HW # 5

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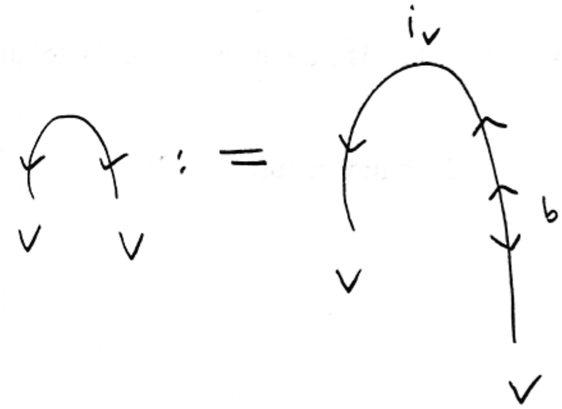
Suppose V is an orthogonal (or symplectic) vector space.

Define



Facts:

① $\#(e_i) = \sum_j g_{ij} e^j$

② $b(e^i) = \sum_j g^{ij} e_j$

Then, calculate:

g or w

Claim:

g^{ij} is the inverse of g_{ij}

First let's prove our claim above:

g^{ij} is the inverse of g_{jk}. ie) $\boxed{\delta^i_k = g^{ij} g_{jk}}$

proof:

$$\delta^i_k e^k = e^i = \#(b(e^i)) = \#(g^{ij} e_j) = g^{ij} g_{jk} e^k$$

* using Einstein's notation

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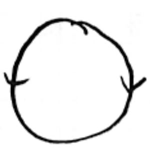
So, $\delta^i_k e^k = g^{ij} g_{jk} e^k$

$$\Rightarrow \delta^i_k = g^{ij} g_{jk} \quad \checkmark$$

Now —

<u>Case 1</u>: V is orthogonal. Then, $g: V \otimes V \to \mathbb{C}$ is symmetric & nondegenerate. Therefore, we can define $\#$, and since g is nondegenerate, $\#$ has an inverse, b. g being symmetric means $g(v,w) = g(w,v)$ or, $g_{ij} = g_{ji}$.

So:



$$1 \xmapsto{1_V} \sum_i e_i \otimes e^i \xmapsto{1_V \otimes b} \sum_i e_i \otimes b(e^i) = \sum_i e_i \otimes \sum_j g^{ij} e_j$$

$$\xmapsto{g} g\Big(\sum_i e_i \otimes \sum_j g^{ij} e_j\Big)$$

$$= \sum_{i,j} g^{ij} \big(g(e_i \otimes e_j)\big)$$

since g is bilinear map

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And, $\sum_{i,j} g^{ij}(g(e_i \otimes e_j))$

$= \sum_{i,j} g^{ij} \cdot g_{ij}$ way g is defined

But since g is symmetric, $g_{ij} = g_{ji}$

So —

$= \sum_{i,j} g^{ij} \cdot g_{ji}$ Now, by our "claim" on the 1st page

$= \sum_i \sum_j g^{ij} \cdot g_{ji} = \sum_i \delta^i_i = \dim V$ ✓

<u>case 2</u>: V is symplectic. Then $\omega : V \otimes V \longrightarrow \mathbb{C}$ is <u>antisymmetric</u> and nondegenerate.

Therefore, $\omega(v,u) = -\omega(u,v)$, or $g_{ij} = -g_{ji}$

so, again we have:

$$1 \xmapsto{i_V} \sum_i e_i \otimes e^i \xmapsto{1_V \otimes b} \sum_i e_i \otimes b(e^i) = \sum_i e_i \otimes \sum_j g^{ij} e_j$$

$$\xmapsto{w} w\left(\sum_i e_i \otimes \sum_j g^{ij} e_j\right)$$

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$$w\left(\sum_i e_i \otimes \sum_j g^{ij} e_j\right)$$

$$= \sum_{i,j} g^{ij} \left(w(e_i \otimes e_j)\right)$$

$$= \sum_{i,j} g^{ij} \cdot g_{ij}$$

but, since w is antisymmetric, $g_{ij} = -g_{ji}$

$$= \sum_{i,j} g^{ij} (-g_{ji})$$

$$= \sum_i \sum_j g^{ij} (-g_{ji}) = \sum_i -\delta_i^i = -\sum_i \delta_i^i = -\dim V$$ ✓

✓ great!