10/3/02

$SL(2, \mathbb{C})$ acts on:
$\llcorner$ 2x2 complex matrices w/ det=1

1) $\mathbb{C}^2$ - "Weyl spinors" via

$$g: \psi \longmapsto g\psi \quad \text{(matrix mult)} \qquad \begin{array}{l} g \in SL(2, \mathbb{C}) \\ \psi \in \mathbb{C}^2 \end{array}$$

2) $\mathcal{H} = \{2\times 2 \text{ Hermitian matrices}\}$ via

Note — need * here

$$g: x \longmapsto gxg^* \in \mathcal{H} \qquad x \in \mathcal{H}$$

(det of member of $\mathcal{H}$ related to Minkowski metric)

Note — every $x \in \mathcal{H}$ is of the form:

$$x = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = \sum_{i=0}^{3} x_i \sigma^i$$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We saw that: $\det(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$

So — $\mathcal{H}$ is Minkowski spacetime.

More precisely $SL(2,\mathbb{C})$ acts on $\mathcal{H}$ preserving det:

$$\det(gxg^*) = \det(x).$$

So we get a homomorphism:

$$p: SL(2,\mathbb{C}) \longrightarrow SO_0(3,1)$$

connected grp

connected component of $SO(3,1)$.

and p is 2-1 and onto.

When $x_0 = 0$ in x, matrix has vanishing trace.

If $\mathcal{H}_0 = \{x \in \mathcal{H} \mid tr(x) = 0\}$ then this is isomorphic to 3-dim'l Euclidean space.

Last time we saw

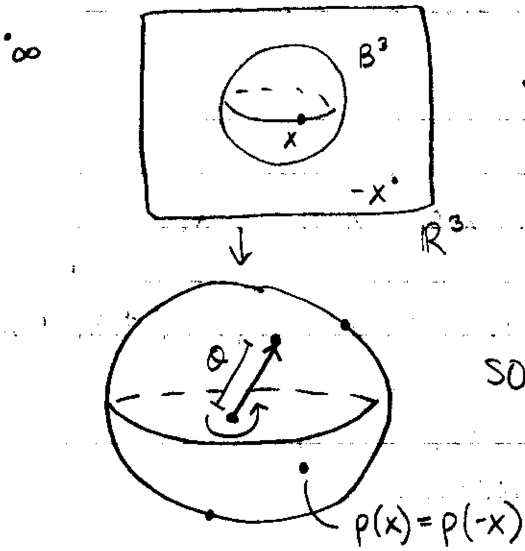
$$SU(2) \subseteq SL(2,\mathbb{C}) \text{ maps } \mathcal{H}_0 \text{ to itself.}$$

2x2 unitary matrices w/ det = 1

so we get: $p: SU(2) \longrightarrow SO(3)$

which is also 2-1 and onto.

SO(3) - grp of rotations in 3-dim'l space
any rotation is rotation about same axis.



S^3

$SO(3) \cong$ 3-ball w/ antipodal pts on boundary identified
$\cong \mathbb{R}P^3$

Always true:

$$\begin{tikzcd} S^3 \arrow[d, two heads, "2\text{-}1"', "\text{onto}"] \\ \mathbb{R}P^3 \end{tikzcd}$$

of spheres to proj. spaces.

same as:

$$\begin{tikzcd} SU(2) \arrow[r, "\sim"] \arrow[d, two heads, "p"', "2\text{-}1"] & S^3 \arrow[d, two heads, "2\text{-}1"] \\ SO(3) \arrow[r, "\sim"] & \mathbb{R}P^3 \end{tikzcd}$$

Why is $SU(2) \cong S^3$?

1) can use fact that $SU(2)$ = unit quaternions
then, it's a unit sphere in 4-dim. space

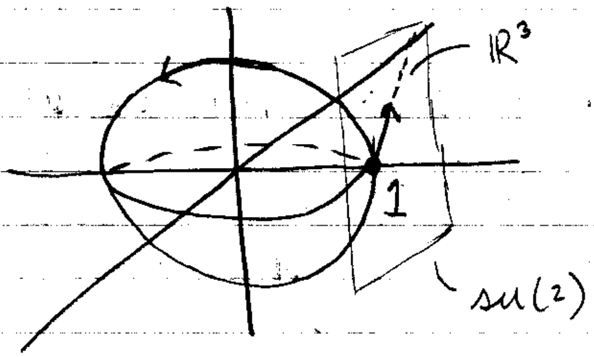
or

σ's – Pauli matrices on prev pg

2) $SU(2) = \{ x_0\sigma^0 + i(x_1\sigma^1 + x_2\sigma^2 + x_3\sigma^3) \mid x_0, \ldots, x_3 \in \mathbb{R} \text{ \& } x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \}$

Check: Check this formula for x holds iff $xx^* = 1$ and $\det(x) = 1$.

(in $\mathbb{R}^4$)



tangent space to $SU(2)$ is Lie alg $\mathfrak{su}(2)$

$\mathfrak{su}(2) = \{ i(x_1\sigma^1 + x_2\sigma^2 + x_3\sigma^3) \mid x_1, x_2, x_3 \in \mathbb{R}^3 \}$

Hermitian matrices w/ trace zero.

$= i\mathcal{H}_0$

* Recall – rotation by 2π in $SO(3)$ is only by π in $SU(2)$ since it's double cover.

Pick a vector in $\mathbb{R}^3$ – Lie alg. We can talk about rotations about this vector. Want to find the elts in $SU(2)$.

· Don't want to trace out a line in the Lie alg, want to turn it into a curve on $SU(2)$.

the elements on great circle (green) correspond to rotations around the vector in Lie alg.

Any $X \in \mathfrak{su}(2)$ gives

$$\exp(X) = 1 + X + \frac{X^2}{2!} + \dots \in SU(2)$$

which does rotation about the vector $X \in \mathbb{R}^3$ (the Lie alg) counterclockwise by angle $2|X|$ as an element of $SO(3)$ via ρ.

*

$SL(2, \mathbb{C})$ acts on $\mathbb{C}^2$ (Weyl spinors) and $\mathcal{H}$ (Minkowski spacetime).

How are they (spinors $\&$ spacetime) related?

The following are all the same:

1) States of a Weyl spinor = unit vectors in $\mathbb{C}^2$ modulo phase
   ↳ unit complex # (elt of $U(1)$)

unit vectors in $\mathbb{C}^2 = \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \; |\psi_1|^2 + |\psi_2|^2 = 1 \right\}$

ψ_1, ψ_2 are complex #'s, so consisting of 4 real #'s.

$\therefore$ unit vectors in $\mathbb{C}^2$ is 3-dim'l sphere! $= S^3$

So – unit vectors in S^3 mod phase is $S^3/U(1)$.

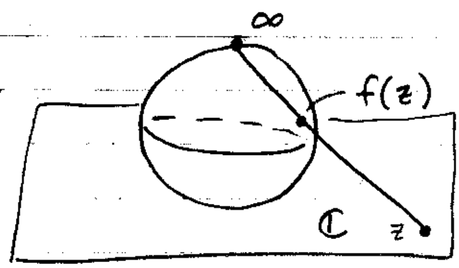
we know this as...

2) The complex projective line: $\mathbb{C}P^1$.

$\mathbb{C}P^1 = \{$ all 1-dim'l complex subspaces of $\mathbb{C}^2 \}$

This is the same since any unit vector (mod phase) determines a 1-dim'l subspace & vice versa.

3) Riemann sphere: $\mathbb{C} \cup \{\infty\}$



stereographic projection (almost onto! hits everything except pt @ infinity)

$f: \mathbb{C} \longrightarrow S^2$ (1-1) & almost onto. preserves angles

We want to show that 2) is equivalent to 3).

Any 1-dim'l subspace of $\mathbb{C}^2$ is of the form:

$$\left\{ \alpha \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

All but one (when $\psi_2 = 0$) of these points can be written as:

(we've divided by ψ_2)

$$\left\{ \alpha \begin{pmatrix} \psi_1 \\ 1 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{C} \right\}$$

But there's one more:

$$\left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

So we have

$$\mathbb{C} \cup \{\infty\} \xrightarrow[\text{onto}]{1\text{-}1} \mathbb{C}P^1$$

$$\psi_1 \in \mathbb{C} \longmapsto \left\{ \alpha \begin{pmatrix} \psi_1 \\ 1 \end{pmatrix} \right\}$$

$$\infty \longmapsto \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

So— states of a spinor are pts on Riemann sphere.

Recall $U(1) \cong S^1$

$$\begin{array}{c} S^3 \\ \Big\downarrow \text{mod out} \\ S^3/U(1) \cong S^2 \end{array}$$

equiv. classes (orbits) which are circles

$S^3/U(1)$ — 2 pts in same orbit if differ by mult by unit complex #.

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d] & S^3 \arrow[d] \\ \{x\} \arrow[r, hook] & S^2 \end{tikzcd}$$

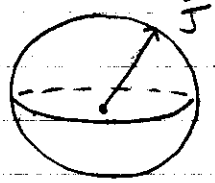
Hopf fibration

- foliating S^3 w/ an S^2's worth of S^1's.

Penrose & Rindler: Spinors & Spacetime vol 1 has a picture

So — all the ways a spinor can spin form S^2.

4) The space of directions a spinor can spin in.



angular momentum

There are angular momentum operators:

$$J_1, J_2, J_3 : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

use them to measure angular momentum of spinors.

$$J_i = \tfrac{1}{2}\sigma_i \quad (i=1,2,3) \quad \text{Pauli matrices}$$

Given a unit spinor ψ, we say its expected angular momentum $\vec{J} \in \mathbb{R}^3$ is the vector w/ components.

hermitian

$$\langle \psi, J_i \psi \rangle$$

expectation value

$\langle\, ,\, \rangle$ is linear in one slot,
conjugate linear in other

eigenvalues of hermitian matrices are all real, so expected value is real.

Check: $\|\vec{J}\| = \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)}$

always the same no matter what ψ is.

Note: $\langle e^{i\theta}\psi, J_i e^{i\theta}\psi\rangle = \langle \psi, J_i \psi\rangle$
(phase doesn't matter)
so we get a map:

$$\underset{\underset{S^2}{\shortparallel}}{\{\text{unit spinors mod phase}\}} \xrightarrow{\cong} \underset{\underset{S^2}{\shortparallel}}{\left\{ \vec{J} \,\middle|\, \|\vec{J}\| = \tfrac{\sqrt{3}}{2} \right\}}$$

5) The set of projections $p: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ w/ $tr(p) = 1$. (Projections onto 1-dim'l subspaces of $\mathbb{C}^2$.)

(project from inner product space onto subspace)

These are linear operators $p: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ w/ $p^2 = p$ & $p^* = p$.
$p^2 = p$ — (that is, we've projected already, so doing it again, we stay where we are). we're also demanding $tr(p) = 1$.

$\left.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right]$ projections

want to get a 1-dim'l subspace

* note - the trace of p = dim. of the space being projected onto.

So we see:

$$\{p : \mathbb{C}^2 \to \mathbb{C}^2 \mid p^2 = p,\ p^* = p,\ tr(p) = 1\} \cong S^2.$$
$$\cap\!|$$
$$\mathcal{H} \text{ (hermitian matrices)}$$

and $\mathcal{H} \cong$ Minkowski spacetime

How do we <u>see</u> this embedding of S^2 in Minkowski spacetime?
We want to see what these p's have in common w/ Minkowski spacetime.

the det. of something in $\mathcal{H}$ is Minkowski metric.
So - let's figure out the determinant of some p?

Recall: $\det(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$

We have $\det(p) = 0$ since p is not 1-1.

$p^2 = p \Rightarrow p$ is id or not 1-1
but $p \neq$ id.

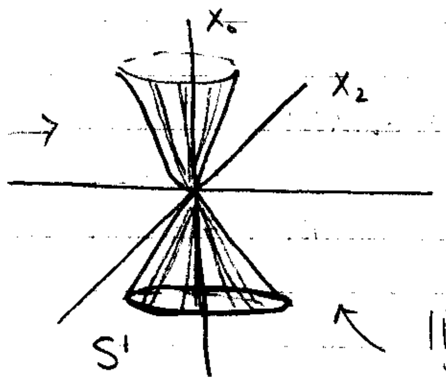
Since $\det(p) = 0$, we're talking about some subset of

$$\{x \in \mathcal{H} \mid x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0\}$$

This is the light cone through origin in Minkowski spacetime.

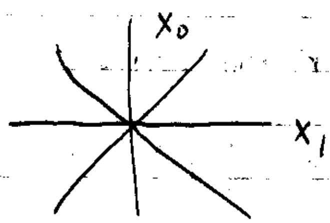
Here we have a circle, S^1, <u>set</u> of all light rays through origin

light rays through origin



light cone – union of light rays through the origin.

2-d lightcone (2 ways for rays of light to go through origin)



$$x_0^2 - x_1^2 = 0$$

Here we have S^0 (2 pts) set of light rays through origin in 2-dim.

In 4-d, the light rays form the set S^2.

<u>Note</u>: In $\mathcal{H}$, the set of light rays through the origin $\cong S^2$,
= "sky" or "heavenly sphere"

Not quite done w/ what we're doing...