09/26/02

Building Spacetime from Spin

- Dirac equation: $(\not{\partial} + i\underset{\text{mass}}{m})\psi = 0$
- describes electrons & positrons

spinor field
$\psi : \mathbb{R}^4 \longrightarrow \mathbb{C}^4$

Minkowski spacetime ($\mathbb{R}^4$)

space of Dirac spinors ($\mathbb{C}^4$)

$\not{\partial}$ is a differential operator: Dirac operator

$$\gamma^0 \partial_0 + \gamma^1 \partial_1 + \gamma^2 \partial_2 + \gamma^3 \partial_3$$

$\not{\partial}$ is a differential operator on spinors — takes spinor fields to another spinor field.

Dirac Spinors

$$\mathbb{C}^4 = \mathbb{C}^2 \oplus (\mathbb{C}^2)^*$$

left and right-handed Weyl Spinors

Note – turning an electron around 360° isn't the same! so we need to distinguish bet. no rotation & 360° rotation.

SPACE | SPACETIME

QUANTUM / CLASSICAL

$$\begin{tikzcd}
SU(2) \arrow[r, hook, "\text{incl.}"] \arrow[d, "\substack{2\text{-}1 \\ \text{map} \\ \text{(onto)}}"'] & SL(2, \mathbb{C}) \arrow[d, "\substack{2\text{-}1 \text{ onto} \\ \text{map}}"] \\
SO(3) \arrow[r, hook, "1\text{-}1 \text{ incl. homo.}"] & SO_0(3,1)
\end{tikzcd}$$

rotation grp

- special orthogonal 3×3 real matrices

orthogonal: $AA^T = I$

$\det(A) = 1$

- orthogonal – preserves inner product

- Lorentz grp
- 4×4 real matrices st $AA^* = 1$

$\det(A) = 1$

- mixes up space & time components

how to define *

Minkowski metric:

$$g(x,y) = \underbrace{-x_0 y_0}_{\text{time}} + \underbrace{x_1 y_1 + x_2 y_2 + x_3 y_3}_{\text{space}}$$

x, y in $\mathbb{R}^4$

and

$$g(Ax, y) = g(x, A^* y) \quad \text{(how we define *)}$$

Note: $SO(3)$ is connected
$SO(3,1)$ is not. Can't get to time & space reversal from ident. matrix. $\begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

SO(3,1) has 2 connected components — one containing id, other containing $\begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

We don't want something that's not connected, so we'll add an "o" at $SO_o(3,1)$ to mean we just want the connected component containing the id.

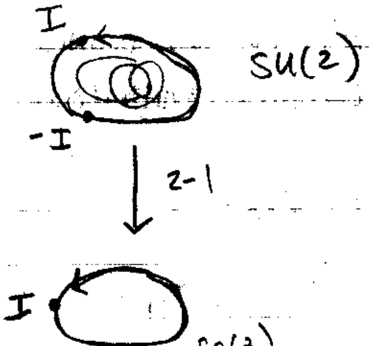
SO(3)

can't contract this path to id (due to arm twist)
(hold something in hand, rotate it 360°, have twist in arm, do again – untwists)

- but – going around again – this path is contractible to id.

Coffee cup trick shows SO(3) has a loop γ starting at I s.t. γ is not contractible; but γ^2 (do γ twice) is!

SU(2) knows the difference bet. a 360° rotation & no rotation, both get mapped to id $\in$ SO(3).



But doing loop twice down in SO(3), we get a loop up in SU(2) from $-I$ to I.

loop around to I, in SU(2) this is a path from $-I$ to I

Adjoint

* So – $SU(2)$ has no non-contractible loops

- $SU(2)$ = 2x2 complex matrices which are <u>unitary</u>: $AA^* = I$ (complex conjugate transpose) and $\det(A) = 1$

- $SL(2, \mathbb{C})$ = 2x2 complex matrices ("L" means linear) and $\det(A) = 1$.

Let's describe the maps in our square on prev page:

* $SO(3) \hookrightarrow SO_0(3,1)$

$$\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & - & - & - \\ 0 & - & - & - \\ 0 & - & - & - \end{pmatrix}$$

* $SU(2) \hookrightarrow SL(2, \mathbb{C})$ is the obvious inclusion since one is subgrp of other

Let's describe $p: SU(2) \longrightarrow SO(3)$

$SO(3)$ is symmetries of $\mathbb{R}^3$ w/ dot prod.

$\mathbb{R}^3$ w/ its usual dot product is isomorphic to the 2x2 <u>Hermitian</u> complex matrices w/ trace = 0. We'll call this set $\mathcal{H}_0$.

$$\mathcal{H}_0 = \left\{ \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \middle| \; x_1, x_2, x_3 \in \mathbb{R} \right\}$$

So $\mathcal{H}_0$ is 3-dim'l (we make 3 choices) and $\mathbb{R}^3$ is 3-dim'l.

$$\det \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} = -(x_1^2 + x_2^2 + x_3^2)$$
$$= -x \cdot x \quad \text{where } x \in \mathbb{R}^3$$

This will allow us to relate $SU(2)$ to $SO(3)$.

To get $g \in SU(2)$ to act as a rotation of $\mathbb{R}^3$ $(\rho(g))$ we just need it to act on $\mathcal{H}_0$, but we need it to be a rotation – must preserve inner product, i.e. must preserve the determinant.

Here's how: if $x \in \mathcal{H}_0$, let $\rho(g)(x) = gxg^{-1}$.

We then check:

1) $gxg^{-1} \in \mathcal{H}_0$

So we need to check that $(gxg^{-1})^* = gxg^{-1}$.

$(gxg^{-1})^* = (g^{-1})^* x^* g^*$ because g is unitary so $g^* = g^{-1}$ or $(g^{-1})^* = g$

$= g x^* g^{-1}$

$= g x g^{-1}$ x is Hermitian

We also must show $tr(gxg^{-1}) = 0$

But $tr(gxg^{-1}) = tr(g^{-1}gx) = tr(x) = 0$.

2) $\det(gxg^{-1}) = \det(g)\det(x)\det(g)^{-1}$

$$= \det(x).$$

* We should check that $p: SU(2) \longrightarrow SO(3)$ is 2-1 and onto.

$p(1)(x) = x$

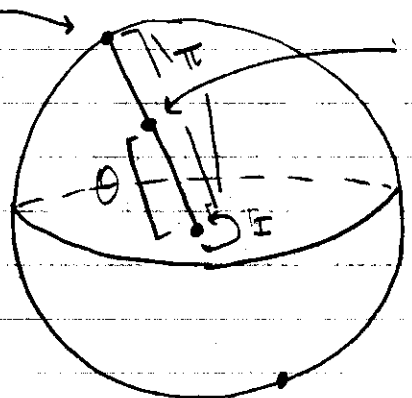
$p(-1)x = -1x(-1)^{-1} = x$

so, $p(1) = p(-1) = x$, so p is at least 2-1, now we have to show it isn't more than 2-1.

Thm: Every rotation in $\mathbb{R}^3$ fixes same axis (is a rotation about same axis by same angle)

pf- uses Hairy Ball Thm or a thm in linear alg about eigenvalues

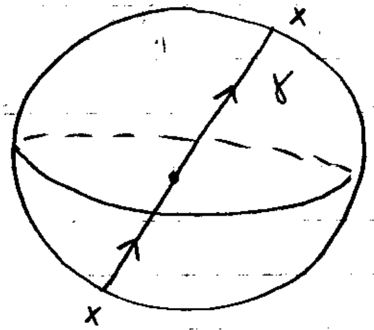
rotation by π around this axis is also



This is rotation by θ counterclockwise around this axis.

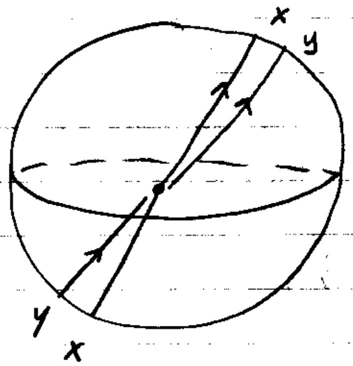
$SO(3) = \mathbb{R}P^3$

the 3-ball w/ antipodal pts on surface identified the coffee cup trick — rotate once w/ coffee cup, that loop looks like: and isn't contractible.



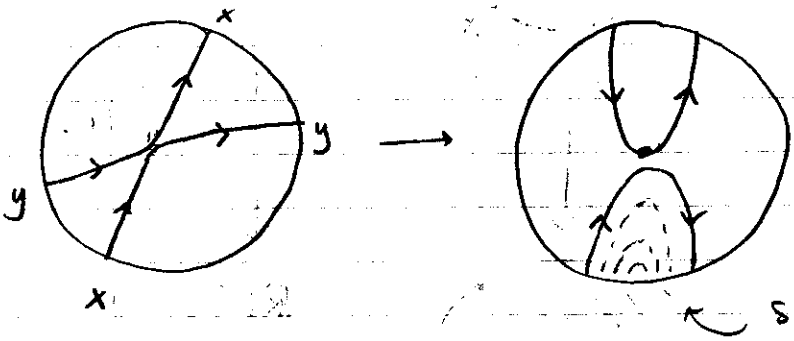
γ isn't contractible

But — doing loop twice, we get something that is contractible.

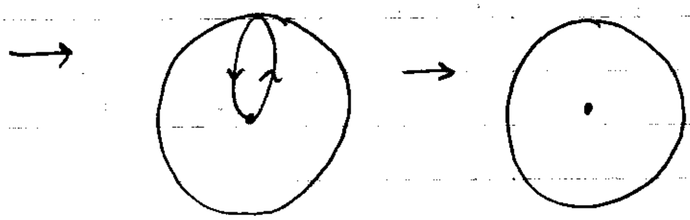
$\gamma^2 =$ 

(we draw it not as loop on top of each other)

take y and pull

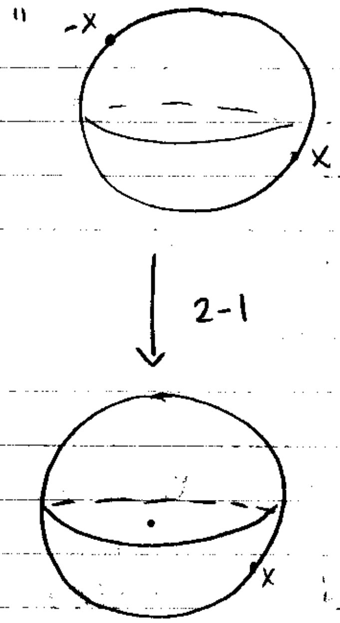


shrink this loop down



This picture is SO(3). Where's SU(2)?
It maps to SO(3) in a 2-1 way.

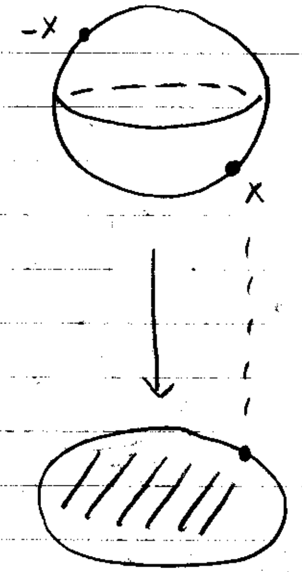
pairs of antipodal pts become identified



$$SU(2) = S^3 \subseteq \mathbb{R}^4 \xrightarrow{p} \mathbb{R}P^3$$

(w/ antipodal pts identified)

1-dimension down, we have:



$$S^2 \xrightarrow{\text{antipodal map}} \mathbb{R}P^2$$

Picture on prev. pg shows that $su(2) = S^3$.

Similarly - $\mathbb{R}^4$ w/ its Minkowski metric is isomorphic to

$$\mathcal{H} = \{ \text{all } 2\times 2 \text{ hermitian complex matrices} \}$$
(drop trace condition)

guys in $\mathcal{H}$ look like:

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$ where $x_i \in \mathbb{R}$

then, det of above matrix

$$\det(X) = (x_0 + x_3)(x_0 - x_3) - (x_1 + ix_2)(x_1 - ix_2)$$
$$= x_0^2 - x_1^2 - x_2^2 - x_3^2$$
$$= -g(x;x)$$ this is Minkowski metric

So - any $g \in SL(2, \mathbb{C})$ acts on $\mathcal{H}$ preserving the Minkowski metric via:

$$\rho(g)x = gxg^*$$ adjoint

before we had "gxg^{-1}" inverse, but we had unitary matrices - ie. $g^* = g^{-1}$.

Check:

1) If $x \in \mathcal{H}$, $g \in SL(2,\mathbb{C})$, $gxg^* \in \mathcal{H}$.

$$(gxg^*)^* = g^{**}x^*g^* = gxg^*$$

2) $$\det(gxg^*) = \underbrace{\det(g)}_{=1}\,\det(x)\,\underbrace{\det(g^*)}_{=\bar{1}}$$ ← conjugate

$$= \det(x)$$

Note: $p(1) = p(-1)$ so p is at least 2-1.
In fact, it's 2-1 and onto.

- If use 2x2 Real matrices – we'd get 3 dimensions, 2x2 Quaternion matrices, we'd get 6, 2x2 octonian matrices 10 dimensions.

Normed division algs:		Weyl Spinors
real	$SL(2,\mathbb{R}) \xrightarrow{2-1} SO_0(2,1)$	$\mathbb{R}^2$
complex	$SL(2,\mathbb{C}) \xrightarrow{2-1} SO_0(3,1)$	$\mathbb{C}^2$
quaternion	$SL(2,\mathbb{H}) \xrightarrow{2-1} SO_0(5,1)$	$\mathbb{H}^2$
octonian	$\star\ SL(2,\mathbb{O}) \xrightarrow{2-1} SO_0(9,1)$	$\mathbb{O}^2$

String theorists like 10-dims.

quaternions – not commut.
octonian – not assoc.