HW#6

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✓ good!

A linear operator $g: \mathbb{C}^2 \to \mathbb{C}^2$ preserves

$\omega: \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}$ where $\omega(e_i \otimes e_i) = 0$, $\omega(e_1 \otimes e_2) = 1$

i.e. $\omega(gv \otimes gw) = \omega(v \otimes w)$ iff $\det g = 1$.
$(\forall v \,\&\, w)$

Recall: the det. of a linear operator is the det. of the matrix associated w/ it.

ie) we look at what g does to basis elements e_1 and e_2 of $\mathbb{C}^2$, and we write $g(e_1), g(e_2)$ as a linear combination of the basis elements.

$$g(e_1) = g_{11} e_1 + g_{12} e_2$$

for some $g_{11}, g_{12}, g_{21}, g_{22}$

$$g(e_2) = g_{21} e_1 + g_{22} e_2$$

the coefficients of $g(e_i)$ w/r/t the basis $\{e_1, e_2\}$ is the matrix of g.

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

So, $\det g = g_{11} g_{22} - g_{12} g_{21}$.

Let $v, u \in \mathbb{C}^2$

$$v = v_1 e_1 + v_2 e_2$$

$$u = u_1 e_1 + u_2 e_2$$

Then $v \otimes u = (v_1 e_1 + v_2 e_2) \otimes (u_1 e_1 + u_2 e_2)$. Tensors distribute over sums.

$$= v_1 u_1 (e_1 \otimes e_1) + v_1 u_2 (e_1 \otimes e_2)$$

$$+ v_2 u_1 (e_2 \otimes e_1) + v_2 u_2 (e_2 \otimes e_2)$$

Now, applying ω, we obtain:

$$\boxed{\omega(v \otimes u) = v_1 u_2 - v_2 u_1} \quad (1)$$

On the other hand —

$$gv = v_1 g(e_1) + v_2 g(e_2)$$

$$gu = u_1 g(e_1) + u_2 g(e_2)$$

$$\otimes g w = v_1 u_1 (g(e_1) \otimes g(e_1)) + v_1 u_2 (g(e_1) \otimes g(e_2))$$

$$+ v_2 u_1 (g(e_2) \otimes g(e_1)) + v_2 u_2 (g(e_2) \otimes g(e_2))$$

Now, applying ω we obtain:

$$\omega(gv \otimes gw) = v_1u_1\,\omega(g(e_1)\otimes g(e_1)) + v_1u_2\,\omega(g(e_1)\otimes g(e_2)) + v_2u_1\,\omega(g(e_2)\otimes g(e_1)) + v_1u_2\,\omega(g(e_2)\otimes g(e_2))$$

However - we have:

$$g(e_1)\otimes g(e_1) = (g_{11}e_1 + g_{12}e_2)\otimes(g_{11}e_1 + g_{12}e_2)$$
$$= g_{11}g_{11}(e_1\otimes e_1) + g_{11}g_{12}(e_1\otimes e_2) + g_{12}g_{11}(e_2\otimes e_1) + g_{12}g_{12}(e_2\otimes e_2)$$

Applying ω:

$$\omega(g(e_1)\otimes g(e_1)) = g_{11}g_{12} - g_{12}g_{11} = 0.$$

Similarly:

$$g(e_1)\otimes g(e_2) = (g_{11}e_1 + g_{12}e_2)\otimes(g_{21}e_1 + g_{22}e_2)$$
$$= g_{11}g_{21}(e_1\otimes e_1) + g_{11}g_{22}(e_1\otimes e_2) + g_{12}g_{21}(e_2\otimes e_1) + g_{12}g_{22}(e_2\otimes e_2)$$

So

$$\omega(g(e_1)\otimes g(e_2)) = g_{11}g_{22} - g_{12}g_{21} = \det g$$

$$g(e_2) \otimes g(e_1) = (g_{21}e_1 + g_{22}e_2) \otimes (g_{11}e_1 + g_{12}e_2)$$
$$= g_{21}g_{11}(e_1 \otimes e_1) + g_{21}g_{12}(e_1 \otimes e_2) + g_{22}g_{11}(e_2 \otimes e_1) + g_{22}g_{12}(e_2 \otimes e_2)$$

$$\omega(g(e_2) \otimes g(e_1)) = g_{21}g_{12} - g_{22}g_{11} = -\det g$$

$$g(e_2) \otimes g(e_2) = (g_{21}e_1 + g_{22}e_2) \otimes (g_{21}e_1 + g_{22}e_2)$$
$$= g_{21}g_{21}(e_1 \otimes e_1) + g_{21}g_{22}(e_1 \otimes e_2) + g_{22}g_{21}(e_2 \otimes e_1) + g_{22}g_{22}(e_2 \otimes e_2)$$

$$\omega(g(e_2) \otimes g(e_2)) = g_{21}g_{22} - g_{22}g_{21} = 0$$

So —

$$\omega(gv \otimes gw) = v_1u_2 \det g - v_2u_1 \det g$$
$$= \det g\,(v_1u_2 - v_2u_1)$$

$$\boxed{\omega(gv \otimes gw) = \det g\,(v_1u_2 - V_2u_1)} \quad (2)$$

Now we're ready to prove the statement!

($\Leftarrow$) $\det g = 1$

This means in (2)

$$\omega(gv \otimes gw) = \underset{\substack{\| \\ (1)}}{\det g}\,(v_1u_2 - v_2u_1)$$

$$\Rightarrow \omega(gv \otimes gw) = v_1u_2 - v_2u_1 \xrightarrow[\text{by (1).}]{} = \omega(v \otimes u)$$

($\Rightarrow$) $v_1u_2 - v_2u_1 \underset{(1)}{=} \omega(v \otimes u) \underset{\text{assumption}}{=} \omega(gv \otimes gw) \underset{(2)}{=} \det g\,(v_1u_2 - v_2u_1)$

$\Rightarrow \det g = 1$, when $(v_1u_2 - v_2u_1) \neq 0$.

But $v_1u_2 - v_2u_1 \neq 0$ iff v (or u) $\neq 0$.

If $v = 0$, the claim: $\omega(gv \otimes gw) = \omega(v \otimes u)$ holds regardless of what g is.