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Renormalized Oscillator Hamiltonians

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Let p_j, q_j, denote the self-adjoint operators $-i\partial/\partial x_j$ and multiplication by x_j on $L^2(\mathbb{R}^n)$, respectively. Let $H_0 = (1/2)\sum_{j=1}^n \omega_j(p_j^2 + q_j^2 - 1)$ where $\omega_j > 0$, and let P be a polynomial in the q's that is bounded below. The existence and uniqueness problems are considered for $H_0 + \lambda P(q)$ renormalized relative to its own ground state. For example, suppose that P has degree $2d$ and $P(q) \geqslant \varepsilon(q_1^2 + \cdots + q_n^2)^d - k$ for some $\varepsilon, k > 0$. Then for some $\lambda_0 > 0$ there exists a unique continuous function $u\colon [0, \lambda_0] \to L^2(\mathbb{R}^n)$ such that $u(\lambda)$ is the nonnegative normalized ground state of $H_0 + \lambda :P(q):_{u(\lambda)}$ and $\langle u(\lambda), P(q)\, u(\lambda)\rangle$ is bounded on $[0, \lambda_0]$. If u is continuous only on $(0, \lambda_0]$ uniqueness may fail. © 1992 Academic Press, Inc.

1. Introduction

Let us recall some of the logical development of constructive quantum field theory, summarizing the more thorough account given in [1]. The objects of study of quantum field theory were originally presumed to be operator-valued functions on space-time, $\mathbb{R}^4$, satisfying nonlinear wave equations such as

$$(\Box + m^2)\,\phi + \lambda\phi^3 = 0,$$

together with the canonical commutation relations

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = [\dot{\phi}(t, \mathbf{x}), \dot{\phi}(t, \mathbf{y})] = 0,$$
$$[\phi(t, \mathbf{x}), \dot{\phi}(t, \mathbf{y})] = i\delta(x - y).$$

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Of course, the appearance of the Dirac delta in the commutation relations implies that ϕ is not a true operator-valued function on $\mathbb{R}^4$, but a generalized function of some kind. This immediately makes the interpretation of nonlinear functions of ϕ problematic.

Thus in practice one begins by considering a "free field" satisfying a linear wave equation such as the Klein–Gordon equation

$$(\Box + m^2)\,\phi_0 = 0.$$

For simplicity let us consider this equation on $\mathbb{R} \times S^1$. The free field ϕ_0 is a distribution with values in self-adjoint operators on a Hilbert space $\mathbf{K}$; it satisfies the canonical commutation relations and the Klein–Gordon equation, with time evolution generated by a nonnegative self-adjoint operator H_0 on $\mathbf{K}$,

$$e^{itH_0}\phi_0(0, x)\,e^{-itH_0} = \phi_0(t, x).$$

The "free Hamiltonian" H_0 has a nondegenerate ground state v_0, the "free vacuum." There exist "Wick" or "normal-ordered" powers of ϕ_0, operator-valued distributions $:\phi_0^n(t, \cdot):$ on S^1 for any t, that satisfy a nonlinear generalization of the canonical commutation relations

$$[:\phi_0^n(t, x):, \phi_0(t, y)] = 0,$$
$$[:\phi_0^n(t, x):, \dot{\phi}_0(t, y)] = in\delta(x-y)\,:\phi_0^{n-1}(t, x):,$$

and have vanishing expectation values relative to the free vacuum

$$\langle v_0, :\phi_0^n(t, x):\, v_0\rangle = 0.$$

Here these equations are to be interpreted in a distributional sense, i.e., they hold upon integrating against smooth functions of $x, y \in S^1$. The Wick squares of $\nabla\phi_0$ and $\dot{\phi}_0$ are also well-defined, and the free Hamiltonian H_0 may be expressed as

$$H_0 = \tfrac{1}{2}\int_{t=0} \{:(\nabla\phi_0)^2: + m^2 :\phi_0^2: + :\dot{\phi}_0^2:\}\,dx.$$

One can proceed to treat nonlinear quantum field equations as follows. Formally, the Hamiltonian for the "interacting field" ϕ satisfying

$$(\Box + m^2)\phi + p(\phi) = 0$$

for some polynomial p is given by

$$\int_{t=0} \tfrac{1}{2}\{(\nabla\phi)^2 + m^2\phi^2 + \dot{\phi}^2\} + P(\phi)\,dx,$$

where P is an antiderivative of p. To make sense of this expression one may use the free field ϕ_0 and its Wick powers, thus interpreting it as

$$\int_{t=0} \tfrac{1}{2}\{:(\nabla \phi_0)^2: + m^2 :\phi_0^2: + :\dot{\phi}_0^2:\} + :P(\phi_0): dx = H_0 + \int_{t=0} :P(\phi_0): dx.$$

It is important, however, to realize that this *ad hoc* manuever is unnatural in at least three respects: (1) There is no *a priori* reason to substitute the free field ϕ_0 into the Hamiltonian for the interacting field. (2) The Wick powers of ϕ_0 are defined in terms of the free vacuum, which has no intrinsic relation to the nonlinear quantum field equation. (3) Integrating over the surface $t=0$ is a purely arbitrary choice, and any other value of t would lead to a different operator.

It is thus somewhat remarkable that anything useful comes of this approach. As it turns out, there is a well-defined operator

$$V = \int_{t=0} :P(\phi_0): dx,$$

for any polynomial P, and if P is bounded below then $H_0 + V$ is essentially self-adjoint. Let H denote the closure of $H_0 + V$. Defining ϕ by

$$\phi(t, \mathbf{x}) = e^{itH}\phi_0(0, \mathbf{x})\, e^{-itH},$$

it is natural to ask whether ϕ satisfies the differential equation

$$(\Box + m^2)\,\phi + :p(\phi): = 0.$$

It does not, but it satisfes a *different* nonlinear wave equation. The Hamiltonian H has a unique ground state v, and one can define renormalized products relative to v with properties analogous to those of the the Wick product. For example, there are operator-valued distributions $:\phi^n(t, x):_v$ satisfying the commutation relations

$$[:\phi^n(t, x):_v, \phi(t, y)] = 0,$$
$$[:\phi^n(t, x):_v, \dot{\phi}(t, y)] = in\delta(x-y) :\phi^{n-1}(t, x):_v,$$

with vanishing expectation values relative to the state v,

$$\langle v, :\phi^n(t, x):_v v\rangle = 0.$$

The Hamiltonian H may be rewritten in terms of these renormalized products as

$$H = \text{closure of} \int_{t=0} \tfrac{1}{2}\{:(\nabla \phi)^2:_v + m^2 :\phi^2:_v + :\dot{\phi}^2:_v\} + :\tilde{P}(\phi):_v\, dx$$

where $\tilde{P}$ is a polynomial different from P, but with the same leading term.

It can then be shown that ϕ satisfies the nonlinear wave equation

$$(\Box + m^2)\,\phi + :\tilde{p}(\phi):_v = 0,$$

where $\tilde{p}$ is the derivative of $\tilde{P}$. We have thus solved a nonlinear quantum field equation different from the one with which we began. However, the three objections raised above may now be answered: (1) The Hamiltonian H for the interacting field ϕ is defined in terms of ϕ itself. (2) The renormalized products appearing in H are defined in terms of the ground state v of H itself, not the free vacuum. (3) The self-adjoint closure of the integral

$$\int_{t=t_0} \tfrac{1}{2}\{(:\nabla\,\phi)^2:_v + m^2 :\phi^2:_v + :\dot{\phi}^2:_v\} + :\tilde{P}(\phi):_v\,dx$$

can be seen to be independent of t_0 as a consequence of (1) and (2).

It remains to consider the existence and uniqueness, for a given polynomial P, of an operator of the form

$$H = \text{closure of} \int_{t=0} \tfrac{1}{2}\{:(\nabla\,\phi)^2:_v + m^2 :\phi^2:_v + :\dot{\phi}^2:_v\} + :P(\phi):_v\,dx$$

renormalized relative to its own ground state v. These are nonlinear problems of a distinctive sort, in which one simultaneously solves for an operator and its ground state in terms of a relation between them. Here we investigate the corresponding problems for perturbations of harmonic oscillator Hamiltonians by polynomial potentials that are bounded below. These arise as Hamiltonians of quantum field theories that have been cut off to a finite number of modes. We will treat the case of infinitely many degrees of freedom in a future paper.

These problems, and a plan of attack, were proposed by Irving Segal [7], whom we thank for many useful discussions. Some of our results are extensions of the work of Friedman [2].

2. Preliminaries

Let $\mathbf{H}$ be a complex Hilbert space. The *free boson field* over $\mathbf{H}$ is a system $(\mathbf{K}, W, \Gamma, v_0)$, unique up to unitary equivalence, such that: (1) $\mathbf{K}$ is a complex Hilbert space; (2) W is a strongly continuous map from $\mathbf{H}$ to unitaries on $\mathbf{K}$ satisfying the Weyl relations

$$W(x)\,W(y) = e^{i\,\mathrm{Im}\langle x,y\rangle/2}\,W(x+y);$$

(3) Γ is a strongly continuous unitary representation of $U(\mathbf{H})$ on $\mathbf{K}$ such that

$$\Gamma(T)\, W(x)\, \Gamma(T)^{-1} = W(Tx)$$

for all $T \in U(\mathbf{H})$ and $x \in \mathbf{H}$, and $d\Gamma(A) \geqslant 0$ for all positive self-adjoint A on $\mathbf{H}$; (4) v_0, the *free vacuum*, is a unit vector in $\mathbf{K}$ invariant under Γ and cyclic for W.

Let κ be a conjugation on $\mathbf{H}$, and let $\mathbf{H}_\kappa$ be the real part of $\mathbf{H}$ fixed by κ. Define $U(x)$ and $V(x)$ for $x \in \mathbf{H}_\kappa$ by

$$U(x) = W(x), \qquad V(x) = W(ix).$$

Then (U, V) is a *Weyl pair*, i.e., a pair of strongly continuous maps from $\mathbf{H}_\kappa$ to $U(\mathbf{K})$ satisfying

$$U(x)\, U(y) = U(x+y), \qquad V(x)\, V(y) = V(x+y),$$
$$U(x)\, V(y) = e^{i\langle x, y\rangle}\, V(y)\, U(x).$$

Throughout this paper we assume that $\mathbf{H}$ is finite-dimensional. Fix a conjugation κ on $\mathbf{H}$ and choose a basis $\{e_i\}$ for $\mathbf{H}_\kappa$ so as to identify it with $\mathbb{R}^n$, with the usual coordinates x_i. In the Schrödinger representation of the free boson field $\mathbf{K}$ is then identifield with $L^2(\mathbb{R}^n)$ (relative to Lebesgue measure), U and V are given by

$$U(x) : f(u) \to f(u+x), \qquad V(x) : f(u) \to e^{iux} f(u),$$

and the free vacuum v_0 is given by

$$v_0(x) = \pi^{-1/4} e^{-\langle x, x\rangle/2}.$$

Let p_j denote the self-adjoint generator of the one-parameter group $U(te_j)$, and let q_j denote the self-adjoint generator of $V(te_j)$. In the Schrödinger representation p_j corresponds to the operator $-i\partial/\partial x_j$, while q_j corresponds to the operator of multiplication by x_j. Let A be the self-adjoint operator on $\mathbf{H}$ given by

$$Ae_j = \omega_j e_j,$$

where $\omega_1, ..., \omega_n > 0$. Let H_0, the *free Hamiltonian*, denote

$$d\Gamma(A) = \tfrac{1}{2} \sum_{j=1}^{n} \omega_i (p_j^2 + q_j^2 - 1).$$

Let **P** denote the vector space of real-valued polynomials of degree $\leqslant 2d$ on $\mathbb{R}^n$, where $d \geqslant 1$. Equip **P** with the vector space topology, so that $P_\alpha \to P$ if and only if all the coefficients of P_α converge to those of P. Let $\mathbf{C} \subset \mathbf{P}$ denote the convex cone consisting of those elements that are bounded below as functions on $\mathbb{R}^n$. While **C** is not closed, it does contain the origin.

The following facts about operators of the form $H_0 + P(q)$ are well-known [5, 7, 8, 10]. For any $P \in \mathbf{C}$, the operator $H_0 + P(q)$ is essentially self-adjoint; let H_P denote its self-adjoint closure. The operator H_P is bounded below, has pure point spectrum, and has a unique nonnegative normalized lowest eigenvector, or *vacuum* for short, which we denote by $v(P)$. Define $E(P) \in \mathbb{R}$ by

$$H_P v(P) = E(P)\, v(P).$$

Note that $E(0) = 0$ and $v(0) = v_0$.

Renormalized products of the p's and q's relative to the free vacuum v_0 are often called 'Wick" or "normal-ordered" products, and they have a simple description in terms of reordering annihilation and creation operators. Renormalized products relative to quite general states have been developed in a series of papers by Segal [7]; an exposition is also given in [1]. In the present context it suffices to introduce renormalized products relative to states in the space $D^\infty(W)$ of C^∞ vectors for the Weyl system W. In the finite-dimensional case at hand, $D^\infty(W)$ is precisely the space of C^∞ vectors for H_0, and in the Schrödinger representation it corresponds to the Schwartz space on $\mathbb{R}^n$. Let **W** denote the infinitesimal Weyl algebra over **H**, that is, the associative algebra with unit generated by $\{p_j, q_j\}_{j=1}^n$ with the relations

$$[p_j, p_k] = [q_j, q_k] = 0, \qquad [p_j, q_k] = i^{-1}\delta_{jk}.$$

By the above remarks, the algebra **W** has a natural representation on $D^\infty(W)$. A *monomial* in **W** is an element of the form $z_1 \cdots z_k$, where $z_1, ..., z_k \in \mathbf{H}$. Given $v \in D^\infty(W)$ there is a unique *renormalization map* from monomials in **W** to **W**, denoted $:\ :_v$, such that

$$\langle v, {:}z_1 \cdots z_k{:}_v\, v \rangle = 0$$

and

$$[z, {:}z_1 \cdots z_k{:}_v] = \sum_{i=1}^{k} [z, z_i]\, {:}z_1 \cdots \hat{z}_i \cdots z_k{:}_v$$

for all $z, z_1, ..., z_k \in \mathbf{H}$. The element ${:}z_1 \cdots z_k{:}_v$ is called the *renormalized product* of $z_1, ..., z_k$ relative to v. The renormalization map extends uniquely to a linear transformation on the subalgebra of $\mathbf{W}$ generated by $\{q_j\}_{j=1}^n$; we also denote this extension by $: :_v$. We write the renormalization map relative to the free vacuum v_0 simply by $: :$.

To apply the above definition of renormalized products to the vacua $v(P)$, note the following:

LEMMA 1. *For all* $P \in \mathbf{C}$, $v(P) \in D^\infty(W)$.

Proof. We shall show that any eigenvector of H_P lies in $D^\infty(W)$. Suppose $u \in \mathbf{K}$ has $H_P u = \lambda u$. Working with the Schrödinger representation, it follows from elliptic regularity that $u \in C^\infty(\mathbb{R}^n)$, and it suffices to show that u is in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. We will prove by induction that for all multi-indices I and all $p > 0$, there exists $c > 0$ such that

$$|\partial^I u(x)| \leqslant c(|x|+1)^{-p}. \tag{1}$$

It is known [9] that for all $p > 0$ there exists $c > 0$ such that $|u(x)| \leqslant c(|x|+1)^{-p}$. Suppose that the induction hypothesis (1) holds for all I with $|I| \leqslant k$. Let Δ denote the operator $\sum \omega_i p_i^2$, which is the (non-negative) Laplacian for some flat metric on $\mathbb{R}^n$. By the induction hypothesis, $\Delta \partial^I u = f$ for some $f \in C^\infty(\mathbb{R}^n)$ such that for all $p > 0$ there exists $c > 0$ with $|f(x)| \leqslant c(|x|+1)^{-p}$. Taking x_0 with $\|x - x_0\| \leqslant 1/2$, we have

$$\partial^I u(x) = \int_{\|x_0 - y\| \leqslant 1} G(x,y) f(y)\, d^n y + \int_{\|x_0 - y\| = 1} \frac{\partial G(x,y)}{\partial n} \partial^I u(y)\, d^{n-1} y,$$

where $G(x, y)$ is the Green's function for Δ on the ball $\|x_0 - y\| \leqslant 1$ with $G(x, y) = 0$ for y on the boundary. We may differentiate this to obtain

$$\partial^i \partial^I u(x) = \int_{\|x_0 - y\| \leqslant 1} \frac{\partial G(x,y)}{\partial x_i} f(y)\, d^n y + \int_{\|x_0 - y\| = 1} \frac{\partial^2 G(x,y)}{\partial x_i\, \partial n} \partial^I u(y)\, d^{n-1} y,$$

noting from explicit formulas [3] that the function $\partial G(x, \cdot)/\partial x_i$ is in L^1_{loc}. It follows that for all $p > 0$ there exists $c > 0$ such that $|\partial^i \partial^I u(x)| \leqslant c(|x|+1)^{-p}$. Thus the induction hypothesis (1) holds for all I with $|I| \leqslant k+1$. ∎

3. Renormalizing the Interaction Hamiltonian

In this section we consider the following problem:

PROBLEM 1. *Given* P *in* $\mathbf{C}$, *find* $u \in \mathbf{K}$ *such that the vacuum of the closure of* $H_0 + :P(q):_u$ *is* u.

We deal with the troublesome circularity of this problem as follows. Given $P \in \mathbf{C}$, let $v(P)$ be the vacuum of the closure of $H_0 + P$. For some $\tilde{P} \in \mathbf{C}$ we have $P(q) = :\tilde{P}(q):_{v(P)}$. Thus $v(P)$ is the vacuum of the closure of $H_0 + :\tilde{P}(q):_{v(P)}$, so $v(P)$ solves Problem 1 for $\tilde{P}$. It is easy to see that all solutions of Problem 1 are of this form, but it is difficult to determine precisely which polynomials $\tilde{P}$ arise in this manner. However, we shall show that the differential of the mapping $P \mapsto \tilde{P}$ at $0 \in \mathbf{C}$ is the identity, so that by the implicit function theorem there exists a solution to Problem 1 for all "sufficiently small" polynomials in the interior of $\mathbf{C}$. More precisely, we have:

THEOREM 1. *Suppose that $P \in \mathbf{P}$ satisfies*

$$P(q) \geqslant \varepsilon(q_1^2 + \cdots + q_n^2)^d - k$$

for some $\varepsilon, k > 0$. Then for some $\lambda_0 > 0$, there is a unique map u: $[0, \lambda_0] \to \mathbf{K}$ such that:

1. *$u(\lambda)$ is the vacuum of $H_0 + \lambda :P:_{u(\lambda)}$.*
2. *u is norm-continuous from $[0, \lambda_0]$ to $\mathbf{K}$.*
3. *$\langle u(\lambda), P(q)\, u(\lambda)\rangle$ is bounded for $\lambda \in [0, \lambda_0]$.*

Proof. Suppose $P \in \mathbf{C}$. By Lemma 1, $v(P) \in D^\infty(W)$, so the renormalization map relative to $v(P)$ is a well-defined linear transformation of $\mathbf{P}$. If $I = (i_1, ..., i_n)$ is a multi-index, or n-tuple of nonnegative integers, let q^I denote the polynomial $q_1^{i_1} \cdots q_n^{i_n}$, and let $|I| = i_1 + \cdots + i_n$. The q^I with $|I| \leqslant 2d$ forms a basis for $\mathbf{P}$. By the general theory of renormalized products [1, 6], $:q^I:_{v(P)} - q^I$ is a polynomial of degree less than $|I|$. The renormalization map relative to $v(P)$ thus can be written in upper triangular form with 1's on the diagonal. Thus there exists a unique $\tilde{P} \in \mathbf{C}$ such that

$$P(q) = :\tilde{P}(q):_{v(P)}.$$

Define the map $T: \mathbf{C} \to \mathbf{C}$ by $T(P) = \tilde{P}$.

We begin the proof with some lemmas on the dependence of $v(P)$ and $E(P)$ on $P \in \mathbf{C}$. We will develop only the bare minimum of properties needed, following well-established techniques (see the references cited above). One technical problem is that there exist sequences $P_j \in \mathbf{C}$ such that $P_j \to 0$ but $v(P_j) \not\to v_0$. One can avoid this problem by working with a slightly smaller cone. Let $\mathbf{C}_0$ be any open convex cone contained in $\mathbf{C}$.

LEMMA 2. *For any $P \in \mathbf{C}_0$ there exist $\varepsilon, k > 0$ such that*

$$P(q) \geqslant \varepsilon(q_1^2 + \cdots + q_n^2)^d - k.$$

Proof. The polynomial $Q(q)=(q_1^2+\cdots+q_n^2)^d$ is in $\mathbf{P}$. Since P is in the open cone $\mathbf{C}_0$, for some $\varepsilon>0$ we have $P-\varepsilon Q\in\mathbf{C}_0$, so that $P-\varepsilon Q$ is bounded below. ∎

Let $\mathbf{D}=D(H_0)\cap D((q_1^2+\cdots+q_n^2)^d)$, which is a Hilbert space with the norm

$$\|\psi\|_{\mathbf{D}}^2=\|H_0\psi\|^2+\|(q_1^2+\cdots+q_n^2)^d\psi\|^2.$$

LEMMA 3. *For all $P\in\mathbf{C}_0$, $D(H_P)=\mathbf{D}$.*

Proof. We follow the method of [8]. As operators on $C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}(H_0+P(q))^2&=H_0^2+P(q)^2+H_0P(q)+P(q)\,H_0\\&=H_0^2+P(q)^2+\sum_i\omega_i\left(2P(q)\,q_i^2-\frac{\partial^2P}{\partial q_i^2}(q)+2p_iP(q)\,p_i\right)\\&\geqslant H_0^2+P(q)^2-\sum\omega_i\frac{\partial^2P}{\partial q_i^2}(q).\end{aligned}$$

By Lemma 1, for any $a>1$ there exists $b>0$ such that

$$P(q)^2-\sum\omega_i\frac{\partial^2P}{\partial q_i^2}(q)>a^{-1}(P(q)^2-b).$$

Thus for $\psi\in C_0^\infty(\mathbb{R}^n)$,

$$\|H_0\psi\|^2+\|P(q)\,\psi\|^2\leqslant a\|H_P\psi\|^2+b\|\psi\|^2. \tag{2}$$

Taking limits, the inequality follows for all $\psi\in\mathbf{D}$. By Lemma 2, $D(H_0)\cap D(P(q))=\mathbf{D}$. Clearly $D(H_0)\cap D(P(q))\subseteq D(H_P)$, and (2) implies the reverse inclusion. ∎

LEMMA 4. *$v(P)\in\mathbf{D}$ and $E(P)\in\mathbb{R}$ are real-analytic functions of P in $\mathbf{C}_0$.*

Proof. By Lemma 2, if $Q\in\mathbf{P}$ is sufficiently close to the origin, $Q(q)$ is bounded relative to H_P with relative bound <1. Since $E(P)$ is an isolated nondegenerate eigenvalue of H_P, Kato's theory of type A perturbations [4] implies that $v(P)$ is analytic from $\mathbf{C}_0$ to $\mathbf{K}$ and $E(P)$ is analytic from $\mathbf{C}_0$ to $\mathbb{R}$. (While Kato's theory is formulated in terms of one-parameter families of perturbations, the relevant results are easily seen to extend to many-parameter families.)

Next we show that $v(P)$ is actually norm-analytic as a function from $\mathbf{C}_0$ to $\mathbf{D}$. Suppose $P\in\mathbf{C}_0$. The operator H_{P+Q} is bounded from $\mathbf{D}$ to $\mathbf{K}$, and norm-analytic from $\mathbf{D}$ to $\mathbf{K}$ as a function of $Q\in\mathbf{P}$. For sufficiently large

$c>0$, $(H_P+c)^{-1}$ exists and by the estimate (2) is bounded from $\mathbf{K}$ to $\mathbf{D}$. Thus $(H_{P+Q}+c)^{-1}: \mathbf{K} \to \mathbf{D}$ is bounded and norm-analytic in Q for Q in a neighborhood of $0 \in \mathbf{P}$. Since

$$v(P+Q) = (E(P+Q)+c)(H_{P+Q}+c)^{-1}\, v(P+Q),$$

it follows from the above that $v(P+Q) \in \mathbf{D}$ is norm-analytic as a function of Q for Q near 0. ▮

LEMMA 5. *As $P \to 0$ in $\mathbf{C}_0$, $E(P) \to 0$ and $v(P) \to v_0$ in $\mathbf{D}$.*

Proof. It follows from results of [10] that $E(P) \to 0$ and

$$v(P) \to v_0, \qquad (q_1^2 + \cdots + q_n^2)^d\, v(P) \to (q_1^2 + \cdots + q_n^2)^d\, v_0,$$

in norm in $\mathbf{K}$ as $P \to 0$. Thus it suffices to show that $H_0 v(P) \to H_0 v_0 = 0$ as $P \to 0$. This is a consequence of the relation

$$H_0 v(P) = -Pv(P) + E(P)\, v(P). \quad ▮$$

LEMMA 6. *Suppose $Q \in \mathbf{P}$. Then the directional derivatives*

$$\left.\frac{d}{d\varepsilon} E(P+\varepsilon Q)\right|_{\varepsilon=0}$$

and

$$\left.\frac{d}{d\varepsilon} v(P+\varepsilon Q)\right|_{\varepsilon=0}$$

converge, the latter in the norm topology on $\mathbf{K}$, *as $P \to 0$ in $\mathbf{C}_0$.*

Proof. By Lemma 4, $E(P) \in \mathbb{R}$ and $v(P) \in \mathbf{D}$ are differentiable functions in $\mathbf{C}_0$. Thus we may differentiate the relationship

$$H_P v(P) = E(P)\, v(P),$$

obtaining

$$(H_P - E(P)) \left.\frac{d}{d\varepsilon} v(P+\varepsilon Q)\right|_{\varepsilon=0} = \left(\left.\frac{d}{d\varepsilon} E(P+\varepsilon Q)\right|_{\varepsilon=0} - Q\right) v(P). \tag{3}$$

Taking the inner product of both sides with $v(P)$, it follows that

$$\left.\frac{d}{d\varepsilon} E(P+\varepsilon Q)\right|_{\varepsilon=0} = \langle v(P), Qv(P)\rangle, \tag{4}$$

so as $P \to 0$,

$$\frac{d}{d\varepsilon} E(P+\varepsilon Q)\bigg|_{\varepsilon=0} \to \langle v_0, Qv_0 \rangle$$

by Lemma 5.

Equations (3) and (4) imply

$$(H_P - E(P)) \frac{d}{d\varepsilon} v(P+\varepsilon Q)\bigg|_{\varepsilon=0} = (\langle v(P), Qv(P) \rangle - Q)\, v(P). \tag{5}$$

Next note that

$$\frac{d}{d\varepsilon} v(P+\varepsilon Q)\bigg|_{\varepsilon=0}$$

is orthogonal to $v(P)$, hence to the kernel of $H_P - E(P)$. Since the resolvent of H_P converges to that of H_0 in norm away from the spectrum of H_0 as $P \to 0$ [10], and the right hand side of (5) converges to $(\langle v_0, Qv_0 \rangle - Q)\, v_0$ as $P \to 0$ by Lemma 5, it follows that

$$\frac{d}{d\varepsilon} v(P+\varepsilon Q)\bigg|_{\varepsilon=0} \to \tilde{H}_0^{-1}(\langle v_0, Qv_0 \rangle - Q)\, v_0$$

as $P \to 0$, where $\tilde{H}_0^{-1}$ is the inverse of the operator $\tilde{H}_0$ that is the restriction of H_0 to the space of vectors orthogonal to v_0. ∎

LEMMA 7. *Suppose $Q \in \mathbf{P}$ and $|I| \leqslant 2d$. The directional derivative*

$$\frac{d}{d\varepsilon} \langle v(P+\varepsilon Q), q^I v(P+\varepsilon Q) \rangle \bigg|_{\varepsilon=0}$$

converges as $P \to 0$ in $\mathbf{C}_0$.

Proof. One notes that

$$\frac{d}{d\varepsilon} \langle v(P+\varepsilon Q), q^I v(P+\varepsilon Q) \rangle \bigg|_{\varepsilon=0} = 2\mathrm{Re} \left\langle \frac{d}{d\varepsilon} v(P+\varepsilon Q), q^I v(P) \right\rangle,$$

and it follows from Lemmas 5 and 6 that the right side converges as $P \to 0$ in $\mathbf{C}_0$. ∎

Next we prove a modified version of the implicit function theorem, dealing with a map defined on a cone, rather than a neighbourhood of a point. Then we show using the above lemmas that T satisfies the hypotheses of this implicit function theorem.

LEMMA 8. *Let C be a convex open cone in a Banach space V. Suppose that $F\colon C \to V$ is C^1, and $F(x) \to 0$ and $\|dF(x) - I\| \to 0$ as $x \to 0$. If u is in C, then for some $\lambda_0 > 0$ there is a unique continuous function $y\colon [0, \lambda_0] \to C \cup \{0\}$ such that*

$$F(y(\lambda)) = \lambda u$$

for $\lambda > 0$, and $y(0) = 0$.

Proof. First we prove existence. We assume without loss of generality that $\|u\| = 1$. Choose $d > 0$ such that $\|x - u\| < d$ implies $x \in C$. By hypothesis there is a monotone increasing continuous function f with $f(0) = 0$ such that

$$\|dF(x) - I\| \leqslant f(\|x\|) \tag{6}$$

for small enough x in C. Choose $\lambda_0 > 0$ such that

$$f(2\lambda_0) \leqslant \min(1/4, d/2). \tag{7}$$

Let

$$X = \{x \in V\colon \|x\| \leqslant 2\lambda, \|x - \lambda u\| \leqslant \lambda d\}.$$

Note that by the choice of the constant d, $X \subseteq C$. Define $G\colon X \to V$ by

$$G(x) = x - F(x) + \lambda u.$$

We claim that G maps X into X contractively for $\lambda \in (0, \lambda_0]$; the fixed point $y(\lambda)$ will then satisfy $F(y(\lambda)) = \lambda u$.

Assume $x \in X$. Then using (6) and (7) we have

$$\begin{aligned} \|G(x)\| &\leqslant \|x - F(x)\| + \lambda \leqslant f(\|x\|)\,\|x\| + \lambda \\ &\leqslant 2\lambda f(2\lambda) + \lambda \leqslant 2\lambda, \end{aligned}$$

and similarly

$$\|G(x) - \lambda u\| = \|x - F(x)\| \leqslant 2\lambda f(2\lambda) \leqslant \lambda d.$$

Thus $G(x) \in X$. To see that G is a contraction note that

$$\|G(x) - G(x')\| = \|(I - F)x - (I - F)x'\| \leqslant f(2\lambda)\,\|x - x'\| \leqslant \tfrac{1}{4}\|x - x'\|.$$

Since $F(y(\lambda)) = \lambda u$ and dF is invertible at $y(\lambda)$ for $\lambda \in (0, \lambda_0]$, the usual implicit function theorem implies that $y(\lambda)$ is continuous for $\lambda \in (0, \lambda_0]$. Defining $y(0) = 0$, continuity at $\lambda = 0$ is easily seen.

As for uniqueness, suppose that $z\colon [0, \lambda_0] \to C$ is continuous, $F(z(\lambda)) = \lambda u$, and $z(0) = 0$. If $z \neq y$, let $\lambda_1 \in [0, \lambda_0)$ be the infimum of λ with $y(\lambda) \neq z(\lambda)$. By continuity $y(\lambda_1) = z(\lambda_1)$, and the implicit function theorem implies that $y(\lambda) = z(\lambda)$ for λ close to λ_1, obtaining a contradiction. ∎

LEMMA 9. *Let* $\mathbf{C}_0$ *be a convex open cone contained in* $\mathbf{C}$. *The map* T *is real-analytic from* $\mathbf{C}_0$ *to* $\mathbf{P}$, $T(P) \to 0$ *and* $\|dT(P) - I\| \to 0$ *as* $P \to 0$ *in* $\mathbf{C}_0$, *and* $T(0) = 0$.

Proof. It is clear that $T(0) = 0$, since the vacuum of H_0 is v_0. Given multi-indices I and J, define $I < J$ to mean that $I \neq J$ and $I_k \leqslant J_k$ for all k. By the general theory of renormalized products, for any $v \in D^\infty(W)$

$$:q^I: = :q^I:_v + \sum_{J<I} c_{IJ}(v) :q^J:_v,$$

where the coefficients $c_{IJ}(v)$ are fixed polynomials (each with vanishing constant term) in the expectation values $\langle v, :q^K: v\rangle$ for $K < I$. Writing $P \in \mathbf{C}_0$ as

$$P(q) = \sum_I a_I :q^I:,$$

it follows that

$$\tilde{P}(q) = P(q) + \sum_I \sum_{J<I} a_I c_{IJ}(v(P))\, q^J.$$

By Lemma 4, the coefficients c_{IJ} are real-analytic in $\mathbf{C}_0$. It follows that T is real-analytic on $\mathbf{C}_0$.

To show that $T(P) \to 0$ and $\|dT(P) - I\| \to 0$ as $P \to 0$ in $\mathbf{C}_0$ it suffices to show that $c_{IJ}(v(P)) \to 0$ as $P \to 0$ in $\mathbf{C}_0$, and that the first derivatives of $c_{IJ}(v(P))$ are bounded for P near 0 in $\mathbf{C}_0$. The first fact is a consequence of Lemma 5. The derivative in the direction Q of $c_{IJ}(v(P))$ is a polynomial in the functions

$$\langle v(P), :q^K: v(P)\rangle$$

and

$$\frac{d}{d\varepsilon} \langle v(P + \varepsilon Q), :q^K: v(P + \varepsilon Q)\rangle \bigg|_{\varepsilon = 0}$$

for $K < I$. These are bounded for P near 0 in $\mathbf{C}_0$ by Lemmas 5 and 7. ∎

To conclude the proof of Theorem 1, suppose that P satisfies the bound given in the hypothesis of the theorem. Then P is in the interior of $\mathbf{C}$, so we may choose a convex open cone $\mathbf{C}_0$ contained in $\mathbf{C}$ with $P \in \mathbf{C}_0$. As a consequence of Lemmas 8 and 9, for some $\lambda_0 > 0$ there is a unique continuous function Q: $[0, \lambda_0] \to \mathbf{C}_0$ such that

$$T(Q(\lambda)) = \lambda P.$$

Define $u(\lambda) \in \mathbf{K}$ by

$$u(\lambda) = v(Q(\lambda)).$$

Then by the definition of T, the closure of $H_0 + \lambda :P(q):_{u(\lambda)}$ is $H_{Q(\lambda)}$, so its vacuum is $u(\lambda)$. Thus $u(\lambda)$ satisfies condition 1 of the theorem. By Lemmas 4 and 5, u: $[0, \lambda_0] \to \mathbf{D}$ is continuous, so $u(\lambda)$ satisfies conditions 2–3, completing the proof of existence.

To prove uniqueness, suppose that w: $[0, \lambda_0] \to \mathbf{K}$ is another function that satisfies conditions 1–3. By conditions 2 and 3 and Lemma 2, $\langle w(\lambda), q^I w(\lambda) \rangle$ is bounded on $[0, \lambda_0]$ for all I with $|I| \leqslant 2d$. It follows from the general theory of renormalized products that $\lambda :P(q):_{w(\lambda)} \to 0$ in some open convex cone $\mathbf{C}_0 \subset \mathbf{C}$ as $\lambda \to 0$. Write $\lambda :P(q):_{w(\lambda)} = (Q(\lambda))(q)$. By condition 1, $w(\lambda) = v(Q(\lambda))$. It follows that $\lambda P = T(Q(\lambda))$. By the uniqueness result of the previous paragraph, it follows that $w(\lambda) = u(\lambda)$ for $\lambda \in [0, \lambda_0]$ if $\lambda_0 > 0$ is taken to be sufficiently small. ∎

The conditions in Theorem 1 hold for any interaction Hamiltonian of the form

$$\int_M P(\phi(0, x))\, d^n x,$$

where M is a complete Riemannian manifold, $\phi(t, x)$ is the free real scalar quantum field of mass m on $\mathrm{R} \times M$ cut off to finitely many modes, and P is a polynomial that is bounded below.

It is an interesting question whether the technical condition that $\langle u(\lambda), P(q)\, u(\lambda) \rangle$ be bounded can be omitted from Theorem 1. The condition that $u(\lambda)$ be continuous from $[0, \lambda_0]$ to $\mathbf{K}$ is essential. The following one-dimensional example adapted from the work of Friedman [2] shows that uniqueness may fail if $u(\lambda)$ is only continuous on a half-open interval $(0, \lambda_0]$.

THEOREM 2. *Let $n = 1$, so $\mathbf{H} = \mathbb{C}$, and let $H_0 = (1/2)(:p^2: + :q^2:)$. For some $\lambda_0 > 0$ there exists u: $(0, \lambda_0] \to \mathbf{K}$ such that*:

1. *$u(\lambda)$ is the vacuum of the closure of $H_0 + \lambda :q^4:_{u(\lambda)}$.*
2. *u is norm-continuous from $(0, \lambda_0]$ to $\mathbf{K}$.*
3. *$u(\lambda)$ converges weakly to zero as $\lambda \to 0^+$.*
4. *$\lim_{\lambda \to 0^+} \langle u(\lambda), q^2 u(\lambda) \rangle = \infty$.*

Proof. Let $G(b)$ be the vacuum of the Hamiltonian

$$H(b) = \tfrac{1}{2}(p^2 + bq^2) + q^4,$$

i.e., $H(b)\, G(b) = E(b)\, G(b)$, where $b \in \mathbb{R}$. Since $G(b)$ is the ground state of an even potential, $G(b)$ is an even function. Completing the square in $H(b)$ yields $E(b) \geqslant -b^2/16$. If $b < 0$, taking $u(x) = \pi^{-1/4} \exp(-(1/2)(x - (1/2)\,|b|^{1/2})^2)$, then

$$E(b) \leqslant \langle u, H(b)\, u \rangle = -\frac{b^2}{16} - \frac{b}{2} + 1.$$

It follows that

$$\lim_{b \to -\infty} \frac{E(b)}{b^2} = -\frac{1}{16}. \tag{8}$$

Using (8), for $b < -10$,

$$\langle G(b), q^2 G(b) \rangle > \frac{2E(b)}{b} \geqslant -\frac{b}{8} - \frac{6}{5}. \tag{9}$$

Since

$$:q^4:_{G(b)} = q^4 - 6\, \langle G(b), q^2 G(b) \rangle\, q^2 - \langle G(b), q^4 G(b) \rangle + 6 \langle G(b), q^2 G(b) \rangle^2,$$

$G(b)$ is the vacuum of

$$\tfrac{1}{2}(p^2 + mq^2) + :q^4:_{G(b)},$$

where

$$m = b + 12 \langle G(b), q^2 G(b) \rangle.$$

By the estimate (9), $m > -(1/2)\, b - 72/5 > 3/5$ for $b \leqslant -30$. Using the scale transformation

$$g(b, x) = m^{-1/8} G(b, m^{-1/4} x),$$

it is easy to check that $g(b)$ satisfies the equation

$$\{\tfrac{1}{2}(p^2 + m^{-1} bq^2) + m^{-1} bq^2) + m^{-3/2} q^4\}\, g(b) = m^{-1/2} E(b)\, g(b), \tag{10}$$

and that $g(b)$ is the vacuum of the Hamiltonian

$$H_0 + m^{-3/2} :q^4:_{g(b)}.$$

Letting $u(\lambda)=g(b)$, where $\lambda=m^{-3/2}$, the continuity of $u(\lambda)$ follows from Lemma 4, and λ_0 can be taken as $(1/2)\sup\{m^{-3/2}\mid b\in(-\infty,-30]\}$. Conditions 1 and 2 are then evident. For condition 4, note that as $b\to-\infty$ we have $m\to+\infty$, so $\langle g(b), q^2g(b)\rangle=m^{1/2}\langle G(b), q^2G(b)\rangle\to\infty$ as $b\to-\infty$. For condition 3 it suffices to show that $\langle g(b), w\rangle\to 0$ as $b\to-\infty$ for all $w\in C_0^\infty(\mathbb{R})$. It follows from Eq. (10) that

$$\langle g(b), \{\tfrac{1}{2}(p^2+m^{-1}bq^2)+m^{-3/2}q^4\}\, w\rangle=m^{-1/2}E(b)\,\langle g(b), w\rangle. \quad (11)$$

The left hand side of Eq. (11) stays bounded as $b\to-\infty$ since b/m is bounded and $m\to\infty$. Note that

$$\begin{aligned} b\langle q^2G(b), G(b)\rangle+\langle q^2G(b), G(b)\rangle^2 &\leqslant b\langle G(b), q^2G(b)\rangle+\langle G(b), q^4G(b)\rangle \\ &=E(b)-\langle pG(b), pG(b)\rangle\leqslant 0 \end{aligned}$$

if $b<0$ and $|b|$ is sufficiently large, by (8). It follows that $\langle G(b), q^2G(b)\rangle\leqslant|b|$. Therefore $|m^{-1/2}E(b)|\geqslant|12b|^{-1/2}\,|E(b)|\to\infty$ as $\lambda\to 0$ by (8), which implies that $\langle g(b), w\rangle\to 0$ by Eq. (11). ∎

It follows that in this 1-dimensional case there exist two distinct solutions $u_1(\lambda)$ and $u_2(\lambda)$ to Problem 1 for λq^4 if $\lambda>0$ is sufficiently small. Both are continuous functions of small $\lambda>0$, but $u_1(\lambda)$ converges to the free vacuum as $\lambda\downarrow 0$, while $u_2(\lambda)$ does not converge to any state.

It follows from the proof of Theorem 2 and Theorem 5 below that there is a one-to-one correspondence between solutions to Problem 1 for λq^4 (where $\lambda>0$) and values of $b\in\mathbb{R}$ such that

$$b+12\langle G(b), q^2G(b)\rangle=m,$$

where $\lambda=m^{-3/2}$. To estimate m, consider the operator

$$\hat{H}(b)=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}+bx^2\right)+x^4$$

on $L^2[-10, 10]$, with vanishing Dirichlet boundary conditions. Let $\hat{E}(b)$ be the lowest eigenvalue of $\hat{H}(b)$ and let $\hat{G}(b)$ be the corresponding eigenfunction. One can show using standard techniques [9] that

$$\int_{-10}^{10} x^2\hat{G}(b,x)^2\,dx\leqslant\int_{-\infty}^{\infty} x^2G(b,x)^2\,dx,$$

so that $m\geqslant\hat{m}=b+12\int_{-10}^{10}x^2\hat{G}(b,x)^2\,dx$. In addition to being a lower bound on m, $\hat{m}$ should be a good approximation to m for $b\in[-30, 0]$. To

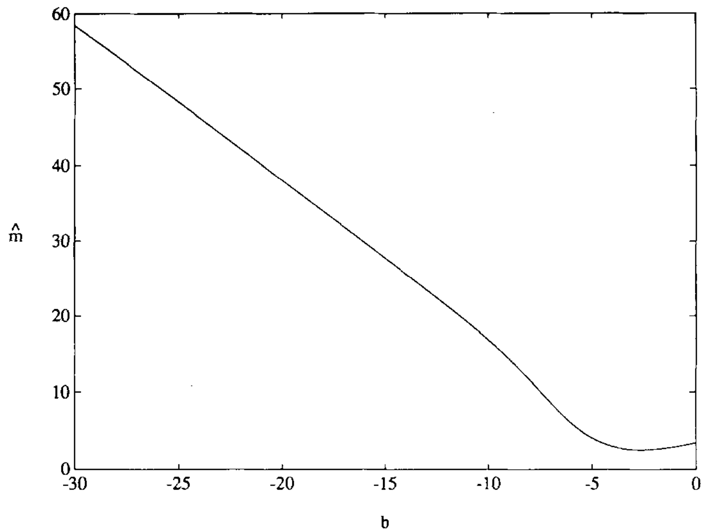

FIG. 1. Numerical approximation to $m = b + 12 \langle G(b), q^2 G(b) \rangle$.

numerically calculate $\hat{m}$, we used the central difference method to discretize the equation $\hat{H}(b)\,\hat{G}(b) = \hat{E}(b)\,\hat{G}(b)$, with step size $h = 0.01$. It is known that the error from the discretization is $O(h^2)$.

Figure 1 is the graph of the numerically calculated $\hat{m}$ as a function of b for $b \in [-30, 0]$, which indicates that

$$\min\{m \mid b \in [-30, 0]\} = m_0 > 0.$$

It is easily seen that there exists $\varepsilon > 0$ such that $m > \varepsilon$ for all $b \geqslant 0$. If $b \leqslant -30$ it follows from (9) that $m \geqslant 3/5$. Thus we conjecture that m has minimum $m_0 > 0$, implying that Problem 1 has a solution for λq^4 only if $\lambda \leqslant \lambda_0 = m_0{}^{-3/2}$. Moreover, we conjecture that m has only one critical point, where it attains its minimum. If this is the case, then there are two continuous solutions $u_1(\lambda)$ and $u_2(\lambda)$ of Problem 1 for $\lambda \in (0, \lambda_0]$; moreover, these are the only solutions, and $u_1(\lambda) \neq u_2(\lambda)$ for $\lambda \in (0, \lambda_0)$, but $u_1(\lambda_0) = u_2(\lambda_0)$.

4. Renormalizing the Total Hamiltonian

In the previous section we renormalized the interaction Hamiltonian relative to the ground state of the total Hamiltonian. We next consider renormalizing the total Hamiltonian relative to its own ground state:

PROBLEM 2. *Given P in $\mathbf{C}$, find $u \in \mathbf{K}$ such that the vacuum of the closure of*

$$\frac{1}{2}\sum_{i=1}^{n} \omega_i(:p_i^2:_u + :q_i^2:_u) + :P(q):_u$$

is u.

The relation between this problem and Problem 1 is given in the following:

LEMMA 10. *Suppose that $u \in \mathbf{K}$ has $\langle u, q_i u\rangle = 0$ for all $1 \leqslant i \leqslant n$. Then u is a solution to Problem 1 for $P \in \mathbf{C}$ if and only if it is a solution to Problem 2 for P.*

Proof. If u satisfies $\langle u, q_i u\rangle = 0$ for all i and is a solution to either Problem 1 or Problem 2 for $P \in \mathbf{C}$, then $(1/2)\sum \omega_i(:p_i^2:_u + :q_i^2:_u)$ differs from H_0 by a constant, by the general theory of renormalized products. Thus u is a solution to both Problems 1 and 2 for P. ∎

Friedman [2] showed that, due to a translational invariance in Problem 2, there does not exist a solution for generic $P \in \mathbf{C}$, and when a solution exists it is never unique. For the convenience of the reader we present these results here, in a generalized form.

THEOREM 3. (Friedman). *Suppose that $u \in \mathbf{K}$ is a solution to Problem 2 for P. Then for any $x \in \mathbb{R}^n$, $U(x)u$ is a solution to Problem 2 for P.*

Proof. Let $T = U(x)$. Note that

$$:p_i^2:_{Tu} = :p_i^2:_u = T:p_i^2:_u T^{-1}$$

by the general theory of renormalized products and the fact that T commutes with p_i. Note also that

$$:Q(q):_{Tu} = T:Q(q):_u T^{-1}$$

for any $Q \in \mathbf{P}$. This can be shown using induction on the degree of Q using the definition of renormalized products and the properties

$$\langle Tu, (T:Q(q):_u T^{-1})\, Tu\rangle = 0,$$

$$\begin{aligned}[q_j, T:Q(q):_u T^{-1}] &= T[T^{-1}q_j T, :Q(q):_u]\, T^{-1}\\ &= T[q_j + x_j, :Q(q):_u]\, T^{-1} = 0,\end{aligned}$$

and

$$[p_j, T:Q(q):_u T^{-1}] = T[p_j, :Q(q):_u]\, T^{-1} = -iT:\frac{\partial Q}{\partial q_i}(q):_u T^{-1}.$$

It follows that

$$\begin{aligned}&\tfrac{1}{2}\sum_{i=1}^{n} \omega_i(:p_i^2:_{Tu} + :q_i^2:_{Tu}) + :P(q):_{Tu}\\ &\qquad = T\left(\tfrac{1}{2}\sum_{i=1}^{n} \omega_i(:p_i^2:_u + :q_i^2:_u) + :P(q):_u\right) T^{-1}\end{aligned}$$

so that the vacuum of the closure of this operator is Tu. ∎

COROLLARY 1 (Friedman). *The set of $P \in \mathbf{C}$ for which there is a solution to Problem 2 is of Lebesgue measure zero in $\mathbf{C}$.*

Proof. This follows from Sard's theorem, see [2]. ∎

We may eliminate the translational degrees of freedom described in the theorem above by requiring that $\langle u, q_i u\rangle = 0$. This device allows us to easily prove an existence and uniqueness theorem for Problem 2 if we restrict ourselves to even polynomials, i.e., polynomials satisfying $P(x) = P(-x)$.

THEOREM 4. *Suppose that $P \in \mathbf{P}$ is even and satisfies*

$$P(q) \geq \varepsilon(q_1^2 + \cdots + q_n^2)^d - k$$

for some $\varepsilon, k > 0$. Then for some $\lambda_0 > 0$ there is a unique map u: $[0, \lambda_0] \to \mathbf{K}$ such that:

1. *$u(\lambda)$ is the vacuum of the closure of*

$$\tfrac{1}{2}\sum_{i=1}^{n} \omega_i(:p_i^2:_{u(\lambda)} + :q_i^2:_{u(\lambda)}) + \lambda:P(q):_{u(\lambda)}.$$

2. *u is norm-continuous from $[0, \lambda_0]$ to $\mathbf{K}$.*
3. *$\langle u(\lambda), q_i u(\lambda)\rangle = 0$ for $1 \leq i \leq n$.*
4. *$\langle u(\lambda), P(q)\, u(\lambda)\rangle$ are bounded for $\lambda \in [0, \lambda_0]$.*

Proof. Note that the map T described in the proof of Theorem 1 maps even polynomials to even polynomials, since an even polynomial in the q's renormalized relative to a state $v \in D^\infty(W)$ that is even in the Schrödinger representation is again even. Defining u: $[0, \lambda_0] \to \mathbf{K}$ as in Theorem 1, it follows that conditions 2–4 hold, so by Lemma 10 condition 1 holds as well. Uniqueness follows similarly from Theorem 1 and Lemma 10. ▮

COROLLARY 2. *Given u as in Theorem 4, then $u(0) = v_0$ and u is norm-continuous from $[0, \lambda_1]$ to $\mathbf{D}$ for some $\lambda_1 > 0$, and norm-analytic from $(0, \lambda_1]$ to $\mathbf{D}$.*

Proof. It follows from Theorem 6 below that any solution u of Problem 2 for $P = 0$ satisfying $\langle u, q_i u \rangle = 0$ for all i has $u = v_0$. The continuity property of u was shown in the proof of Theorem 1. The analyticity property follows from Lemmas 4 and 9 and the proof of Theorem 1. ▮

As with Problem 1, uniqueness need not hold if we do not require continuity at $\lambda = 0$:

COROLLARY 3. *Let $n = 1$, so $\mathbf{H} = \mathbb{C}$. For some $\lambda_0 > 0$ there exists a function u: $(0, \lambda_0] \to \mathbf{K}$ such that:*

1. *$u(\lambda)$ is the vacuum of $(1/2)(:p^2:_{u(\lambda)} + :q^2:_{u(\lambda)}) + \lambda :q^4:_{u(\lambda)}$.*
2. *u is norm-continuous from $(0, \lambda_0]$ to $\mathbf{K}$.*
3. *$\langle u(\lambda), qu(\lambda) \rangle = 0$.*
4. *$u(\lambda)$ converges weakly to zero as $\lambda \to 0^+$.*
5. *$\lim_{\lambda \to 0^+} \langle u(\lambda), q^2 u(\lambda) \rangle = \infty$.*

Proof. Construct $u(\lambda)$ as in Theorem 2. Since $u(\lambda)$ is even, $\langle u(\lambda), q_i u(\lambda) \rangle = 0$ for $1 \leqslant i \leqslant n$, proving that condition 3 holds. Condition 1 then follows from Lemma 10, and the rest follow from Theorem 2. ▮

5. POLYNOMIALS OF DEGREE FOUR

In this section we present some nonperturbative results for polynomial interactions of degree four.

LEMMA 11. *Suppose that $Q \in \mathbf{P}$ has the form $Q = Q_e + \sum b_i q_i$, where Q_e is an even polynomial, and suppose that $\langle v(Q), q_i v(Q) \rangle = 0$ for all $1 \leqslant i \leqslant n$. Then $v(Q) \in L^2(\mathbb{R}^n)$ is even, and $b_i = 0$ for $1 \leqslant i \leqslant n$; i.e., Q is even.*

Proof. Let $H = H_Q$ and $H_e = H_{Q_e}$ with lowest eigenvalues $E = E(Q)$ and $E_e = E(Q_e)$. From the definition of E and E_e, we have

$$E = \langle v(Q), Hv(Q) \rangle = \langle v(Q), H_e v(Q) \rangle \geqslant E_e$$

since $\langle v(Q), q_i v(Q)\rangle = 0$. On the other hand, the nondegeneracy of the lowest eigenvalue implies that $v(Q_e)$ is an even function of q, so

$$E_e = \langle v(Q_e), H_e v(Q_e)\rangle = \langle v(Q_e), Hv(Q_e)\rangle \geqslant E.$$

Combining these two inequalities, it follows that $E = E_e$ and $\langle v(Q_e), Hv(Q_e)\rangle = E$, so $v(Q_e) = v(Q)$ by the uniqueness of the vacuum, and $v(Q)$ is an even function of q. From $H_e v(Q) - Ev(Q) = -\sum_i b_i q_i v(Q)$, the right hand side of the equation is even, hence $b_i = 0$. ▮

THEOREM 5. *Suppose that $P \in \mathbf{C}$, and suppose u is the vacuum of the closure of $H_0 + {:}P(q){:}_u$. If the degree of P is 4 and P is even, then u is even.*

Proof. Let $c_i = \langle u, q_i u\rangle$ for $1 \leqslant i \leqslant n$. Using Lemma 11 it is enough to show that $c_i = 0$ by noting that

$${:}P(q){:}_u = P(q) + \sum_{i,j,k,l} a_{ijkl} c_i q_j q_k q_l + Q_e(q) + \sum_j b_j q_j,$$

where Q_e is an even quadratic polynomial of q.

Let $w(q_1, ..., q_n) = u(q_1 + c_1, ..., q_n + c_n)$. It is easy to check that $\langle w, q_i w\rangle = 0$, and by Theorem 3, w is the vacuum of the closure of

$$H_0 + {:}P(q){:}_w - \sum \omega_i c_i q_i = H_0 + R(q) + \sum a_i q_i,$$

where R is an even polynomial of degree 4. It follows from Lemma 11 that w is an even function and $a_i = 0$ for all $1 \leqslant i \leqslant n$. Thus by the general theory of renormalized products ${:}P(q){:}_w$ is an even polynomial, so $c_i = a_i/\omega_i = 0$, completing the proof. ▮

THEOREM 6. *Suppose that u is the vacuum of the closure of*

$$\tfrac{1}{2} \sum_{i=1}^{n} \omega_i({:}p_i^2{:}_u + {:}q_i^2{:}_u) + {:}P(q){:}_u$$

and $\langle u, q_i u\rangle = 0$ for $1 \leqslant i \leqslant n$. If the degree of the polynomial P is 4 and P is even, then u is even.

Proof. Since $\langle u, q_i u\rangle = 0$,

$$\sum \omega_i({:}p_i^2{:}_u + {:}q_i^2{:}_u) + {:}P(q){:}_u = H_0 + Q(q) + \sum_i b_i q_i,$$

where Q is an even polynomial of degree 4. It follows immediately from Lemma 11 that u is even. ▮

COROLLARY 4. *If the degree of $P \in C$ is 4 and P is even, $u \in \mathbf{K}$ is a solution to Problem 1 if and only if it is a solution to Problem 2 and $\langle u, q_i u \rangle = 0$ for all $1 \leqslant i \leqslant n$.*

Proof. This follows from Theorems 5 and 6, together with Lemma 10. ∎

It is an interesting problem to try to extend these results to polynomials of higher degree.

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