10/8/02

The following are the same:

① $\{\text{States of a Weyl spinor}\} = \{\text{unit vectors in } \mathbb{C}^2\} / \underbrace{U(1)}_{\text{phase}}$

② $\mathbb{C}P^1 = \{\text{1-dim'l subspaces of } \mathbb{C}^2\}$

③ Riemann sphere $= \mathbb{C} \cup \{\infty\}$

④ The sphere of angular momenta

$$= \{\vec{S} \text{ w/ } S_i = \langle \psi, J_i \psi \rangle, \ \psi \in \mathbb{C}^2, \ \|\psi\| = 1\}$$

⑤ $\{p : \mathbb{C}^2 \xrightarrow[\text{linear}]{} \mathbb{C}^2 \mid p^2 = p, \ p^+ = p, \ \operatorname{tr}(p) = 1\}$

$= \{\text{projections onto 1-dim'l subspaces of } \mathbb{C}^2\}$

⑥ Heavenly sphere $= \{$light rays through origin in $\underbrace{\text{4-d Minkowski spacetime}}$ $\}$

$\quad\quad\quad\quad\quad\quad\quad\quad$ ∥

space of $2 \times 2$ Hermitian matrices

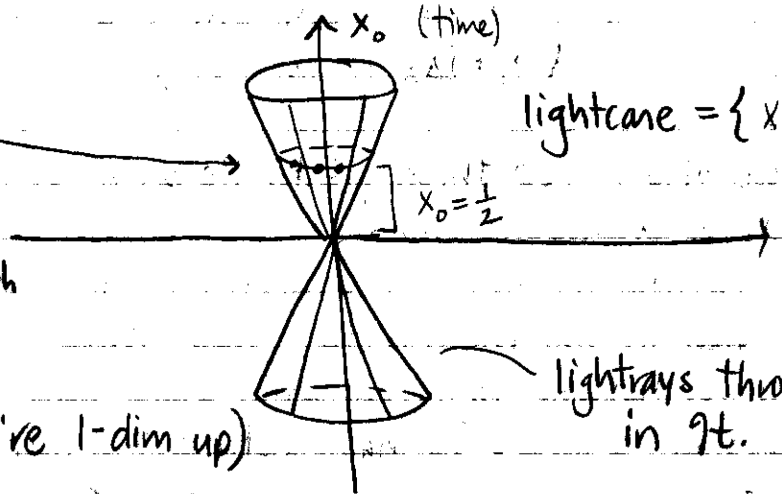
∥

$$\mathcal{H} = \{p : \mathbb{C}^2 \xrightarrow[\text{lin}]{} \mathbb{C}^2 \mid p = p^*\}$$

We want to show ⑤ is the same as ⑥.

So — every projection onto a 1-dim'l subspace of $\mathbb{C}^2$ "is" a point in Minkowski spacetime.

The light rays through the origin in Minkowski spacetime form a cone.




Here they are. One on each light ray through origin. Form a sphere (since we're 1-dim up)

Suppose $p \in \mathcal{H}$.

$p^2 = p \iff$ all (both) the eigenvalues of $p$, $\lambda$ have $\lambda^2 = \lambda$, so $\lambda = 0, 1$.

$p^2 = p$ and $\mathrm{tr}(p) = 1 \iff \lambda = 0, 1$ and sum of both $\lambda$'s is 1.

(Hermitian matrices – trace is sum of eigenvalues)

$\iff$ $p$ has one 0 and one 1 eigenvalue

$\iff$ $p$ is proj. onto 1-dim'l subspace.

· det is product of eigenvalues

$\Rightarrow \det p = 0$ (since det is product of eigenvalues)

So - all these proj. lie on lightcone.

Note -

$$\operatorname{tr}\begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = 2x_0$$

So we also know these points lie on a sphere where $x_0 = \frac{1}{2}$

($\Leftarrow$) If you lie on the circle, your trace $= 1$, $\det = 0$

Conversely, if $p \in \mathcal{H}$ has $\det(p) = 0$, $\operatorname{tr}(p) = 1$ then product of eigenvalues is $0$,
sum " " is $1$

then $p^2 = p$.

$$\begin{aligned} \lambda_1 + \lambda_2 &= 1 \\ \lambda_1 \lambda_2 &= 0 \end{aligned} \Rightarrow \begin{aligned} \text{one } \lambda &= 0 \\ \lambda &= 1 \end{aligned}$$

This means we have a projection since $0^2 = 0$, $1^2 = 1 \Rightarrow p^2 = p$. ☒

Since we've described the sphere just in terms of $\mathcal{H}$ and det, we know $SL(2, \mathbb{C})$ acts on this sphere.

use –
How $SL(2,\mathbb{C})$ acts on $\mathcal{H}$
translate to:

$\boxed{HW}$: Show that if we think of this sphere as Riemann sphere $\mathbb{C} \cup \{\infty\}$

$g \in SL(2,\mathbb{C})$ acts by:

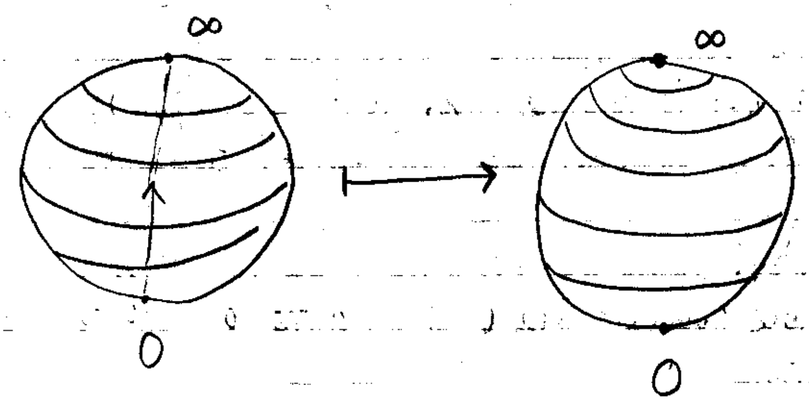
$$z \longmapsto \frac{az+b}{cz+d}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathbb{C} \cup \{0\}$

In complex analysis, we learn these are all the conformal transformations of Riemann sphere.

So: if you whiz along at high speeds, you'll see the constellations distorted, but by an angle-preserving transformation.

We're traveling ↑




so we see this!

Here, $g = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ $\alpha > 1$

$g: z \longmapsto \alpha^2 z$ $\alpha^2 > 1$

Dilation of $\mathbb{C}^2$

But Riemann sphere (squashing up)

All of ① - ⑥ generalize straightforwardly if we replace "$\mathbb{C}$" by any normed division algebra.

(Can divide, doesn't need to be assoc. or commut, has $|xy| = |x||y|$)

| | |
|---|---|
| 1 dim'l | – $\mathbb{R}$ |
| 2 dim'l | $\mathbb{C}$ |
| 4 dim'l | $\mathbb{H}$ |
| 8 dim'l | $\mathbb{O}$ |

$\mathbb{C}^2$ is Weyl spinor

$SL(2,\mathbb{C})$ acts on $\mathbb{C}^2$
↪ double cover of $SO_0(3,1)$

$\mathbb{C}^2$ are "Weyl spinors" are reps of $SL(2,\mathbb{C})$, which is a double cover of $SO_0(3,1)$ (connected component of $SO(3,1)$), and $SO_0(3,1)$ which acts on 4d Minkowski spacetime.

$\mathbb{R}^2$ are Weyl spinors, are reps of $SL(2,\mathbb{R})$
$SL(2,\mathbb{R})$ is double cover of $SO_0(2,1)$,
$SO_0(2,1)$ acts on 3d Minkowski spacetime.

$\mathbb{H}^2$ - Weyl spinors, reps of $SL(2,\mathbb{H})$
double covers $SO_0(5,1)$, acts on 6-d Mink spacetime

$\mathbb{O}^2$ – Weyl spinors, reps of $SL(2,\mathbb{O})$, double covers $SO_0(9,1)$, acts on 10-d Mink spacetime.

◯ #'s are 1 more than dim. of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Consider $\{$1-dim'l subspaces of $\mathbb{C}^2\}$ = the sphere.

Any point in it is: $\left\{ \alpha \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \, \alpha \in \mathbb{C} \right\}$ $\quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \neq 0$

$$\|$$

Divide by $\psi_1$ $\quad \left\{ \alpha \begin{pmatrix} 1 \\ z \end{pmatrix} \middle| \, \alpha \in \mathbb{C} \right\}$ $\quad z = \dfrac{\psi_2}{\psi_1}$

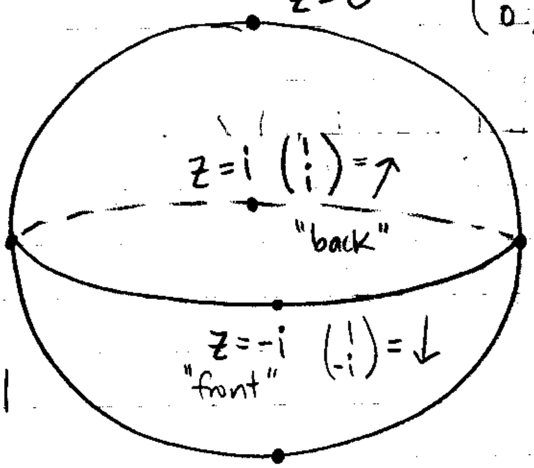
This is how we think of a point in this space ($z$).

Unless – if $\psi_1 = 0$, in which case we get

$$\left\{ \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle| \, \alpha \in \mathbb{C} \right\}$$

We call $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ an "up" spinor. (angular momentum points up)

$z = 0$ $\quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftarrow$ a vector that spans this 1-dim'l subspace




$z = 1$ $\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $= \longrightarrow$ $\quad$ (not normalized, not a unit vector)

$z = -1$

$\longleftarrow = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$z = \infty$ $\quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$ "down" $= \downarrow$

$z = 1$ gives $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

sphere of angular momenta in 3-d space

Given one of these, say $\begin{pmatrix}1\\1\end{pmatrix}$, we can work out the angular momentum vector $\vec{S} \in \mathbb{R}^3$.

Normalize: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \psi$; then work out

check this!

$$S_i = \langle \psi, J_i \psi \rangle$$

$$J_1 = \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad J_2 = \frac{1}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$$

$$J_3 = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad \text{(Pauli matrices)}$$

$J_1$ - angular momentum in $x$-direction.

Let's work it out:

$$S_1 = \frac{1}{2}\left\langle \begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\rangle$$

$$= \frac{1}{4}\left\langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\rangle = \frac{1}{2}$$

$$\Rightarrow S_1 = \frac{1}{2}$$

$$\frac{1}{4}\left\langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\rangle$$

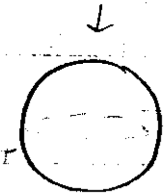
$$= \frac{1}{4}\left\langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-i\\i\end{pmatrix}\right\rangle = \frac{1}{4}(-i+i) = 0 \Rightarrow S_2 = 0$$

and $S_3 = 0$

So – for $\begin{pmatrix}1\\1\end{pmatrix}$, $S = \frac{1}{2}(1,0,0)$

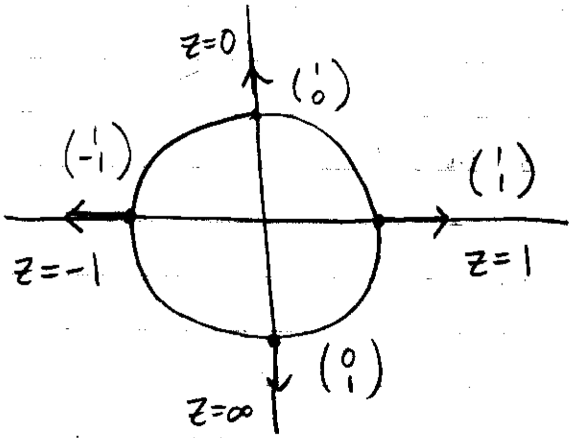
meaning in pts in x-direction.

Our sphere lives in $\mathbb{R}^3$, but the circle lives in $\mathbb{R}^2$, we have a circle's worth of angular momenta.



We could copy the whole story w/ $\mathbb{R}$ instead of $\mathbb{C}$ and get "Riemann circle".

sphere of angular momenta in 2-d space




This is related to 3-d Mink. spacetime.

$SL(2,\mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$ via $z \longmapsto \frac{az+b}{cz+d}$

$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{R})$

W/ quaternions, $\mathbb{H}$, we'd have $0, \infty, 1, -1, i, -i, j, -j, k, -k$.

So we get a sphere

$\mathbb{H} \cup \{\infty\}$ (a 4-sphere sitting in 5 dim'l space) which is 6-dim'l Mink. spacetime.

cross-product in $\mathbb{R}^7$ comes from octonians?
in $\mathbb{R}^3$ comes from quaternions

For $\mathbb{O}$, we get sphere $\mathbb{O} \cup \{\infty\}$ in 9d space;
10 d Mink. spacetime.

<u>Problem</u>: ④ may not really generalize!

e.g. for $\mathbb{O}$ are there really "ang. mom. ops"
$J_1, \ldots, J_7$.