Operads in Algebra, Topology and Physics
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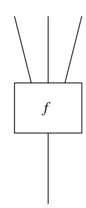
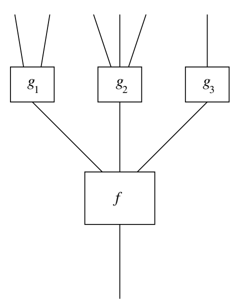
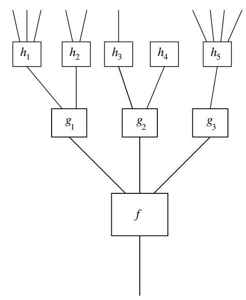
Operads are powerful tools, and this is the book to read about them. However, if you're like most mathematicians, your first question will be: *what is an operad?* Luckily, the answer is simple. An operad O consists of a set O_n of abstract 'n-ary operations' for each n, together with rules for composing these operations. We can think of an n-ary operation as a little black box with n wires coming in and one wire coming out:



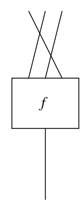
We are allowed to compose these operations as follows:



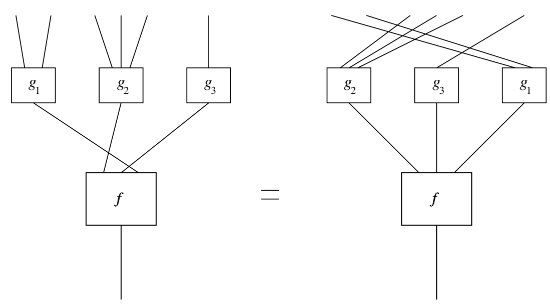
feeding the outputs of the operations $g_1, \ldots, g_n$ into the inputs of the n-ary operation f, and obtaining a new operation which we call $f \circ (g_1, \ldots, g_n)$. We demand that there be a unary operation serving as the identity for composition, and impose an 'associative law' that makes a composite of composites like this well-defined:



We can permute the inputs of an n-ary operation f and get a new operation:



which we call $f\sigma$ if σ is the the permutation of the inputs. We demand that this give a right action of each permutation group S_n on each set O_n. Finally, we demand that these actions be compatible with composition. For example:



That's all! Unfortunately, if one writes these definitions using equations, they look quite formidable. There must be hundreds of mathematicians roaming the earth who think that operads are difficult and abstruse, because they've seen the definitions without any pictures. These people should be grateful that they weren't taught how to tie their shoes in an equally wrong-headed manner.

With this answered, your next question is probably: *why should I care about these things?* There are many reasons, but historically, the first comes from topology. In homotopy theory, the main way to probe a space X is by looking at maps $f : S^k \to X$. We define the 'kth loop space' of X, $\Omega^k X$, to be the space of all such maps sending the north pole to a chosen point $* \in X$. The set of connected components of $\Omega^k X$ is called the 'kth homotopy group' of X; this is a group for $k > 0$ and an abelian group for $k > 1$.

Most homotopy theorists would gladly sell their souls for the ability to compute the homotopy groups of an arbitrary space. However, there is extra information lurking in the space $\Omega^k X$ that gets lost when we consider only its connected components. Starting in the late 1950s, a large number of excellent topologists including Adams and MacLane, Stasheff, Boardman and Vogt, and May struggled to understand *all* the structure possessed by an k-fold loop space. For example, $\Omega^1 X$ is something like a topological group, thanks to our ability to 'compose' loops. However, the usual group laws such as associativity hold only up to homotopy. To make matters even trickier, these homotopies satisfy certain laws of their own, but only up to homotopy — and so on *ad infinitum.* Similarly, $\Omega^k X$ is something like an abelian topological group for $k > 1$, but again only up to homotopies that themselves satisfy certain laws up to homotopy, and so on — and in a manner that gets ever more complicated for higher k.

After more than decade of hard work, it became clear that operads are the easiest way to organize all these higher homotopies. Just as a group can act on a set, so can an operad O, each abstract operation $f \in O_n$ being realized as actual n-ary operation on the set in a manner preserving composition, the identity, and the permutation group actions. A set equipped with an action of the operad O is usually called an 'algebra over O'. It turns out that the structure of a k-fold loop space is completely captured by saying that it is an algebra over a certain operad! Even better, if we choose this operad O to be 'cofibrant', any space equipped with a homotopy equivalence to a k-fold loop space will also become an algebra over O. This is the simplest example of how operads are used to describe 'homotopy invariant algebraic structures', in which all laws hold up to an infinite sequence of higher homotopies.

For an operad to do this job, it must really have a *topological space* of operations O_n for each n, since the fact that various laws hold up to homotopy is expressed by the existence of certain continuous paths in these spaces. Similarly, composition and the permutation group actions should be *continuous maps.* Finally, we should only consider algebras that are topological spaces on which the operad acts continuously.

In short, topology really requires operads and their algebras in the category of topological spaces rather than sets. The ability to transplant the theory of operads to various different categories is an important aspect of their power. After a long pedagogical warmup, the authors of this book wisely treat operads in an arbitrary symmetric monoidal category. They also prove the worth of this level of generality by discussing many examples in detail. For example, they describe how operads in the category of chain complexes have been used to study deformation quantization — and also string theory, where the operations of gluing together Riemann surfaces are important. Indeed, these physics applications have led to a kind of renaissance in the theory of operads!

The plan of the book is as follows. Part I gives a nice introductory tour of operads and their applications. Part II starts by describing operads in a symmetric monoidal category, focusing on their relation to trees and introducing the associahedron operad as a key example. The authors then consider applications to homotopy theory, leading up to a general theory of homotopy-invariant algebraic structures. The next section treats a variety of algebraic topics including the theory of 'quadratic operads', which are operads in the category of vector spaces generated by binary operations satisfying ternary relations. Three classic examples are the operads whose algebras are associative algebras, commutative algebras and Lie algebras. The authors describe a (co)homology theory for the algebras of any quadratic operad, which generalizes Hochschild (co)homology for assocative algebras, Harrison (co)homology for commutative algebras, and the usual (co)homology of Lie algebras. Then the authors turn to operads coming from geometry, particularly certain moduli spaces and configuration spaces. Finally, they discuss some generalizations including cyclic operads, which show up in cyclic cohomology, and modular operads, which show up in closed string field theory. The book does not cover the applications of operads to the theory of n-categories. Luckily, Tom Leinster is writing a book on that subject. For all other aspects of operad theory, this book by Markl, Shnider and Stasheff is the place to start.