(x)$. To make this precise requires a bit of technical work.

We begin with some remarks on the functoriality of the slice operad construction. Suppose O is an S-operad, O' is an S'-operad, and $F\colon S \to S'$ is a function. By Proposition 11 we obtain a lax monoidal functor $F^*\colon \mathrm{sig}(S') \to \mathrm{sig}(S)$. As in Section 2.2 this allows us to speak of morphisms from S'-operads to S-operads, but we can also define morphisms going the other way. Namely, we define an *operad morphism $f\colon O \to O'$ riding F* to be an operad homomorphism $f\colon O \to F^*(O')$.

Given such an operad morphism there is an obvious function from from $|\mathrm{elt}(O)|$ to $|\mathrm{elt}(O')|$, which we call F^+. We also obtain a operad morphism $f^+\colon O^+ \to O'^+$

riding this function. To see this it is easiest to use metatree notation: an operation of O^+ is given by a 1-dimensional O-metatree, and using $f\colon O \to O'$ one can convert this to a 1-dimensional O'-metatree, which specifies an operation of O' and thus of $F^{+*}(O')$. One can then check this defines an operad morphism $f^+\colon O^+ \to O'^+$.

Now suppose that Y is a set over S and Y' is a set over S'. We define a *function* $\phi\colon Y \to Y'$ *over* $F\colon S \to S'$ to be a function making the following diagram commute:

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[d] & Y' \arrow[d] \\ S \arrow[r, "F"] & S' \end{tikzcd}$$

Given an operad morphism $f\colon O \to O'$ riding F, there is an obvious operad morphism from O_Y to $O'_{Y'}$ riding ϕ, which we call f_ϕ.

Finally, suppose that X is an O-opetopic set and X' is an O'-opetopic set. Suppose that $f\colon O \to O'$ is an operad morphism riding $F\colon S \to S'$. We define an *opetopic map* $\phi\colon X \to X'$ *riding* f to consist of, for each $n \geq 0$, a function

$$\phi_n\colon X(n) \to X'(n)$$

over the function

$$F_n\colon S(n) \to S'(n)$$

given as follows. We set $F_0 = F$, and define F_n for higher n recursively, along with a sequence of operad morphisms

$$f_n\colon O(n) \to O'(n),$$

starting with $f_0 = f$. To do so, we let

$$f_{n+1} = ((f_n)_{\phi_n})^+$$

and note that this operad morphism gives a map from $S(n+1)$ to $S'(n+1)$, which we take as F_{n+1}. Unrolling this recursive construction one sees that, fixing f and F, the morphism $\phi\colon X \to X'$ is completely determined by the functions ϕ_n sending n-cells of X to n-cells of X'.

Definition 35. *Given an S-operad O, we define the* category of O-opetopic sets *to be that with O-opetopic sets as objects and opetopic morphisms riding the identity function as morphisms.*

In fact, this category is equivalent to the category of presheaves on a certain category of O-opetopes. To save space we shall not prove this here, but only seek to make it plausible by showing that every n-cell of an O-opetopic set X has some n-dimensional opetope as its 'shape'. This is trivial in the case $n = 0$, so we assume $n \geq 1$.

Recall that every n-dimensional cell of X is in some opening, which is an operation of $O(n-1)$. On the other hand, each n-dimensional opetope is an operation of $O^{(n-1)+}$. Thus to associate an n-dimensional opetope to each n-cell of X, we construct, for all $n \geq 0$, an operad morphism

$$p_n \colon O(n) \to O^{n+}.$$

Since $O(0) = O$, we take p_n to be the identity when $n = 0$. Given p_n, to define p_{n+1} we first form the composite

$$O(n)_{X(n)} \longrightarrow O(n) \xrightarrow{p_n} O^{n+}$$

where the first arrow is the tautologous morphism from a pullback of an operad to the operad itself. Taking the '+' of this composite, we then obtain p_{n+1}.

4 n-Categories

In Section 4.1 we define 'n-coherent O-algebras'. The basic idea is that for any operad O, an n-coherent O-algebra is an n times categorified analog of an O-algebra. For example, just an I-algebra is a set, an n-coherent I-algebra is an n-category. Other examples are also interesting: just as an I^+-algebra is a monoid, an n-coherent I^+-algebra is a 'monoidal n-category', and just as T-operad is a commutative monoid, an n-coherent T-algebra is a 'stable n-category'. Stable n-categories play an important role in the program sketched in HDA0, and also in the foundations of n-category theory itself, since the $(n+1)$-category of all n-categories will be a stable $(n+1)$-category.

In Section 4.2 we define 'k-ary virtual n-functors' to be n-coherent F_k-algebras, where F_k is the free operad on one k-ary operation. This concept allows us to reinterpret and clarify some of the previous material. For example, in Theorem 53 we use them to give a recursive characterization of n-coherent O-algebras that is often more useful than the original definition. We also use them in Propositions 54 and 55 to characterize the concepts of 'balanced' punctured niche and 'universal' niche-occupant, introduced in the previous section. Finally, in Section 4.3 we give a rather general precise statement of the 'microcosm principle'.

4.1 n-Coherent O-algebras

In what follows we fix a nonnegative integer n and define the notion of 'n-coherent O-algebra', which will be an O-opetopic set with certain properties. To do so, we need the notions of 'balanced punctured niche' and 'universal niche-occupant', which we define in a recursively interlocking way.

As the definitions are a bit complicated, let us first explain them in a heuristic way. We shall see in Section 4.2 that in an n-coherent O-algebra, any m-dimensional punctured niche

$$(a_1, \ldots, a_{j-1}, ?, a_{j+1}, \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

determines a 'virtual $(n-m)$-functor'. In Proposition 54 we show that the punctured niche is balanced if and only if this virtual $(n-m)$-functor is an 'equivalence'. On the other hand, for a niche-occupant

$$(c_1, \ldots, c_k) \xrightarrow{\quad u \quad} d$$

to be 'universal' means roughly that any other occupant of the same niche factors through the given one — at least 'up to equivalence'. We make this precise in Proposition 55.

The definitions are as follows:

Definition 36. *For an m-dimensional opening o, a punctured o-niche:*

$$(a_1, \ldots, a_{j-1}, ?, a_{j+1}, \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

is said to be balanced *if and only if $m > n+1$ or:*

1. *any extension*

$$(a_1, \ldots, a_{j-1}, ?, a_{j+1}, \ldots, a_k) \xrightarrow{\quad ? \quad} b$$

extends further to:

$$(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_k) \xrightarrow{\quad u \quad} b$$

with u universal in its niche, and

2. *for any occupant*

$$(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_k) \xrightarrow{\quad u \quad} b$$

universal in its niche, and frame-competitor a'_j of a_j, the $(m+1)$-dimensional punctured niches:

$$\begin{array}{c} (a'_j \xrightarrow{?} a_j,\ (a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_k) \xrightarrow{u} b) \\ \Big\downarrow{\scriptstyle ?} \\ (a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_k) \xrightarrow{?} b \end{array}$$

and

$$\begin{array}{c}((a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_k) \xrightarrow{u} b,\ a'_j \xrightarrow{?} a_j) \\ \Big\downarrow{\scriptstyle ?} \\ (a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_k) \xrightarrow{?} b\end{array}$$

are balanced.

Definition 37. *An m-dimensional niche-occupant:*

$$(c_1, \ldots, c_k) \xrightarrow{\quad u \quad} d$$

is said to be universal *if and only if $m > n$ and u is its own unique niche-competitor, or $m \leq n$ and for any frame-competitor d' of d, the $(m+1)$-dimensional punctured niches:*

$$\begin{array}{c}((c_1, \ldots, c_k) \xrightarrow{u} d,\ d \xrightarrow{?} d') \\ \Big\downarrow{\scriptstyle ?} \\ (c_1, \ldots, c_k) \xrightarrow{?} d'\end{array}$$

and

$$\begin{array}{c}(d \xrightarrow{?} d',\ (c_1, \ldots, c_k) \xrightarrow{u} d) \\ \Big\downarrow{\scriptstyle ?} \\ (c_1, \ldots, c_k) \xrightarrow{?} d'\end{array}$$

are balanced.

Definition 38. *Given a universal o-cell:*

$$(a_1, \ldots, a_k) \xrightarrow{\quad u \quad} b$$

we call b a composite *of $(a_1, \ldots, a_k)$, or o-*composite *if we need to be more specific.*

Definition 39. *An n-*coherent *O-*algebra *is an O-opetopic set such that 1) every niche has a universal occupant, and 2) composites of universal cells are universal.*

The dependence on n in this definition is implicit in how the definition of 'universal' depends on n. Note that in an n-coherent O-algebra, for $m > n$ every m-dimensional niche has a unique occupant, which is automatically universal, and for $m > n+1$ every m-dimensional punctured niche is balanced. One can also check that for $m > n+1$ every m-dimensional frame has a unique occupant. This is analogous to how a Kan complex represents an n-groupoid if, for $m > n+1$, any configuration in which all the faces of m-simplex are filled in by $(m-1)$-dimensional cells in a consistent way can be uniquely extended to a m-dimensional cell.

A 0-coherent O-algebra is essentially the same thing as an O-algebra. Given a 0-coherent O-algebra A, the types of O are the 0-dimensional frames of A, so for any type s there is a set $\tilde{A}(s)$ of 0-cells of A having s as frame. For any operation f of O with profile $(s_1, \ldots, s_k, s')$, and any 0-cells $a_i \in \tilde{A}(s_i)$, the 1-dimensional niche

$$(a_1, \ldots, a_k) \xrightarrow{\;?\;} ?$$

has a unique occupant

$$(a_1, \ldots, a_k) \xrightarrow{\;u\;} a'.$$

Thus one can check that there is an O-algebra $\tilde{A}$ with the sets $\tilde{A}(s)$ given as above, and with the operation f acting by

$$f(a_1, \ldots, a_k) = a'.$$

In fact, one can check that this construction gives an equivalence between the category of O-algebras and the category of 0-coherent O-algebras in which morphisms are defined as follows:

Definition 40. *Let O be an S-operad and let A, A' be n-coherent O-algebras. Then a morphism of O-opetopic sets $f\colon A \to A'$ is called an* n-coherent O-algebra morphism *if it preserves universality of niche-occupants.*

We study n-coherent O-algebras for higher n in the following two sections. In Theorem 53 we recursively describe n-coherent O-algebras in terms of $(n-1)$-coherent O-algebras. In Theorem 58 we use this to give a concrete description of 1-coherent O-algebras.

The simplest sort of operad algebra is an I-algebra, which by Example 16 is just a set. Similarly, the simplest sort of n-coherent O-algebra is an n-category:

Definition 41. *An* n-category *is an n-coherent I-algebra. An* n-functor *is a morphism of n-coherent I-algebras.*

Example 42. *1-categories as categories.* A 1-coherent I-algebra C has a set $C(0)$ of 0-cells, and given 0-dimensional cells c and c' we may denote the set of occupants of the frame

$$c \xrightarrow{?} c'$$

as $\hom(c, c')$. Given a 0-cell c the 2-dimensional niche

$$\begin{tikzcd} {} \arrow[d, "?"] \\ c \xrightarrow{?} c \end{tikzcd}$$

has a unique occupant

$$\begin{tikzcd} {} \arrow[d, "u"] \\ c \xrightarrow{1_c} c \end{tikzcd}$$

so we have $1_c \in \hom(c, c)$. Similarly, given 0-cells c, c', c'', the 2-dimensional niche

$$\begin{tikzcd} (c \xrightarrow{f} c',\ c' \xrightarrow{g} c'') \arrow[d, "?"] \\ c \xrightarrow{?} c'' \end{tikzcd}$$

has a unique occupant

$$\begin{tikzcd} c \xrightarrow{f} c',\ c' \xrightarrow{g} c'' \arrow[d, "u"] \\ c \xrightarrow{fg} c'' \end{tikzcd}$$

so given $f \in \hom(c, c'), g \in \hom(c', c'')$ we get $fg \in \hom(c, c'')$. By examining the 3-dimensional cells of C one can check that these operations give a category $\tilde{C}$ with

$C(0)$ as its set of objects and the sets $\hom(c, c')$ as hom-sets. One can also check that this construction gives an equivalence between the category with 1-categories as objects and 1-functors as morphisms, and the category with small categories as objects and functors as morphisms.

In Examples 17 and 19 we saw that an I^+-algebra is a monoid, and a T-algebra is a commutative monoid. By analogy we make the following definitions:

Definition 43. *A* monoidal n-category *is an n-coherent I^+-algebra.*

Definition 44. *A* stable n-category *is an n-coherent T-algebra.*

Since there are unique operad homomorphisms from I to I^+ and from I^+ to T, the following result lets us extract an n-category from any monoidal n-category, and a monoidal n-category from any stable n-category.

Proposition 45. *Suppose O is an S-operad, O' is an S'-operad, $F\colon S \to S'$ is a function, and $f\colon O \to O'$ is an operad morphism riding F. Suppose X' is an O'-opetopic set and $X = f^*X'$ is the pullback O-opetopic set. Then a punctured niche in X is balanced if and only if the corresponding punctured niche in X' is balanced, and a niche-occupant in X is universal if and only the corresponding niche-occupant in X' is universal. Thus X is an n-coherent O-algebra if X' is an n-coherent O'-algebra.*

Proof - The proof is a straightforward verification once we have clarified the notion of 'pullback' used here. Suppose that O is an S-operad and O' is an S'-operad. Given an operad morphism $f\colon O \to O'$ riding a function $F\colon S \to S'$, the pullback $X = f^*X'$ of an O'-opetopic set X', which is an O-opetopic set. The set $X(0)$ over S is defined to be the pullback of the set $X'(0)$ over S, and the underlying $(O_{X(0)})^+$-opetopic set of X is defined (recursively) to be the pullback of the underlying $(O_{X'(0)})^+$-opetopic set of X'. $\square$

In a future paper we plan to discuss the stable $(n+1)$-category of n-categories, nCat. This is needed for most of the interesting applications of n-category theory. The 1-cells in nCat are 'k-ary virtual functors'. We study a version of these in the following section, defined in a way that is convenient now but not necessarily best in the long run.

4.2 k-ary virtual n-functors

As we saw in Example 29, the free operad on one k-ary operation, F_k, is the operad for k-ary multi-functions. By analogy we make the following definition:

Definition 46. *A* k-ary virtual n-functor *is an n-coherent F_k-algebra. We omit the term 'k-ary' if $k = 1$, and the reference to n if $n = 1$.*

Suppose that A is k-ary virtual n-functor. Recall that F_k has one operation f of type $(x_1, \ldots, x_k, x')$, together with $k+1$ identity operations $1_{x_1}, \ldots, 1_{x_k}$, and $1_{x'}$. Thus there are $k+1$ operad morphisms from I to F_k, and by Proposition 45, the pullback of A along any one of these is an n-category. Calling these n-categories $C_1, \ldots, C_k$ and C', respectively, we say that A is a k-ary virtual n-functor *from* $C_1 \times \ldots \times C_k$ *to* C', and write

$$A\colon C_1 \times \ldots \times C_k \rightharpoonup C'.$$

Example 47. *Virtual functors as saturated anafunctors.* A virtual functor is essentially the same as what Makkai [23] calls a 'saturated anafunctor', which may be viewed as a special sort of distributor. A *distributor* A from the category C to the category D is a functor $A\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$, and A is a *saturated anafunctor* if for every object $c \in C$, the functor $A(c, \cdot)$ is naturally isomorphic to $\hom(d, \cdot)$ for some object $d \in D$. Thus, in keeping with the philosophy of this paper, a saturated anafunctor does not specify a unique object $d \in D$ for each object $c \in C$. Instead, it specifies a universal property, which automatically determines an object $d \in D$ up to a specified isomorphism.

Suppose that $A\colon C \rightharpoonup D$ is a virtual functor. Then we obtain 1-categories C and D, which by Example 42 we may think of as categories. Given objects $c \in C, d \in D$, we denote the set of occupants of the f-frame

$$c \xrightarrow{?} d$$

by $\tilde{A}(c, d)$. Since 1-cells in a 1-coherent O-algebra have unique composites, any morphism $f\colon c \to c'$ in C gives a function

$$\tilde{A}(c', d) \to \tilde{A}(c, d)$$

for each $d \in D$, and any morphism $f\colon d \to d'$ gives a function

$$\tilde{A}(c, d) \to \tilde{A}(c, d')$$

for each $c \in C$. Thus $\tilde{A}$ can be thought of as a distributor from C to D. Because every f-niche

$$c \xrightarrow{?} ?$$

has a universal occupant, $\tilde{A}$ is a saturated anafunctor. Conversely, every saturated anafunctor can be thought of as a virtual functor.

Example 48. *0-ary virtual functors as representable presheaves.* Generalizing the previous example, one can show that k-ary virtual functors are essentially the same as 'k-ary saturated anafunctors'. The case $k = 0$ is particularly interesting. A 0-ary virtual functor with codomain C is just a functor $P\colon C \to \mathrm{Set}$ that is naturally isomorphic to $\hom(c, \cdot)$ for some $c \in C$. This is also called a 'representable presheaf' on $C^{\rm op}$.

Recall from Example 25 that we gave the operad F_0 another name, K. This stands for 'constant', since a K-algebra is just a pointed set. Here we see that a 1-coherent K-algebra, or in other words a representable presheaf, is a categorified version of a pointed set: it is a category equipped, not quite with a distinguished object, but with a universal property that determines an object up to natural isomorphism.

Generalizing, we call an n-coherent K-algebra a *representable n-prestack*. It follows from Theorem 53 that an n-prestack P may be regarded as a special sort of *n-prestack*, which we define as an $(n-1)$-coherent $(K_{P(0)})^+$-algebra. We expect that prestacks are to the 'stacks' sought by Grothendieck [19] as presheaves are to sheaves.

As noted earlier, the concept of balanced punctured niche is closely related to the concept of 'equivalence'. We can now begin to make this more precise:

Definition 49. *A virtual n-functor $A: C \rightharpoonup C'$ is an* n-equivalence*, or simply an* equivalence*, if the punctured f-niche*

$$? \xrightarrow{?} ?$$

is balanced.

A functor is an equivalence if and only if it is essentially surjective and fully faithful. The same is true for virtual n-functors. Note the similarity of the following two definitions to the two clauses in the definition of 'balanced':

Definition 50. *A virtual n-functor $A: C \rightharpoonup C'$ is* essentially surjective *if any extension*

$$? \xrightarrow{?} c'$$

of the punctured f-niche extends further to

$$c \xrightarrow{u} c'$$

with u universal in its niche.

Definition 51. *A virtual n-functor $A: C \rightharpoonup C'$ is* fully faithful *if for any universal occupant*

$$c \xrightarrow{u} c'$$

of the punctured f-niche, and any niche-competitor b of c, the punctured niches

$$\begin{array}{c} (b \xrightarrow{?} c,\ c \xrightarrow{u} c') \\ \Big\downarrow {\scriptstyle ?} \\ b \xrightarrow{?} c' \end{array}$$

and

$$\begin{tikzcd}
(c \xrightarrow{u} c',\ b \xrightarrow{?} c) \arrow[d, "?"] \\
b \xrightarrow{?} c'
\end{tikzcd}$$

are balanced.

Proposition 52. *A virtual n-functor is an equivalence if and only if it is essentially surjective and fully faithful.*

Proof - A straightforward consequence of the definitions. $\square$

To further explain the relation between balanced punctured niches and equivalences, we need the following characterization of n-coherent O-algebras. Recall that if O is an S-operad, an O-opetopic set A consists of a set $A(0)$ over S together with an $(O_{A(0)})^+$-opetopic set.

Theorem 53. *Suppose that O is an S-operad. For any $n \geq 1$, an O-opetopic set A is an n-coherent O-algebra if and only if:*

1. *The underlying $(O_{A(0)})^+$-opetopic set of A is an $(n-1)$-coherent $(O_{A(0)})^+$-algebra.*
2. *For any k-ary operation of O, the pullback of A along the resulting operad morphism from F_k to O is a k-ary virtual n-functor.*
3. *Composites of universal 1-cells in A are universal.*

Proof - We denote the underlying $(O_{A(0)})^+$-opetopic set of A by A^-. Suppose that A is an n-coherent O-algebra: in other words, every niche of A has a universal occupant, and composites of universal niche-occupants are universal. One can check using the formalism developed in Section 3.5 that for $m \geq 1$, the m-dimensional frames (resp. cells) of A correspond to the $(m-1)$-dimensional frame (resp. cells) of A^-. The same is also true for openings, niches and punctured niches when $m \geq 2$. Also, the definitions of 'balanced' and 'universal' are set up so that an m-dimensional punctured niche of A is balanced if and only if the corresponding punctured niche of A^- is balanced, and an m-dimensional niche-occupant of A is universal if and only if the corresponding niche-occupant of A^- is universal. Thus 1 holds. Proposition 45 implies 2, and 3 is immediate.

Conversely, suppose that 1, 2, and 3 hold. By 1, for $m \geq 2$ every m-dimensional niche of A has a universal occupant, and composites of m-dimensional universal niche-occupants are universal. The former also holds for $m = 1$ by 2, and the latter holds for $m = 1$ by 3. $\square$

Let O be an S-operad and let A be an n-coherent O-algebra. Given $m \leq n$, we now describe how:

1. Every m-dimensional frame in A determines an $(n-m)$-category.
2. For $m \geq 1$, every m-dimensional opening in A determines a k-ary virtual $(n-m+1)$-functor.
3. For $m \geq 1$, every m-dimensional punctured niche in A determines a virtual $(n-m+1)$-functor.
4. For $m \geq 1$, every m-dimensional niche in A determines a representable $(n-m+1)$-prestack.

For example, when A is an n-category there is a unique 1-dimensional frame for any pair of 0-cells a, b in A, and we denote the corresponding $(n-1)$-category by $\hom(a,b)$.

As in Section 3.5, let

$$S(0) = S, \qquad S(i+1) = \mathrm{elt}(O(i)_{A(i)}),$$

where $O(i)$ is the $S(i)$-operad given by

$$O(0) = O, \qquad O(i+1) = (O(i)_{A(i)})^+.$$

Also let $A^{0-} = A$, and let $A^{(i+1)-}$ be the underlying $O(i+1)$-opetopic set of the $O(i)$-opetopic set A^{i-}. By Theorem 53, A^{i-} is an $(n-i)$-coherent $O(i)$-algebra if $i \leq n$. By remarks in the proof of the theorem, the m-dimensional cells (resp. frames) of A correspond to the $(m-i)$-dimensional cells (resp. frames) of A^{i-} if $i \leq m$, and the same is true for openings, niches, and punctured niches if $i < m$.

Using this 'level-shifting' trick, to deal with 1-4 above it suffices to explain how:

1. Every 0-dimensional frame in A determines an n-category.
2. Every 1-dimensional opening in A determines a k-ary virtual n-functor.
3. Every 1-dimensional punctured niche in A determines a virtual n-functor.
4. Every 1-dimensional niche in A determines a representable n-prestack.

For 1, note that a 0-dimensional frame in A is just an element s of the set S of types of O. This determines a unique operad morphism from I to O riding the function $F\colon 1 \to S$ that sends the one element of 1 to s. The pullback of A under this morphism is the desired n-category.

For 2, recall that a 1-dimensional opening in A is simply an operation of O. As noted in Theorem 53, any k-ary operation o of O determines an operad morphism from F_k to O, and the pullback of A under this morphism is a k-ary virtual n-functor, say

$$G\colon C_1 \times \cdots \times C_k \rightharpoonup C'$$

For 3 and 4, note that if we fix an operation o of O, an o-niche then consists of a choice of one 0-cell from each of the n-categories C_i, while a punctured o-niche consists of a choice of 0-cells from all but one of the C_i. Thus it suffices to explain how to extract a $(k-\ell)$-ary virtual n-functor from G by choosing 0-cells in ℓ of the n-categories C_i. By induction it suffices to consider the case $\ell = 1$, so supposing without loss of generality that we have chosen a 0-cell $c_k \in C_k$, let us construct a $(k-1)$-ary virtual n-functor

$$H\colon C_1 \times \cdots \times C_{k-1} \rightharpoonup C'.$$

By Theorem 53, G gives an $(n-1)$-coherent $((F_k)_{G(0)})^+$-algebra G^-. Concretely, $G(0)$ is the $(k+1)$-tuple of disjoint sets $(C_1(0), \ldots, C_k(0), C'(0))$, where each $C_i(0)$ is the set of 0-cells of the corresponding n-category C_i, and $C'(0)$ is the set of 0-cells of the n-category C'. To construct H, we first construct an $(n-1)$-coherent $((F_{k-1})_{H(0)})^+$-algebra H^-, where $H(0)$ is the k-tuple of disjoint sets $(C_1(0), \ldots, C_{k-1}(0), C'(0))$. Note that there is a unique operad morphism

$$f\colon (F_{k-1})_{H(0)} \to (F_k)_{G(0)}$$

sending each operation with profile $(c_1, \ldots, c_{k-1}, c')$ to the unique operation with profile $(c_1, \ldots, c_k, c')$. This gives an operad morphism

$$f^+\colon ((F_{k-1})_{H(0)})^+ \to ((F_k)_{G(0)})^+,$$

and we define

$$H^- = (f^+)^* G^-.$$

Together with $H(0)$, H^- defines an F_{k-1}-opetopic set H. To see that H is an n-coherent F_{k-1}-algebra, it suffices to check that 1-dimensional niches have universal occupants, and that composites of universal 1-cells are universal. These follow from the corresponding properties for G. Thus H is a $(k-1)$-ary virtual n-functor as desired.

Now we can finish clarifying the relationship between balanced punctured niches and equivalences:

Proposition 54. *Suppose that A is an n-coherent O-algebra. Then an m-dimensional punctured niche in A is balanced if and only if the $(n-m)$-functor it defines is an $(n-m)$-equivalence.*

Proof - Suppose that an m-dimensional punctured o-niche p in A defines the $(n-m)$-functor G. Then one can check that p is balanced if and only if the punctured f-niche of G is balanced, that is, if and only if G is an $(n-m)$-equivalence. $\square$

We conclude by explaining the sense in which a given niche-occupant is universal if and only if any of its niche-competitors factors through it, up to equivalence. Recall that associated to any m-dimensional o-frame

$$(a_1, \ldots, a_k) \xrightarrow{\quad ? \quad} b \ ,$$

in A there is an $(n-m)$-category. We denote this by $\hom_o(a_1, \ldots, a_k, b)$, though when o is an identity operation, we may follow more traditional practice and omit it.

Suppose that in above situation b' is a frame-competitor of b. Then there is an $(n-m)$-category $\hom(b, b')$. Given any 0-cell $x \in \hom_o(a_1, \ldots, a_k, b)$, there are two virtual $(n-m)$-functors

$$x_1^*,\ x_2^*\colon \hom(b, b') \to \hom_o(a_1, \ldots, a_k, b')$$

either one of which we may think of as 'composition with x'. The first is the virtual $(n-m)$-functor determined by the $(m+1)$-dimensional punctured niche in A,

$$\begin{array}{c} ((a_1, \ldots, a_k) \xrightarrow{u} b,\ b \xrightarrow{?} b') \\ \Big\downarrow{\scriptstyle ?} \\ (a_1, \ldots, a_k) \xrightarrow{?} b' \end{array}$$

The second is the one determined by the punctured niche

$$\begin{array}{c} (b \xrightarrow{?} b',\ (a_1, \ldots, a_k) \xrightarrow{u} b) \\ \Big\downarrow{\scriptstyle ?} \\ (a_1, \ldots, a_k) \xrightarrow{?} b' \end{array}$$

We now show that x is universal in its niche if and only if both these are equivalences — i.e., heuristically speaking, all the niche-competitors of x factor through x, up to equivalence.

Proposition 55. *Suppose that A is an n-coherent O-algebra. Let*

$$(a_1, \ldots, a_k) \xrightarrow{\quad x \quad} b$$

be an occupant of an m-dimensional o-niche

$$(a_1, \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

Then x is universal if and only if for any frame-competitor b' of b, the virtual $(n-m)$-functors x_1^* and x_2^* are $(n-m)$-equivalences.

Proof - By definition x is universal if and only the punctured niches corresponding to x_1^* and x_2^* above are balanced, or equivalently, by Proposition 54, if x_1^* and x_2^* are equivalences. $\square$

It is a bit annoying to have two virtual $(n-m)$-functors with an equal claim to being 'composition with x', but it is not very surprising in the present context. In fact we conjecture that x_1^* is an equivalence if and only if x_2^* is.

4.3 The microcosm principle

In Section 2.2 we gave a rough statement of the microcosm principle as follows: *certain algebraic structures can be defined in any category equipped with a categorified version of the same structure.* To make this more precise one needs to work with some particular class of algebraic structures. Since our approach to n-categories is especially suited to studying operad algebras, we work with these.

Recall that for any S-operad O, a 1-coherent O-algebra can be thought of as a categorified analog of an O-algebra. Here we show the following version of the microcosm principle: *O-algebra objects can be defined in any 1-coherent O-algebra.* For example, monoid objects can be defined in any monoidal 1-category, and commutative monoid objects can be defined in any stable 1-category. Another example is the fact that we may define morphisms 'riding' virtual functors. These are simply F_1-algebra objects in 1-coherent F_1-algebras.

More generally, we show that *n-coherent O-algebra objects can be defined in any $(n+1)$-coherent O-algebra.* For example, 'monoidal n-category objects' can be defined in any monoidal $(n+1)$-category, and 'stable n-category objects' can be defined in any stable $(n+1)$-category.

Proposition 56. *Let O be an S-operad. There exists a terminal n-coherent O-algebra τ, that is, one such that for any n-coherent O-algebra A, there is a unique n-coherent O-algebra morphism $f\colon A \to \tau$.*

Proof - Let τ be a terminal O-opetopic set, that is, one having only one cell occupying each frame, and thus one cell for each O-opetope. We prove that τ is an n-coherent O-algebra by showing inductively 'from the top down' that every niche-occupant in τ is universal, so that every niche has a universal occupant and composites of universal cells are universal. It then follows that τ is universal as an n-coherent O-algebra, since is already terminal as an O-opetopic set.

We claim that every occupant of an m-dimensional niche is universal, and every m-dimensional punctured niche is balanced. By the definition of n-coherent O-algebra, both of these are true if $m > n+1$. Supposing they are true for a given m, let us show they hold for $m-1$. Given an $(m-1)$-dimensional punctured niche, condition 1 in the definition of 'balanced' holds because every frame has an occupant, while condition 2 holds by our inductive hypothesis. Similarly, every $(m-1)$-dimensional niche-occupant is universal by our inductive hypothesis. $\square$

Definition 57. *Let O be an S-operad, let A be an $(n+1)$-coherent O-algebra, and let τ be the terminal $(n+1)$-coherent O-algebra. Then we define an* n-coherent O-algebra object in A *to be a morphism of O-opetopic sets $a\colon \tau \to A$. If $n = 0$, we call this simply an* O-algebra object in A.

Since τ has one cell for each O-opetope, we see that an n-coherent O-algebra object in A gives:

1. a 0-cell of A for each type of O
2. a 1-cell of A for each operation of O
3. a 2-cell of A for each reduction law of O
4. a 3-cell of A for each way of combining reduction laws of O to obtain another reduction law

and so on, satisfying certain conditions. We can work out what this amounts to quite explicitly in the case $n = 1$. First we give a 'nuts-and-bolts' description of 1-coherent O-algebras:

Theorem 58. *A 1-coherent O-algebra A consists of:*

1. *for each type x of O, a category $A(x)$*
2. *for each nondegenerate operation g of O with profile $(x_1, \ldots, x_k, x')$, a k-ary virtual functor*
$$A(g)\colon A(x_1) \times \cdots \times A(x_k) \rightharpoonup (x')$$
3. *for each nondegenerate reduction law of O — that is, for each nondegenerate operation G of O^+ with profile $(g_1, \ldots, g_k, g')$ — a natural isomorphism*
$$A(G)\colon G(A(g_1), \ldots, A(g_k)) \to A(g')$$

4. *for each nondegenerate way of combining reduction laws of O to obtain another reduction law — that is, for each operation $\mathcal{G}$ of O^{++} with profile $(G_1, \ldots, G_k, G')$ — an equation*

$$\mathcal{G}(A(G_1), \ldots, A(G_k)) = A(G')$$

Proof - Note from Example 47 that a k-ary virtual functor $F: C_1 \times \cdots \times C_k \rightharpoonup C'$ is a special sort of set-valued functor on $C_1^{\mathrm{op}} \times \ldots \times C_k^{\mathrm{op}} \times C'$, so the concept of 'natural isomorphism' between k-ary virtual functors makes sense. Also, much as in Theorem 31, metatree notation makes it clear how to compose the k_i-ary virtual functors $A(g_i)$ in a tree-like pattern specified by the operation G of O^+ to obtain $G(A(g_1), \ldots, A(g_k))$, and how to compose the natural isomorphisms $A(G_i)$ in a tree-like pattern specified by the operation $\mathcal{G}$ of O^{++} to obtain a natural isomorphism $\mathcal{G}(A(G_1), \ldots, A(G_k))$.

By item 1 of Theorem 53, the 1-coherent O-algebra A has an underlying 0-coherent $(O_{A(0)})^+$-algebra, which we may think of as simply an $(O_{A(0)})^+$-algebra. Theorem 31 implies that such an algebra consists of 1-4 as above, but with a k-ary distributor $A(g)$ for each nondegenerate k-ary operation g of O, and a natural transformation $A(G)$ between k-ary distributors for each nondegenerate operation G of O^+. Item 2 of Theorem 53 implies that the $A(g)$ are k-ary virtual functors, and item 3 of that theorem implies that the $A(G)$ are natural isomorphisms. Conversely, one can show that 1-4 as above give a 1-coherent O-algebra. $\square$

In particular, we see that monoidal 1-categories and stable 1-categories are almost the same as monoidal categories and symmetric monoidal categories, respectively, though there is a bit of work required to translate between our concepts and the traditional ones.

To describe O-algebra objects in the language of Theorem 58, it is convenient to define a *k-ary morphism* $b: c_1 \times \cdots \times c_k \to c'$ *riding* the k-ary virtual functor $B: C_1 \times \cdots \times C_k \to C'$ to be an F_k-algebra object b in the 1-coherent F_k-algebra H. Concretely, this amounts to a choice of objects $c_i \in C_i$ and $c' \in C'$, together with a 0-cell b in $\hom_f(c_1, \ldots, c_k, c')$.

Theorem 59. *Given a 1-coherent O-algebra A, an O-algebra object a in A consists of:*

1. *for each type x of O, an object $a(x)$ in the category $A(x)$*

2. *for each nondegenerate operation g of O with profile $(x_1, \ldots, x_k, x')$, a k-ary morphism $a(g): a(x_1) \times \cdots \times a(x_k) \to a(x')$ riding the k-ary virtual functor $A(g)$*

3. *for each nondegenerate reduction law G of O with profile $(g_1, \ldots, g_k, g')$, an equation*

$$A(G)(a(g_1), \ldots, a(g_k)) = a(g')$$

Proof - This is straightforward except that item 3 may need some clarification. Given ℓ_i-ary morphisms $a(g_i)$ riding the ℓ_i-ary virtual functors $A(g_i)$, and a reduction law G of O with profile $(g_1, \ldots, g_k, g')$, one obtains an ℓ'-ary morphism riding the ℓ'-ary virtual functor $G(A(g_1), \ldots, A(g_k))$. Applying the natural isomorphism $A(G)$ to this we obtain an ℓ'-ary morphism riding $A(g')$, which we call $A(G)(a(g_1), \ldots, a(g_k))$. In item 3 we require this to equal $a(g')$. $\square$

5 Conclusions

In addition to our approach to weak n-categories, there are a number of others. We have already mentioned Street's original simplicial approach [29]. After a sketch of our definition appeared [5], Makkai has begun studying it, and a modified version has been developed by Makkai, Hermida, and Power, but the details of this have not yet been published. Independently, Tamsamani [30] developed an approach using multisimplicial sets: simplicial objects in the category of simplicial objects in the category of simplical objects in the category of ... sets. More recently, Batanin [7] has developed a globular approach to weak ω-categories, and thus in particular weak n-categories, using the notion of an 'ω-operad'. We expect that as time goes by even more definitions will be proposed.

The question thus arises of when two definitions of weak n-category may be considered 'equivalent'. This question was already raised, and a solution proposed, in Grothendieck's 600-page letter to Quillen [19]. Suppose that for all n we have two different definitions of weak n-category, say 'n-category$_1$' and 'n-category$_2$'. Then we should try to construct the $(n+1)$-category$_1$ of all n-categories$_1$ and the $(n+1)$-category$_1$ of all n-categories$_2$ and see if these are equivalent as objects of the $(n+2)$-category$_1$ of all $(n+1)$-categories$_1$. If so, we may say the two definitions are equivalent as seen from the viewpoint of the first definition. Of course, there are some 'size' issues involved here, but they should not be a serious problem. More importantly, there is some freedom of choice involved in constructing the two $(n+1)$-categories$_1$ in question. Also, we would be in an embarrassing position if we got a different answer for the question with the roles of the two definitions reversed. Nonetheless, it should be interesting to compare different definitions of weak n-category in this way.

A second solution is suggested by homotopy theory, where many superficially different approaches turn out to be fundamentally equivalent. Different approaches use objects from different 'model categories' to represent homotopy types: compactly generated topological spaces, CW complexes, Kan complexes, and so on [8, 11]. These categories are not equivalent, but each one is equipped with a class of morphisms playing the role of homotopy equivalences. Given a category C equipped with a specified class of morphisms called 'equivalences', under mild assumptions one can 'localize' C with respect to this class, which amounts to adjoining inverses for these morphisms [17]. The resulting category is called the 'homotopy category' of C. Two categories with specified equivalences may be considered the same for the purposes of homotopy theory if their homotopy categories are equivalent. All the model categories above are the same in this sense.

It is natural to adopt the same attitude in n-category theory. (Indeed, this attitude is also implicit in Grothendieck's letter to Quillen, which was in part inspired by the latter's work on model categories [27].) Thus we propose the following homotopy category of n-categories. We define an n-functor $F: C \to D$ to be an *equivalence* if:

1. Every 0-cell in C is connected to a 0-cell in the image of F by a universal 1-cell.
2. For any 0-cells c, c' in C, the restriction of F to the $(n-1)$-category $\hom(c, c')$ is an equivalence.

where to ground this recursive definition we define equivalences between 0-categories to be bijections, using the identification of 0-categories with sets. Condition 1 above says that F is 'essentially surjective', while condition 2 says that F is 'fully faithful'. We then define the following category:

Definition 60. *The* homotopy category of n-categories *is the localization of the category of n-categories and n-functors with respect to the equivalences.*

We regard any other definition of n-category as fundamentally 'the same' as ours if it gives an equivalent homotopy category of n-categories.

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