Probabilities versus Amplitudes

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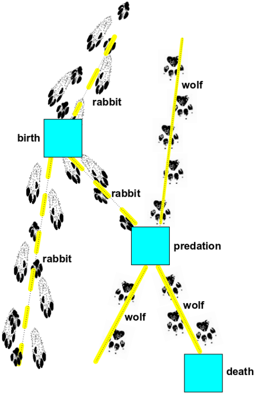
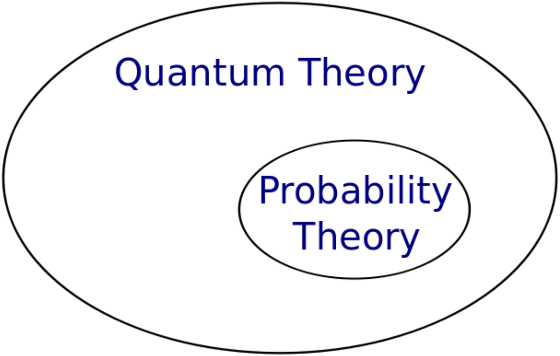


for more, see:

http://math.ucr.edu/home/baez/networks/

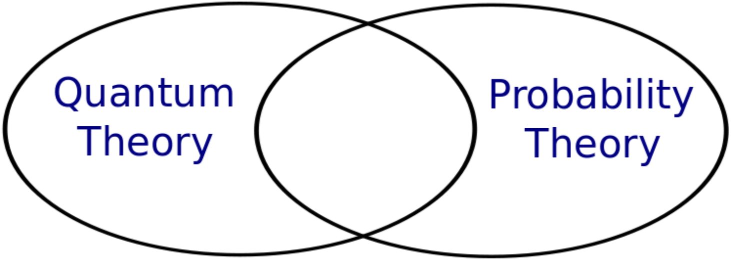
The usual picture:



Quantum Theory

Probability Theory

A different picture:



Quantum Theory

Probability Theory

It doesn’t contradict the other; it’s just another outlook.

Suppose we have a system with n possibilities:

$$X = \{1, \ldots, n\}$$

In quantum theory we consider **quantum states**:

$$\psi \colon X \to \mathbb{C}$$

with

$$\sum_{i \in X} |\psi_i|^2 = 1$$

In probability theory we consider **stochastic states**:

$$\psi \colon X \to \mathbb{R}$$

with

$$\sum_{i \in X} \psi_i = 1 \text{ and } \psi_i \geq 0$$

An operator $U\colon \mathbb{C}^n \to \mathbb{C}^n$ that sends quantum states to quantum states is called **unitary.**

An operator $U\colon \mathbb{R}^n \to \mathbb{R}^n$ that sends stochastic states to stochastic states is called **stochastic.**

Concretely, U is stochastic iff

$$\sum_i U_{ij} = 1 \text{ and } U_{ij} \geq 0$$

An operator $H\colon \mathbb{C}^n \to \mathbb{C}^n$ for which $\exp(-itH)$ is unitary for all $t \in \mathbb{R}$ is called **self-adjoint.**

An operator $H\colon \mathbb{R}^n \to \mathbb{R}^n$ for which $\exp(tH)$ is stochastic for all $t \in [0, \infty)$ is called **infinitesimal stochastic.**

Concretely, H is infinitesimal stochastic iff

$$\sum_i H_{ij} = 0 \text{ and } H_{ij} \geq 0 \text{ if } i \neq j$$

If *H* is self-adjoint, we can describe time evolution of quantum states using **Schrödinger's equation**:

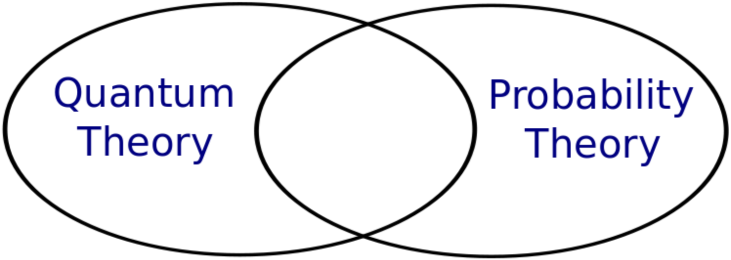
$$\frac{d}{dt}\psi(t) = -iH\psi(t)$$

If *H* is infinitesimal stochastic, we can describe time evolultion of stochastic states using the **master equation**:

$$\frac{d}{dt}\psi(t) = H\psi(t)$$

Unitary operators have unitary inverses; stochastic operators rarely have stochastic inverses! So, we only evolve forwards in time in stochastic physics.

Some operators are both self-adjoint and infinitesimal stochastic. The most famous example is the Laplacian. However, the overlap here is small:



So, the main use of this picture is to pass ideas from quantum theory to probability theory, or vice versa.

Suppose ψ_n is the probability of having n amoebas in a test tube. We can summarize this information in a power series:

$$\Psi(z) = \sum_{n=0}^{\infty} \psi_n z^n$$

The creation operator $a^\dagger$ creates an amoeba:

$$a^\dagger \Psi = z\Psi$$

The annihilation operator a destroys one:

$$a\Psi = \frac{d}{dz}\Psi$$

We have $az^n = nz^{n-1}$ since there are n amoebas to choose from.

We're used to

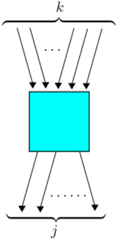
$$aa^\dagger - a^\dagger a = 1$$

for indistinguishable bosons. Can this be right for classical objects if we use probabilities instead of amplitudes?

Yes! There's one more way to create an amoeba and then kill one, than to kill one and then create one.

But let's try some examples.

We would like a Hamiltonian for a process that destroys k amoebas and creates j of them:



The obvious guess is ${a^\dagger}^j a^k$. But this is not infinitesimal stochastic! The right answer has a ‘correction term’:

$${a^\dagger}^j a^k - {a^\dagger}^k a^k$$

$H = a^\dagger - 1$ describes the random 'creation' of amoebas. The master equation

$$\frac{d}{dt}\Psi(t) = H\Psi(t)$$

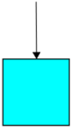
has this solution:

$$\Psi(t) = e^{t(z-1)}\Psi(0)$$

If we start with the 'vacuum state' $\Psi(0) = 1$, where there are no amoebas, at time t we have

$$\Psi(t) = e^{t(z-1)} = e^{-t}\sum_n \frac{t^n}{n!} z^n$$

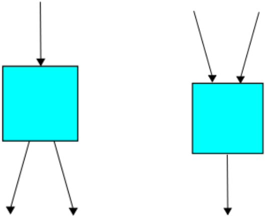
so the probability of having n amoebas is $e^{-t}\frac{t^n}{n!}$. This is just what we expect: a Poisson process.



$H = a - a^\dagger a$ describes the random ‘annihilation’ of amoebas.

Using this Hamiltonian, the master equation predicts that *the expected number of amoebas decays exponentially*. Again, this is just right.

A more interesting example combines *fission* and *competition*:



$$H = \alpha({a^\dagger}^2 a - a^\dagger a) \;\; + \;\; \beta(a^\dagger a^2 - {a^\dagger}^2 a^2)$$

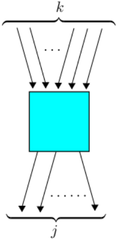
Here we can show that in the 'classical limit' where ψ_n is very sharply peaked near some very large number, the expected number of amoebas:

$$\langle N(t) \rangle = \sum_n n\, \psi_n(t)$$

obeys the logistic equation:

$$\frac{d}{dt} \langle N(t) \rangle = \alpha \langle N(t) \rangle - \beta \langle N(t) \rangle^2$$

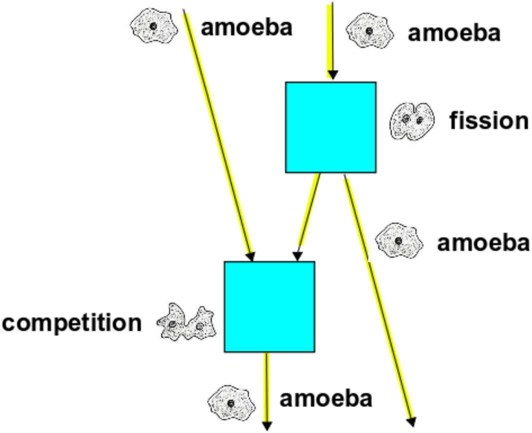
In fact, for *any* Hamiltonian that's a linear combination of terms like this:



$$a^{\dagger j} a^k - a^{\dagger k} a^k$$

it is easy to write a differential equation describing how the expected number of particles $\langle N(t) \rangle$ changes with time in the classical limit. This is called the **rate equation.**

Moreover, one can always express the time evolution operator $\exp(tH)$ as a sum over Feynman diagrams:



In some ways stochastic mechanics works ‘better’ than quantum mechanics. It might seem hard to find a stationary state

$$H\Psi = 0$$

for our Hamiltonian

$$H = \alpha({a^\dagger}^2 a - a^\dagger a) \;\; + \;\; \beta(a^\dagger a^2 - {a^\dagger}^2 a^2)$$

describing amoeba reproduction and competition. But we can do it using the Anderson–Craciun–Kurtz theorem. This applies to a large class of Hamiltonians of the sort we’re considering.

First, find a stationary solution of the rate equation:

$$\frac{d}{dt}\langle N(t)\rangle = \alpha\langle N(t)\rangle - \beta\langle N(t)\rangle^2$$

This is easy:

$$\alpha\langle N(t)\rangle - \beta\langle N(t)\rangle^2 = 0$$

$$\langle N(t)\rangle = \alpha/\beta$$

Then, form the ‘coherent state’ where the expected number of amoebas takes this value:

$$\Psi(z) = \frac{e^{(\alpha/\beta)z}}{e^{\alpha/\beta}} = \frac{1}{e^{\alpha/\beta}} \sum_{n=0}^{\infty} \frac{(\alpha/\beta)^n}{n!} z^n$$

This has

$$H\Psi = 0$$

All these ideas generalize to situations with more than one type of particle, e.g. chemical reactions treated stochastically:

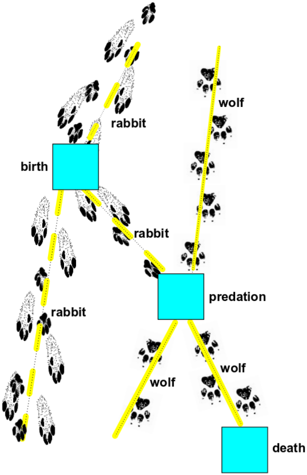
$$\begin{array}{rcl} \mathrm{H_2O} & \leftrightarrow & \mathrm{H^+} + \mathrm{OH^-} \\ \mathrm{H^+} + \mathrm{H_2O} & \leftrightarrow & \mathrm{H_3O^+} \\ \mathrm{H_2O_2} & \to & 2\mathrm{H_2O} + \mathrm{O_2} \end{array}$$

with different chemical species (molecules and ions) as our types of particle.

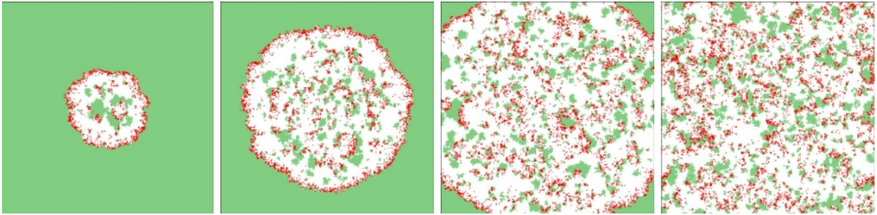
Indeed, the Anderson–Craciun–Kurtz theorem was first proved by experts on chemistry, who did not know their work could be interpreted in terms of annihilation and creation operators!

Now we have a proof using operators.

Stochastic models with several species also show up in biology:



These ideas generalize straightforwardly to *stochastic field theory*, where particles also have locations, e.g. on a lattice. Stochastic field theory is also increasingly important in population biology:



From ‘Beyond the mean field in host-pathogen spatial ecology’ by Stacey, Gros and Bar-Yam.

For more see:

- Baez, Biamonte and Fong, Network Theory, http://math.ucr.edu/home/baez/networks/
- Stacey, Gros and Bar-Yam, Beyond the mean field in host-pathogen spatial ecology, arXiv:1110.3845.

and many references therein.