SL(2, C) acts on 2x2 complex matrices with det=1

1) $C^2$ - "Weyl spinors" via

$$g : \psi \rightarrow g\psi \quad (\text{matrix mult}) \quad g \in SL(2, C) \quad \psi \in C^2$$

2) $\mathcal{H} = \{2 \times 2 \text{ Hermitian matrices}\}$ via

$$g : x \rightarrow gxg^* \in \mathcal{H} \quad x \in \mathcal{H}$$

(det of member of $\mathcal{H}$ related to Minkowski metric)

Note - every $x \in \mathcal{H}$ is of the form:

$$x = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} = \sum_{i=0}^{3} x_i \sigma^i$$

$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We saw that: $\det(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$

So - $\mathcal{H}$ is Minkowski spacetime.
More precisely, $\text{SL}(2, \mathbb{C})$ acts on $\mathcal{H}$ preserving $\det$:
\[
\det(g \times g^*) = \det(x).
\]
So we get a homomorphism:
\[
p : \text{SL}(2, \mathbb{C}) \rightarrow \text{SO}_0(3,1)
\]
\[
\text{connected component of } \text{SO}(3,1)
\]
and $p$ is 2-1 and onto.

When $x_0 = 0$ in $x$, matrix has vanishing trace.

If $\mathcal{H}_0 = \{ x \in \mathcal{H} \mid \text{tr}(x) = 0 \}$ then this is isomorphic to 3-diml. Euclidean space.

Last time we saw
\[
\text{SU}(2) \leq \text{SL}(2, \mathbb{C}) \quad \text{maps } \mathcal{H}_0 \text{ to itself:}
\]
2x2 unitary matrices
w/ $\det = 1$

so we get:
\[
p : \text{SU}(2) \rightarrow \text{SO}(3)
\]
which is also 2-1 and onto.
$SO(3)$ - group of rotations in 3-dim' space
any rotation is rotation about same axis.

$SO(3) \cong 3$-ball w/ antipodal
pts on boundary identified
$= \mathbb{R}P^3$.

Always true:
$S^3$
$\downarrow$
$\mathbb{R}P^3$
$\cong \frac{2\pi}{\alpha}$

of spheres to proj.
spaces.

same as:
$SU(2) \stackrel{\sim}{\rightarrow} S^3$
$p \downarrow \frac{2\pi}{\alpha}$
$\mathbb{R}P^3$

$SO(3) \stackrel{\sim}{\rightarrow} \mathbb{R}P^3$
Why is $\text{SU}(2) \cong S^3$?

1) can use fact that $\text{SU}(2) = \text{unit quaternions}$ then, it's a unit sphere in 4-dim. space

or

2) $\text{SU}(2) = \{ x_0 \sigma^0 + i (x_1 \sigma^1 + x_2 \sigma^2 + x_3 \sigma^3) \mid x_0, \ldots, x_3 \in \mathbb{R}, \frac{1}{2} (x_0^2 + x_1^2 + x_2^2 + x_3^2) = 1 \}$

Check: Check this formula for $x$ holds iff $xx^* = 1$ and $\det(x) = 1$.

$(\text{in } \mathbb{R}^4)$

tangent space to $\text{SU}(2)$ is Lie alg $\text{su}(2)$

$\text{su}(2) = \{ i (x_1 \sigma^1 + x_2 \sigma^2 + x_3 \sigma^3) \mid x_1, x_2, x_3 \in \mathbb{R}^3 \}$

Hermitian matrices w/ trace zero.

$= \ i \mathcal{H}_0$
Recall - rotation by 2\pi in SO(3) is only by \pi in SU(2) since its double cover.

Pick a vector in \(\mathbb{R}^3\) - Lie alg. We can talk about rotations about this vector. Want to find the elts in SU(2).

Don't want to trace out a line in the Lie alg, want to turn it into a curve on SU(2).

the elements on great circle (green) correspond to rotations around the vector in Lie alg.

Any \(x \in su(2)\) gives

\[
\exp(x) = 1 + x + \frac{x^2}{2!} + \ldots \in SU(2)
\]

which does rotation about the vector \(x \in \mathbb{R}^3\) (the Lie alg) counterclockwise by angle \(2 \cdot 1\cdot 1\) as an element of SO(3) via \(p\).

\(SL(2, \mathbb{C})\) acts on \(\mathbb{C}^2\) (Weyl spinors) and \(H\) (Minkowski spacetime).

How are they (spinors vs. spacetime) related?

The following are all the same:

1) States of a Weyl spinor: = unit vectors in \(\mathbb{C}^2\) modulo phase

unit complex \# (elt of U(1))
unit vectors in \( \mathbb{C}^2 = \{ (y_1, y_2) \mid |y_1|^2 + |y_2|^2 = 1 \} \)

\( y_1, y_2 \) are complex #s, so consisting of 4 real #s.

unit vectors in \( \mathbb{C}^2 \) is 3-dim'l sphere, \( S^3 \)

So, unit vectors in \( S^3 \) mod phase is \( S^3 / U(1) \).

2) The complex projective line: \( \mathbb{C}P^1 \).

\( \mathbb{C}P^1 = \{ \text{all 1-dim'l complex subspaces of } \mathbb{C}^2 \} \)

This is the same since any unit vector (mod phase) determines a 1-dim'l subspace \( \iota \), vice versa.

3) Riemann sphere: \( \mathbb{C} \cup \{ \infty \} \)

stereographic projection (almost onto! hits everything except pt @ infinity) preserves angles

\[ f: \mathbb{C} \rightarrow S^2 \quad (1-1) \text{ at almost onto.} \]

We want to show that 2) is equivalent to 3).
Any 1-dim' subspace of $\mathbb{C}^2$ is of the form:

$$\left\{ \alpha \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) : \alpha \in \mathbb{C} \right\}$$

All but one (when $\psi_2 = 0$) of these points can be written as:

$$\left\{ \alpha \left( \begin{array}{c} \psi_1 \\ 0 \end{array} \right) : \alpha \in \mathbb{C} \right\}$$ (we've divided by $\psi_2$)

But there's one more:

$$\left\{ \alpha \left( \begin{array}{c} 1 \\ 0 \end{array} \right) : \alpha \in \mathbb{C} \right\}$$

So we have:

$$\mathbb{C} \cup \{ \infty \} \xrightarrow{\text{out}} \mathbb{C} P^1$$

$$\psi_1 \in \mathbb{C} \mapsto \left\{ \alpha \left( \begin{array}{c} \psi_1 \\ 0 \end{array} \right) \right\}$$

$$\infty \mapsto \left\{ \alpha \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right\}$$

So states of a spinor are pts on Riemann sphere.

Recall $\text{U}(1) \cong S^1$

$$\mathbb{S}^3 \xrightarrow{\text{mod out}} \mathbb{S}^2$$

$\mathbb{S}^3/\text{U}(1) \cong \mathbb{S}^2$

equiv. classes (orbits) which are circles
$S^3 / \text{U}(1)$ - 2 pts in same orbit if differ by unit complex #.

\[ S^1 \hookrightarrow S^3 \xrightarrow{\text{Hopf fibration}} \{x\} \hookrightarrow S^2 \]

- foliating $S^3$ w/ an $S^2$'s worth of $S^1$'s.

Penrose & Rindler: Spinors & Spacetime vol 1 has a picture

So- all the ways a spinor can spin form $S^2$.

4) The space of directions a spinor can spin in.

![Angular momentum](image)

There are angular momentum operators:

\[ J_1, J_2, J_3 : \mathbb{C} \rightarrow \mathbb{C}^2 \]

use them to measure angular momentum of spinors.

\[ J_i = \frac{1}{2} \sigma_i \quad (i = 1, 2, 3) \quad \text{Pauli matrices} \]

Given a unit spinor $\psi$, we say its angular momentum $J \in \mathbb{R}^3$ is the vector w/ components.

\[ \left< \psi, \hat{J} \psi \right> \quad \text{hermitian} \]
is linear in one slot, conjugate linear in other

eigenvalues of hermitian matrices are all real, so expected value is real.

Check: $\| \overrightarrow{J} \| = \sqrt{\frac{3}{2} (\frac{1}{2} + 1^2)}$

always the same no matter what $\mathbf{y}$ is.

Note: $\langle e^{i \mathbf{a} \cdot \mathbf{y}} \cdot \mathbf{J} \cdot e^{-i \mathbf{a} \cdot \mathbf{y}} \rangle = \langle \mathbf{y} \cdot \mathbf{J} \cdot \mathbf{y} \rangle$

(phase doesn't matter)

so we get a map:

$\begin{align*}
\{ \text{unit spinors mod phase} \} & \longrightarrow \{ \overrightarrow{J} \mid \| \overrightarrow{J} \| = \frac{\sqrt{3}}{2} \}\nS^2 & \rightarrow S^2
\end{align*}$

5) The set of projections $p : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ w/ $\text{tr}(p) = 1$. (Projections onto 1-dim'l subspaces of $\mathbb{C}^2$)

(project from inner product space onto subspace)

These are linear operators $p : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ w/ $p^2 = p \in \mathbb{p}$, $p^* = p$.

(that is, we've projected already, so doing it again, we stay where we are). We're also demanding $\text{tr}(p) = 1$.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projections

* note - the trace of $p$ = dim. of the space being projected onto.

want to get a 1-dim'l subspace
So we see:

\[ \{ \rho : \mathbb{C}^2 \to \mathbb{C}^2 \mid \rho^2 = \rho, \rho^* = \rho, \text{tr}(\rho) = 1 \} \cong S^2. \]

\( \mathcal{H} \) (hermitian matrices)

and \( \mathcal{H} \cong \) Minkowski spacetime

How do we see this embedding of \( S^2 \) in Minkowski spacetime?

We want to see what these \( \rho \)'s have in common w/ Minkowski spacetime.

the det. of something in \( \mathcal{H} \) is Minkowski metric.

So let's figure out the determinant of some \( \rho \)?

Recall: \( \det(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2 \)

We have \( \det(\rho) = 0 \) since \( \rho \) is not 1-1.

\( \rho^2 = \rho \Rightarrow \rho \text{ is id or not 1-1} \)

but \( \rho \neq \text{id} \)

Since \( \det(\rho) = 0 \), we're talking about some subset of

\( \{ x \in \mathcal{H} \mid x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0 \} \)

This is the light cone through origin in Minkowski Spacetime.
Here we have a circle, \( S^1 \), set of all light rays through origin.

Light cone – union of light rays through the origin.

2-d lightcone (2 ways for rays of light to go through origin)

\[ x_0^2 - x_1^2 = 0 \]

Here we have \( S^0 \) (2 pts) set of light rays through origin in 2-dim.

In 4-d, the light rays form the set \( S^2 \).

Note: In \( \mathcal{H} \), the set of light rays through the origin \( \cong S^2 \) = "sky" or "heavenly sphere"

Not quite done w/ what we're doing...